in Top: $X \xrightarrow{f} Y$ is a weak equiv. if $f_*: \pi_* X \to \pi_* Y$ is isomorphism

Ch: $C_\bullet \xrightarrow{f} D_\bullet$ " $f_*: H_* C_\bullet \to H_* D_\bullet$ is an isomorphism

$D: I \to \mathcal{C} = \mathrm{Top}$ or Ch

diagram

$\Longrightarrow$ can construct $\underset{I}{\mathrm{hocolim}}\, D \in \mathcal{C}$

point of hocolims (as opposed to colim):

compatibility with weak equivalences:

$D \xrightarrow{T} D'$ is w.e. <u>objectwise</u> $\Longrightarrow$ $\underset{I}{\mathrm{hocolim}}\, D \xrightarrow{\text{w.e.}} \underset{I}{\mathrm{hocolim}}\, D'$

$D(i) \xrightarrow[\text{w.e.}]{T(i)} D'(i) \quad \forall i \in I$

Correction: this condition is not needed!

[assuming that D, D' are objectwise cofibrant, i.e. $D(i)$, $D'(i)$ are cofibrant, i.e. these are CW-complexes resp. projective chaincxs.

here: Ch is the category of chaincx.

$C_\bullet$: $\leftarrow C_i \leftarrow C_{i+1} \leftarrow C_{i+2} \leftarrow$

where C_i is a module over a comm. ring R.

$C_\bullet$ is projective $:\Leftrightarrow$ each C_i is a projective, f.g. R-module

direct summand of a free module.

Recall: A prefactorization algebra

$F : \mathrm{Open}(M) \longrightarrow \mathcal{C}$ is called

a (strict) factorization algebra if :

- multiplicative property:

  U_1, U_2 disjoint, $U = U_1 \sqcup U_2$

  $m^{U}_{U_1, U_2} : F(U_1) \otimes F(U_2) \xrightarrow{\cong} F(U)$

  tensor prod in the symm. monoidal cat. $\mathcal{C}$

- locality / Weiss-descent / Weiss cosheaf property: $F(U) \xleftarrow{\cong} \mathrm{coeq}\left(\coprod_a F(U_a) \leftleftarrows \coprod_{a,b} F(U_a \cap U_b) \right)$

  if $\{U_a\}_{a \in A}$ Weiss cover of $U \subset M$

to modify the descent condition:

$\{U_a\}_{a\in A}$ cover of U, then obtain a diagram

$$(*) \quad \bigoplus_{a_0\in A} F(U_{a_0}) \rightleftarrows \bigoplus_{a_0,a_1} F(U_{a_0}\cap U_{a_1}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \bigoplus_{a_0,a_1,a_2} F(U_{a_0}\cap U_{a_1}\cap U_{a_2}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}$$

shape of this diagram: Δ^{op}

<u>Def:</u> <u>simplex category</u> Δ

<u>objects</u>: $\{0,1,\dots,n\} =: [n]$ $\quad n=0,1,2,\dots$

$\{$<u>morphisms</u> from $[m]$ to $[n]\} \doteq \{ f:[m]\to[n] \ / \ f$ order preserving$\}$

picture of Δ:

$$[0] \;\substack{\xrightarrow{\delta_0}\\ \longleftarrow\\ \xrightarrow[\delta_1]{}}\; [1] \;\substack{\xrightarrow{\delta_0}\\ \longleftarrow\\ \xrightarrow{\delta_1}\\ \longleftarrow\\ \xrightarrow[\delta_2]{}}\; [2] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; [3]$$

$\delta_i : [n] \rightarrowtail [n+1]$

$i \notin$ image of δ_i

$\sigma_i : [n] \twoheadrightarrow [n-1]$ $\qquad \sigma_i(i) = \sigma_i(i+1) = i$

<u>Def</u>: a diagram of shape Δ^{op}, i.e., a functor $F : \Delta^{op} \longrightarrow \mathcal{C}$ is a <u>simplicial object</u> in $\mathcal{C}$.

For example, $(*)$ is a simplicial object in $\mathcal{C}$.

