<u>Def:</u> A presheaf on a manifold M is a functor ← contravariant

$$F : \mathrm{Open}(M) \longrightarrow \mathcal{C}$$ category

<u>obj:</u> $U \subset_{open} M$

<u>mor:</u> $i_U^V : U \hookrightarrow V$

If $\mathcal{C} = \mathrm{Set}, \mathrm{Top}, \mathrm{Vect}$

Set ↑ cat. of sets; Top ↑ top. spaces

$V \subset_{open} M$ $\quad \{U_a\}_{a\in A}$ open cover of V

$i_a = i_{U_a}^V : U_a \hookrightarrow V \qquad i_{ab}^a : U_a \cap U_b \hookrightarrow U_a$

$$F(V) \xrightarrow[(*)]{\prod F(i_a)} \mathrm{equal}\left(\prod_{a\in A} F(U_a) \underset{\prod_{a<b} F(i_{a,b}^b)\circ p_b}{\overset{\prod_{a<b} F(i_{a,b}^a)\circ p_a}{\rightrightarrows}} \prod_{a<b} F(U_a \cap U_b) \right)$$

$$p_a : \prod_{a\in A} F(U_a) \to F(U_a)$$

$$\mathrm{equal}\left(X \overset{f}{\underset{g}{\rightrightarrows}} Y\right) \underset{\text{in Vect}}{\cong} \mathrm{Ker}(f-g : X \to Y)$$

$$\| \quad \{x \in X \mid f(x) = g(x)\}$$

F is a <u>sheaf</u> if for <u>all</u> open covers of V,

the map $(*)$ is an isomorphisms.

<u>sheaf condition</u> or <u>locality</u> or <u>descent condition</u>

<u>observation</u>: if $V = \bigcup_{a \in A} U_a$ with U_a's mutually disjoint,

then $F(V) \cong \prod_{a \in A} F(U_a)$.

<u>claim</u>: $F(U) = \Omega^k_{exact}(U)$ is <u>not</u> a sheaf.

<u>example</u>: $M = S^1$ $\quad S^1 = \{U_1, U_2\}$ open cover $\quad U_1 = S^1 \setminus \{1\}$, $U_2 = S^1 \setminus \{-1\}$

$$\Omega^1_{ex}(S^1) \xrightarrow{(*)} \mathrm{equal}\left(\Omega^1_{ex}(U_1)\times\Omega^1_{ex}(U_2) \rightrightarrows \Omega^1_{ex}(U_{12})\right)$$

$\omega \in \Omega^1(S^1)$ s.t. $\int_{S^1}\omega \neq 0$

$\Rightarrow \omega \notin \Omega^1_{ex}(S^1)$

$$\mathrm{Ker}\left(\Omega^1_{ex}(U_1)\times\Omega^1_{ex}(U_2) \xrightarrow{(i^1_{12})^* - (i^2_{12})^*} \, u\right)$$

$$(\omega_{|U_1}, \omega_{|U_2}) \longmapsto 0$$

exact

but $(\omega_{|U_1}, \omega_{|U_1})$ is not in the image of $(*)$

recall: $\Omega^k(M) = \Gamma(M, \Lambda^k T^*M)$

$$\downarrow d = \text{deRham diff}$$

$$\Omega^{k+1}(M)$$

$$\Omega^k_{ex}(M) = \mathrm{im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right)$$

$$H^k_{dR}(M) = \frac{\Omega^k_{closed}(M)}{\Omega^k_{exact}(M)}$$

Question: Is $F(U) = H^k_{dR}(U)$ a sheaf?

