<u>Recall</u>: prefactorization alg. $\mathcal{F}$ on mfd M consist of

<u>data</u>: $M \supset_{\text{open}} U \longmapsto \mathcal{F}(U) \in \mathrm{Vect}$

$U \hookrightarrow V \longmapsto \mathcal{F}(U) \xrightarrow[\text{linear}]{} \mathcal{F}(V)$

$\underbrace{U_1, \ldots, U_n}_{\text{mutually disjoint}} \subset V \longmapsto m^V_{U_1 \ldots U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$

<u>condition</u>: associativity.

- multiplicative: $U = U_1 \amalg U_2$ $\quad m^U_{U_1, U_2} : \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \xrightarrow{\cong} \mathcal{F}(U)$
- Weiss-cosheaf property.

pref. alg + multiplicative + Weiss decent $=:$ factorization algebra

<u>Def</u>: a prefactorization algebra $\mathcal{F}$ on an n-mfd M is <u>locally constant</u> if $U \overset{i}{\hookrightarrow} V$ inclusion

of n-disks in M (homeomorphic to $\mathring{D}^n$; or a halfdisk)

Then $m_U^V : \mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism.

<u>Ex:</u> let $\mathcal{F}$ be a loc. constant prefact. alg. on $\mathbb{R}$

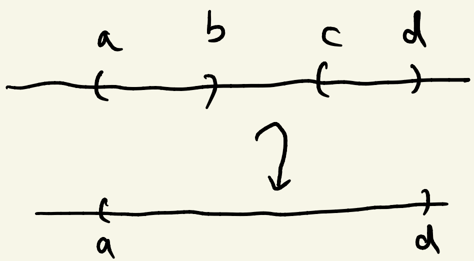
$A := \mathcal{F}(\mathbb{R})$

$(a,b) \overset{i}{\hookrightarrow} (c,d)$

$$\begin{tikzcd}
\mathcal{F}((a,b)) \arrow[r, "\cong"] \arrow[d, "i_*"', "\cong"] & \mathcal{F}(\mathbb{R}) = A \arrow[d, equal] \\
\mathcal{F}((c,d)) \arrow[r, "\cong"] & \mathcal{F}(\mathbb{R}) = A
\end{tikzcd}$$

$\mathcal{F}$ is loc. const. commutative

So can identify $\mathcal{F}((a,b))$ with A.

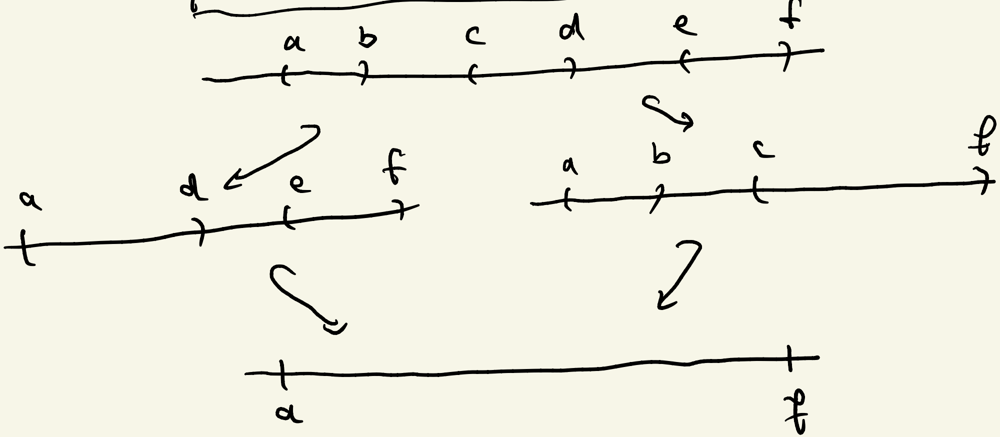


$$\begin{tikzcd}
\mathcal{F}((a,b) \sqcup (c,d)) \arrow[d] & \mathcal{F}((a,b)) \otimes \mathcal{F}((c,d)) = A \otimes A \arrow[l] \arrow[d] \\
\mathcal{F}(a,d) & \mathcal{F}(a,d) = A \arrow[l, equal]
\end{tikzcd}$$

<u>claim:</u> The map is independent of these intervals.

Pf: homework

Proof of associativity:



$$
\begin{tikzcd}
 & \mathcal{F}(a,b)\otimes\mathcal{F}(c,d)\otimes\mathcal{F}(e,f) = A\otimes A\otimes A \arrow[dl, "\mu\otimes 1"'] \arrow[dr, "1\otimes\mu"] \arrow[dd] & \\
\mathcal{F}(a,d)\otimes\mathcal{F}(e,f) = A\otimes A \arrow[dr, "\mu"'] & & \mathcal{F}(a,b)\otimes\mathcal{F}(c,f) = A\otimes A \arrow[dl, "\mu"] \\
 & \mathcal{F}(a,f) = A &
\end{tikzcd}
$$

$\Rightarrow$ multiplication is associative.

