Recall: classical field theory on spacetime M

$\rightsquigarrow$ vector space of classical observables for $U \subset M$

$$\mathrm{Obs}(U) := \mathcal{O}(\underbrace{EL(U)}_{\text{classical fields in } U})$$

functions on $EL(U)$

$\mathrm{Obs} : \mathrm{Open}(M) \longrightarrow \mathrm{Vect}$ covariant.

$$\mathrm{Obs}(U_1 \amalg U_2) = \mathcal{O}(EL(U_1 \amalg U_2)) = \mathcal{O}(EL(U_1) \times EL(U_2))$$

$$\overset{\text{"}}{\mathcal{O}(EL(U_1)) \otimes \mathcal{O}(EL(U_2))}$$

$$\overset{\text{"}}{\mathrm{Obs}(U_1) \otimes \mathrm{Obs}(U_2)}$$

slogan: $\mathcal{O}(X) \otimes \mathcal{O}(Y) \xrightarrow[\text{typically}]{\cong} \mathcal{O}(X \times Y)$

for "suitable" tensor product

e.g. i) $\underbrace{P(X)}_{\text{polyn. function on vector space } X}$ $\qquad P(X) \otimes P(Y) \xrightarrow{\cong} P(X \times Y)$

ii) $L^2(X)$ $\qquad \underbrace{L^2 X \otimes L^2 Y}_{H} \xrightarrow{\cong} L^2(X \times Y)$

measure space

(iii) $C^\infty(X)$ ← standard Frechet topology
$\uparrow$ mfd

$C^\infty(X) \otimes_\pi C^\infty(Y) \to C^\infty(X \times Y)$

← Hilbert space tensor product

projective tensor product

Q: Is $\mathrm{Obs} : \mathrm{Open}(M) \to \mathrm{Vect}$ a cosheaf?

No!

Q: What locality properties does Obs have?

(e.g. does the cosheaf property work for "special" covers?)

Simpler example: $\mathcal{F}(U) := \mathcal{O}(C^\infty(U)) = P(C^\infty(U))$

$$= \bigoplus_{k=0}^{\infty} P_k(C^\infty(U))$$

$$P_k(C^\infty(U)) = \mathrm{Sym}^k(C^\infty(U)^\vee)$$
$$= (C^\infty(U)^\vee \otimes \cdots \otimes C^\infty(U)^\vee)/S_k$$

simplify: $\mathcal{F}(\mathcal{U}) = \underbrace{C^\infty(\mathcal{U})^\vee \otimes \cdots \otimes C^\infty(\mathcal{U})^\vee}_{k}$

$$C^\infty(\mathcal{U})^\vee = \left\{ \begin{array}{c} \text{distributions} \\ \text{on } \mathcal{U} \end{array} \right\} \longleftarrow C_c^\infty(\mathcal{U})$$

$$\left(f \longmapsto \int_{\mathcal{U}} f g \, \mathrm{vol} \right) \longleftarrow g \in C_c^\infty(\mathcal{U})$$

vol: volume form

simplify further: $\mathcal{F}(\mathcal{U}) := \underbrace{C_c^\infty(\mathcal{U}) \otimes_\pi \cdots \otimes_\pi C_c^\infty(\mathcal{U})}_{k}$

$$= C_c^\infty(\underbrace{\underbrace{\mathcal{U} \times \cdots \times \mathcal{U}}_{k}}_{\mathcal{U}^k})$$

$\mathcal{F}(\mathcal{U}) := C_c^\infty(\mathcal{U}^k)$

locality properties of this functor?

let $\{\mathcal{U}_a\}_{a \in A}$ be an open cover for $\mathcal{U}$.

$$
\mathcal{F}(\mathcal{U}) \xleftarrow{\overset{?}{\cong}} \operatorname{coeq}\Big(\bigoplus_a \mathcal{F}(\mathcal{U}_a) \leftleftarrows \bigoplus_{a<b} \mathcal{F}(\mathcal{U}_a \cap \mathcal{U}_b) \Big)
$$

$$
\Vert \qquad\qquad\qquad \Vert \qquad\qquad\qquad \cong
$$

$$
C_c^\infty(\mathcal{U}^k) \xleftarrow{\cong} \operatorname{coeq}\Big(\bigoplus C_c^\infty(\mathcal{U}_a^k) \leftleftarrows \bigoplus C_c^\infty(\mathcal{U}_a^k \cap \mathcal{U}_b^k) \Big) \quad \check{C}(\mathcal{F}, \mathfrak{U})
$$

$$
C_c^\infty(\mathcal{U}_a^k \cap \mathcal{U}_b^k) = C_c^\infty((\mathcal{U}_a \cap \mathcal{U}_b)^k)
$$

Čech object for $\mathfrak{U} = \{\mathcal{U}_a\}_{a \in A}$

note $\{\mathcal{U}_a^k\}_{a\in A}$ is a collection of open subsets of $\mathcal{U}^k$.