(it is a presheaf)

First test: $V = U_1 \sqcup U_2$ ← disjoint union

$$H^k_{dR}(M) \cong H^k_{dR}(U_1) \times H^k_{dR}(U_2)$$

second test: $V = \mathcal{U}_1 \cup \mathcal{U}_2$

$$H^k_{dR}(V) \xrightarrow{(*)} \operatorname{Ker}\left(H^k_{dR}(\mathcal{U}_1) \times H^k_{dR}(\mathcal{U}_2) \xrightarrow{i_1^* - i_2^*} H^k_{dR}(\mathcal{U}_1 \cap \mathcal{U}_2)\right)$$

equal (. ----)

$i_1 : \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathcal{U}_1$

$\xrightarrow{i_2} \mathcal{U}_2$

(*) is an isom.

$\Updownarrow$

$$H^k_{dR}(V) \overset{(*)}{\rightarrowtail} H^k_{dR}(\mathcal{U}_1) \times H^k_{dR}(\mathcal{U}_2) \xrightarrow{i_1^* - i_2^*} H^k_{dR}(\mathcal{U}_1 \cap \mathcal{U}_2)$$

injectivity + sequence is exact at

In general, this map is <u>not</u> injective;

$$\operatorname{Ker}(*) = \operatorname{im}\left(\delta : H^{k-1}_{dR}(\mathcal{U}_1 \cap \mathcal{U}_2) \to H^k_{dR}(V)\right)$$

<u>Rem</u>: $F : \operatorname{Open}(M) \longrightarrow Ch \leftarrow$ category of chain cx.

obj: chain cx

mor: " maps

The construction of products + equalizers for vector spaces extends to Ch.

$$\to C_1 \xrightarrow{d} C_2 \xrightarrow{d} C_3 \xrightarrow{d} C_4 \to$$

$$\updownarrow$$

$$C = \bigoplus_{i \in \mathbb{Z}} C_i \qquad d: C \to C \text{ has degree } +1$$

$\mathbb{Z}$-graded vector space / ab. group

Q: $F(U) := (\underbrace{\Omega^*(U), d}_{\text{de Rham complex}}) \in Ch$

is that a sheaf?

A: Yes, since $\Omega^k(U)$ is!

Q: Is there a condition on $F: Open(M) \to Ch$ that guarantees that you get Mayer-Vietoris sequences for $V = U_1 \cup U_2$?

A: need that $F: Open(M) \to Ch$ is a homotp. sheaf.

<u>digression</u>: product + equalizers in general categories $\mathcal{C}$

Set = cat. of sets

$X_a \in Set$ collection of sets $a \in A$

product: $X = \prod_{a\in A} X_a \xrightarrow[\text{projection}]{p_a} X_a$

$$Z \xrightarrow{f} X \longmapsto \underbrace{p_a \circ f : Z \to X_a}_{\text{component functions of } f}$$

$$Set(Z, X) \xrightarrow{\cong} \prod_{a\in A} Set(Z, X_a)$$

morphisms in Set from Z to X $\quad f \longmapsto p_a \circ f$

this is the universal property of $\prod_{a\in A} X_a$.

Def: $\mathcal{C}$ category $X_a \in \operatorname{ob}\mathcal{C}$, $a \in A$

The <u>categorical product</u> of the X_a is an object $X \in \operatorname{ob}\mathcal{C}$, together with $X \xrightarrow{p_a} X_a$ s.t.

$$\mathcal{C}(Z, X) \xrightarrow{\cong} \underbrace{\prod_{a \in A} \mathcal{C}(Z, X_a)}_{\text{product of } \underline{\text{sets}}}$$

this is a set!

$Z \in \operatorname{ob}\mathcal{C}$ $\qquad f \longmapsto (p_a \circ f)_a$

<u>Notation:</u> $X := \prod_{a \in A} X_a$

<u>Remarks:</u>
1) a cat. product might not exist.
2) if categorical products exist, say (X, p_a), (X', p_a'), then they are <u>unique</u> up to isomorphism,

i.e.

$$\begin{tikzcd} X \arrow[r, "p_a"] \arrow[d, "f"', "\cong"] & X_a \\ X' \arrow[ur, "p'_a"'] & \end{tikzcd}$$

commutes for every $a \in A$.

Q: Cat product in Open(M)?