$B = \mathcal{F}(\emptyset) = ?$

$$\begin{tikzcd}
\emptyset \sqcup \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\
(a,b) \sqcup (c,d) \arrow[r] & (a,d)
\end{tikzcd}$$

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\sim"'] \arrow[d, phantom, "\shortparallel"] & & B \arrow[d, phantom, "\shortparallel"] \\
\mathcal{F}(\emptyset) \otimes \mathcal{F}(\emptyset) \arrow[r] \arrow[d] & \mathcal{F}(\emptyset \sqcup \emptyset) \arrow[r] \arrow[d] & \mathcal{F}(\emptyset) \arrow[d] \\
\mathcal{F}((a,b)) \otimes \mathcal{F}((c,d)) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{F}((a,b) \sqcup (c,d)) \arrow[r] & \mathcal{F}((a,d)) \arrow[d, phantom, "\shortparallel"] \\
A \otimes A \arrow[rr, "\mu"'] & & A
\end{tikzcd}$$

$\Rightarrow$ B is a algebra

\+

$$\begin{tikzcd}
B \arrow[r] \arrow[d, phantom, "\shortparallel"] & A \arrow[d, phantom, "\shortparallel"] \\
\mathcal{F}(\emptyset) & \mathcal{F}(\mathbb{R})
\end{tikzcd}$$

is algebra homom.

B is a comm. alg.

e.g. $B = \mathbb{R}[[\hbar]]$

$$\begin{tikzcd}
(a,b) \sqcup \emptyset \arrow[r] & (a,b) \\
\mathcal{F}(a,b) \otimes \mathcal{F}(\emptyset) \arrow[r, "\cong"] \arrow[d, phantom, "\shortparallel"] & \mathcal{F}(a,b) \arrow[d, phantom, "\shortparallel"] \\
A \otimes B \arrow[r] & A
\end{tikzcd}$$

$\Longleftarrow$ A is algebra over B.

now assume: multiplicativity for the pref. alg. $\mathcal{F}$

$\emptyset \sqcup \emptyset = \emptyset$

$\mathcal{F}(\emptyset) \otimes \mathcal{F}(\emptyset) \xrightarrow[\cong]{} \mathcal{F}(\emptyset)$

isom. by mult. axiom

$\Downarrow$

$\mathcal{F}(\emptyset) \cong \mathbb{R}$ or $\{0\}$

$(a,b) \sqcup \emptyset \hookrightarrow (a,b)$

$\mathcal{F}((a,b)) \otimes \mathcal{F}(\emptyset) \xrightarrow[\cong]{} \mathcal{F}((a,b)) \implies \mathcal{F}(\emptyset) \cong \mathbb{R}$

by mult. (unless $\mathcal{F}((a,b)) \cong \{0\}$.)

in $\mathcal{F}(\emptyset) \longrightarrow \mathcal{F}((a,b)) = A$

$\mathbb{R} \ni 1 \longmapsto u \in A$

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homework: u is a unit for A.

Upshot: a mult. prefact alg on $\mathbb{R}$ amounts to a unital algebra

Ex: $\mathcal{F}$ mult. prefact alg. on $I = [a,b]$.

locally const

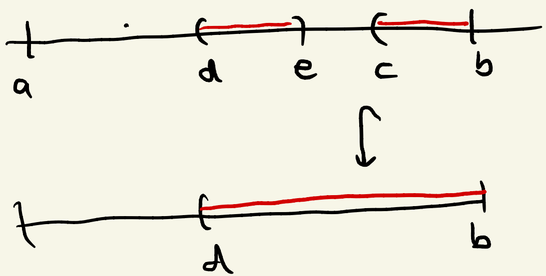
$(c,d) \subset I$

$$\mathcal{F}(c,d) \xrightarrow{\simeq} \mathcal{F}((a,b)) =: A$$

unital algebra

$$\mathcal{F}([a,c)) \xrightarrow{\simeq} \mathcal{F}([a,b)) =: M$$

$$\mathcal{F}((c,b]) \xrightarrow{\simeq} \mathcal{F}((a,b]) = N$$



$$\begin{tikzcd}
\mathcal{F}((d,e)) \otimes \mathcal{F}((c,b]) \arrow[d] & = & A \otimes N \arrow[d, "\mu^N"] \\
\mathcal{F}((d,b]) & = & N
\end{tikzcd}$$

this give N the structure of of left A-module

similarly, M is a <u>right</u> A-module