If it is a cover, then cosheaf property for $C_c^\infty(-)$ implies $\cong$.

let $(x_1, \dots, x_k) \in \mathcal{U}^k$

$$
(x_1, \dots, x_k) \in \mathcal{U}_a^k
$$

$$
\Updownarrow
$$

$$
\{x_1, \dots, x_k\} \subset \mathcal{U}_a
$$

i.e. this is a cover, provided for each finite cardinality k subset S of $\mathcal{U}$ is contained in some $\mathcal{U}_a$.

Def: A Weiss cover of $\mathcal{U}$ is a collection $\{\mathcal{U}_a\}_{a \in A}$ of open subsets s.t. for any finite $S \subset \mathcal{U}$ $\exists$ some a s.t. $S \subset \mathcal{U}_a$.

Examples of Weiss covers:

(1) M Riem. mfd, $\varepsilon > 0$

collection $\mathcal{U}$ consists of finite disjoint union of balls of radius $< \varepsilon$.

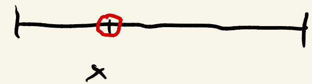
$\mathcal{U}$ is a Weiss cover!

U = union of these balls.

(ii) $M = [a,b] \subset \mathbb{R}$

Weiss cover of M: $\mathcal{U} = \{\mathcal{U}_x\}_{x \in (a,b)}$

$\mathcal{U}_x := [a,b] \setminus \{x\}$



(iii) non-example: $\mathcal{U} = \mathcal{U}_1 \sqcup \mathcal{U}_2$

$\{\mathcal{U}_1, \mathcal{U}_2\}$ is <u>not</u> a Weiss cover.

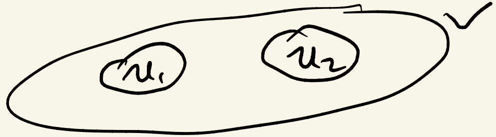
<u>Def:</u> A <u>prefactorization algebra $\mathcal{F}$</u> on a top. space M is a assignment:

<u>data:</u>

$$M \underset{\text{open}}{\supset} \mathcal{U} \longmapsto \mathcal{F}(\mathcal{U}) \in \mathrm{Vect}$$

$$\mathcal{U} \hookrightarrow V \longmapsto m_{\mathcal{U}}^{V} : \mathcal{F}(\mathcal{U}) \xrightarrow[\text{linear map}]{} \mathcal{F}(V)$$

$$\mathcal{U}_1 \sqcup \dots \sqcup \mathcal{U}_n \hookrightarrow V \longmapsto m_{\mathcal{U}_1,\dots,\mathcal{U}_n}^{V} \; \mathcal{F}(\mathcal{U}_1) \otimes \dots \otimes \mathcal{F}(\mathcal{U}_n) \xrightarrow[\text{linear}]{} \mathcal{F}(V)$$



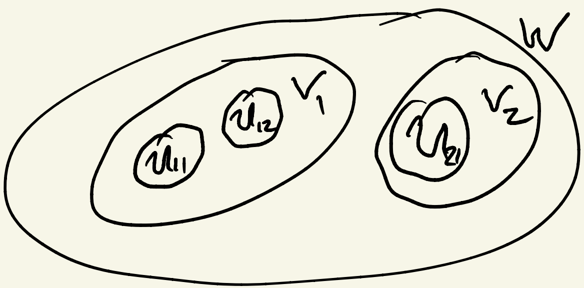
$\longmapsto \quad \mathcal{F}(U_1)\otimes\mathcal{F}(U_2) \to \mathcal{F}(V)$

<u>condition:</u> functoriality = associativity for the "structure maps m":

$V_1 \sqcup \dots \sqcup V_n \subset W$

$U_{i1} \sqcup \dots \sqcup U_{im} \subset V_i$

$$\begin{tikzcd}
\bigotimes_i \bigotimes_j F(U_{ij}) \arrow[d] \arrow[dr] & \\
\bigotimes_i F(V_i) \arrow[r] & F(W)
\end{tikzcd}$$



$$\begin{tikzcd}
F(U_{11})\otimes F(U_{12})\otimes F(U_{21}) \arrow[d, "m^{V_1}_{U_{11}\sqcup U_{12}} \otimes m^{V_2}_{U_{21}}"'] \arrow[dr, "m^W_{U_{11},\dots,U_{21}}"] \arrow[dr, phantom, "\circlearrowright"] & \\
F(V_1)\otimes F(V_2) \arrow[r, "m^W_{V_1\sqcup V_2}"] & F(W)
\end{tikzcd}$$

A pref. alg. is <u>multiplicative</u> if

for $V = U_1 \sqcup U_2$ $\quad m^V_{U_1,U_2} : F(U_1)\otimes F(U_2) \xrightarrow{\cong} F(V)$

A factorization algebra is a pref. alg. which is a cosheaf w.r.t. Weiss covers, i.e. for any Weiss cover $\{U_a\}_{a\in A}$ of U

strict

$$\mathcal{F}(U) \xleftarrow{\cong} \check{C}(\mathcal{F}; \mathfrak{U})$$