# Factorization algebras

$M$ mfd of dim $n$

construction: $M \underset{\text{open}}{\supset} \mathcal{U} \longmapsto F(\mathcal{U})$ math. object
e.g. set, vector space
top. space,
chain cx.

for $\mathcal{U} \hookrightarrow V \rightsquigarrow$ map between $F(\mathcal{U})$ and $F(V)$

<u>Q</u>: suppose $\{\mathcal{U}_a\}_{a\in A}$ is an open covering of $V \subset M$

Can you recover $F(V)$ from knowing $F(\text{intersections of } \mathcal{U})$ and maps between these induced by inclusions?

list of examples of functors $F : \mathrm{Open}(M) \longrightarrow \mathcal{C}$

general category

objects : $\mathcal{U} \underset{\text{open}}{\subset} M$

unique morph. $\mathcal{U} \to V$

$\Updownarrow$

$\mathcal{U} \subset V$

1) $F(\mathcal{U}) = C^\infty(\mathcal{U})$ vector space

2) $X$ smooth mfd ; $F(\mathcal{U}) = C^\infty(\mathcal{U}, X) = \{ f : \mathcal{U} \to X \,/\, \text{smooth} \}$ set

3) $F(\mathcal{U}) = C_c^\infty(\mathcal{U}) = \{ \text{compactly supported functions} \}$ vector space covariant!

$\mathrm{supp}(f : \mathcal{U} \to \mathbb{R}) = \text{closure}(\{ x \in \mathcal{U} \,/\, f(x) \neq 0 \})$

4) $X$ top. space , basepoint $* \in X$

$F(\mathcal{U}) = \mathrm{map}_c(\mathcal{U}, X)$ top. space ( compact-open topology)

$\mathrm{supp}(f : \mathcal{U} \to X) = \mathrm{clos.}(\{ x \in \mathcal{U} \,/\, f(x) \neq * \})$ covariant!

5) $E \xrightarrow{p} M$ vector bdl

$$F(\mathcal{U}) := \Gamma(\mathcal{U}, E) = \left\{ \text{smooth } f : \mathcal{U} \to E \;\middle|\; \begin{array}{l} f \text{ is a } \underline{\text{section}}, \\ \text{i.e. } p \circ f = \mathrm{id}_{\mathcal{U}} \end{array} \right\}$$

vector space

$$F(\mathcal{U}) = \Omega^k(\mathcal{U}) = \Gamma(\mathcal{U}, \Lambda^k T^* M)$$ k-forms

$$F(\mathcal{U}) = (\Omega^*(\mathcal{U}), d)$$ ← chain cx (the dR-complex of $\mathcal{U}$)

$$F(\mathcal{U}) = H^k_{dR}(\mathcal{U})$$ k-th de Rham cohomology

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6) $F(\mathcal{U}) = \mathrm{Conf}(\mathcal{U}) = \left\{ \begin{array}{c} \text{configurations of} \\ \text{finitely many points} \\ \text{in } \mathcal{U} \end{array} \right\}$ top. space

$$F(\mathcal{U}) = C_*(\mathrm{Conf}(\mathcal{U}))$$

↑ singular chain. cx.

$$F(\mathcal{U}) = H_*(\mathrm{Conf}(\mathcal{U}))$$

7) $F(\mathcal{U}) = \mathrm{Obs}(\mathcal{U}) = \left\{ \begin{array}{l} \text{observables measurable} \\ \text{in } \mathcal{U}_i \subset M \\ \qquad\qquad\quad \llcorner \text{ "spacetime"} \end{array} \right\}$

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Ex 1 in detail: $F(\mathcal{U}) = C^\infty(\mathcal{U})$ vector space

$\mathcal{U}_1, \mathcal{U}_2 \in \mathrm{Open}(M)$

$$\begin{tikzcd}
V = \mathcal{U}_1 \cup \mathcal{U}_2 & \mathcal{U}_1 \arrow[l, "j_1"'] & \\
& & \mathcal{U}_1 \cap \mathcal{U}_2 \arrow[ul, "i_1"'] \arrow[dl, "i_2"] \\
& \mathcal{U}_2 \arrow[uul, "j_2"] &
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
& C^\infty(\mathcal{U}_1) \arrow[dr, "i_1^*"] & \\
C^\infty(V) \arrow[ur, "j_1^*"] \arrow[dr, "j_2^*"'] & & C^\infty(\mathcal{U}_1 \cap \mathcal{U}_2) \\
& C^\infty(\mathcal{U}_2) \arrow[ur, "i_2^*"'] &
\end{tikzcd}$$