Suppose $\mathcal{C} = \mathrm{Vect}$.

$$\operatorname*{colim}_{\Delta^{op}}(*) = \operatorname{colim}\Big(\bigoplus_{a_0} F(\mathcal{U}_{a_0}) \leftleftarrows \bigoplus_{a_0, a_1} F(\mathcal{U}_{a_0} \cap \mathcal{U}_{a_1})\Big)$$

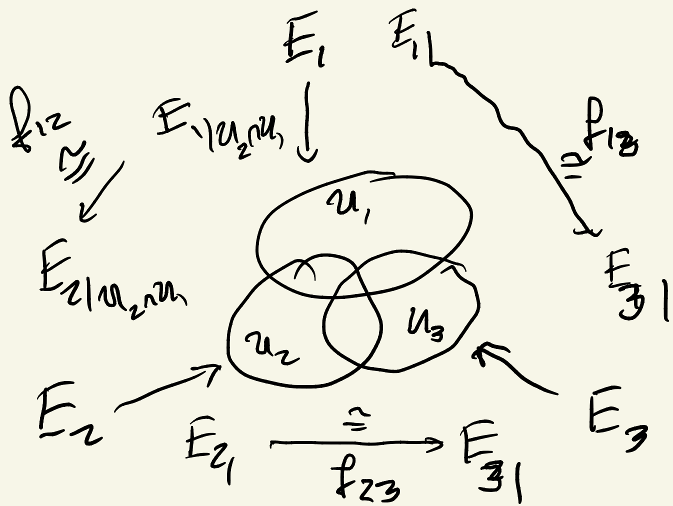
$$\qquad\qquad\qquad\quad \| \qquad\qquad\qquad\qquad\qquad\quad \|$$

$$\operatorname{coeq}\Big(\qquad\qquad\qquad\qquad \Big)$$

Q: Why do we need higher intersections?

Q: How can we construct the groupoid of vectorbdls on $\mathcal{U}$ in terms of the groupoids of vectorbdls over $\mathcal{U}_a$, $\mathcal{U}_{a_0} \cap \mathcal{U}_{a_1}, \ldots$?

$\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$



you need compatibility over $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$:

$$f_{13} = f_{23} \circ f_{12}$$

on $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$.

$E_i \to \mathcal{U}_i$ vectorbdl.

Def: $F : \mathrm{Open}(M) \to \mathcal{C} = \mathrm{Top}$ or Ch

prefactorization algebra

is a factorization algebra if

- multiplicative property: $\mathcal{U} = \mathcal{U}_1 \sqcup \mathcal{U}_2$

$$m^{\mathcal{U}}_{\mathcal{U}_1, \mathcal{U}_2} : F(\mathcal{U}_1) \otimes F(\mathcal{U}_2) \xrightarrow[\text{w.e.}]{\sim} F(\mathcal{U})$$

for Top this is the Cartesian product

- Weiss cosheaf property

if $\mathfrak{U} = \{\mathcal{U}_a\}_{a \in A}$ is a Weiss cover of $\mathcal{U}$, then

simplicial obj. assoc. to $\mathfrak{U}$

$$F(\mathcal{U}) \xleftarrow[\text{w.e.}]{\sim} \operatorname*{hocolim}_{\Delta^{op}} (*)$$

Digression: $\underset{\Delta^{op}}{\mathrm{hocolim}}\, X_\bullet$ in Top

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$X_\bullet : \Delta^{op} \to \mathrm{Top}$ simplicial space.

Def: geometric realization

$$|X_\bullet| = \Big(\coprod_{n=0,1,\dots} X_n \times \Delta^n \Big) \Big/ \sim$$

Δ^n = standard n-simplex = convex hull of $e_0, \dots, e_n \in \mathbb{R}^\infty$

$[m] \xrightarrow{f} [n]$ morphism in Δ $\Rightarrow$ $\Delta^m \xrightarrow{f_*} \Delta^n$, $e_i \longmapsto e_{f(i)}$, $X_m \xleftarrow{f^*} X_n$

$x \in X_n$, $t \in \Delta^m$

$$\underset{X_m \times \Delta^m}{\underset{\cap}{(f^* x, t)}} \sim \underset{X_n \times \Delta^n}{\underset{\cap}{(x, f_* t)}} \quad \text{for every } x, t, f.$$

$$|X_\bullet| = \mathrm{coeq}\left(\coprod_n X_n \times \Delta^n \leftleftarrows \coprod_{[m]\xrightarrow{f}[n]} X_n \times \Delta^m \right)$$

$$(f^*x, t) \longleftarrow (f, x, t)$$

$$(x, f_*t) \longleftarrow (f, x, t)$$