$\Rightarrow$ exact sequence: $0 \to C^\infty(V) \xrightarrow{j_1^* \times j_2^*} C^\infty(\mathcal{U}_1) \times C^\infty(\mathcal{U}_2) \xrightarrow{i_1^* - i_2^*} C^\infty(\mathcal{U}_1 \cap \mathcal{U}_2) \to 0$

$$C^\infty(V) \xrightarrow{\cong} \operatorname{Ker}\left( C^\infty(\mathcal{U}_1) \times C^\infty(\mathcal{U}_2) \xrightarrow{i_1^* - i_2^*} C^\infty(\mathcal{U}_1 \cap \mathcal{U}_2)\right)$$

generally : $\{\mathcal{U}_a\}_{a \in A}$ open cover of $V$

totally ordered

$$C^\infty(V) \xrightarrow{\text{restriction}} \prod_{a \in A} C^\infty(\mathcal{U}_a) \overset{g}{\underset{h}{\rightrightarrows}} \prod_{\substack{a<b \\ a,b \in A}} C^\infty(\mathcal{U}_a \cap \mathcal{U}_b)$$

$$f \longmapsto (f|\mathcal{U}_a)_{a \in A}$$

$$(f|\mathcal{U}_a)_{a \in A} \in \ker(g-h)$$

two maps given by

$$(f_a)_{a \in A} \longmapsto (g_{ab})_{a<b}$$

$$(f_a)_{a \in A} \longmapsto (h_{ab})_{a<b}$$

$$f_a \in C^\infty(\mathcal{U}_a)$$

$$C^\infty(V) \xrightarrow{\cong} \ker(g-h)$$

$$g_{ab} := f_a|\mathcal{U}_a \cap \mathcal{U}_b$$

$$h_{ab} := f_b|\mathcal{U}_a \cap \mathcal{U}_b$$

More generally if $F(\mathcal{U}) = C^\infty(\mathcal{U}, X)$ ← smooth mfld (Ex 2)

then
$$C^\infty(V, X) \xrightarrow{\cong} \text{equal}\left(\prod_{a\in A} C^\infty(\mathcal{U}_a, X) \underset{h}{\overset{g}{\rightrightarrows}} \prod_{a<b} C^\infty(\mathcal{U}_a \cap \mathcal{U}_b, X)\right)$$

<u>Def:</u> $A, B$ sets $g, h : A \to B$

<u>equalizer</u> of $g$ and $h$ is defined by

$$\text{equal}\left(A \underset{h}{\overset{g}{\rightrightarrows}} B\right) = \{a \in A \,/\, g(a) = h(a)\}$$

<u>Def:</u> A <u>presheaf on M with values in a category $\mathcal{C}$</u> is a contravariant functor

$$F : \text{Open}(M) \longrightarrow \mathcal{C}$$

A presheaf is a sheaf if for any cover $\{\mathcal{U}_a\}_{a\in A}$ of $V \subset M$ the map

$$F(V) \longrightarrow \mathrm{equal}\Big(\prod_{a\in A} F(\mathcal{U}_a) \rightrightarrows \prod_{a<b} F(\mathcal{U}_a \cap \mathcal{U}_b)\Big)$$

is an isomorphism.

This requires that the category $\mathcal{C}$ admits categorical products + equalizers (we constructed them explicitly in $\underbrace{\text{Set \& Vect}}_{\text{categories}}$; in general they are characterized by universal properties)

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Examples, non-examples:

1. $F(\mathcal{U}) := \Gamma(\mathcal{U}, E)$ $\qquad E \to M$ vector bdl.
   sheaf $\qquad$ in particular: $\Omega^k(\mathcal{U}) = F(\mathcal{U})$
   sheaf of $k$-forms on $M$

2. $\Omega^k_{\text{closed}}(\mathcal{U}) = \{ \omega \in \Omega^k(\mathcal{U}) \mid d\omega = 0 \}$

de Rham diff

$\Omega^k_{\text{exact}}(\mathcal{U}) = \text{image}\left( d : \Omega^{k-1}(\mathcal{U}) \to \Omega^k(\mathcal{U}) \right)$

sheaves? $\Omega^k_{\text{closed}}(\mathcal{U})$ is sheaf

$\Omega^k_{\text{exact}}(\mathcal{U})$ is not!

counterexample: $V = S^1$ open cover of $V$: $\mathcal{U}_1 = S^1 \setminus \{1\}$
$\mathcal{U}_2 = S^1 \setminus \{-1\}$

$\mathbb{R} = H^1_{dR}(S^1)$