	Classical field theory on a spacetime M	Example of a classical FT: nonlinear σ-model with target X	special case $M=\mathbb{R}$
Space of fields $\mathcal{F}(U)$ for $U \subset M$	$\mathcal{F}(U)$ manifold $U \mapsto \mathcal{F}(U)$ sheaf	$\mathcal{F}(U) = C^\infty(U, X)$	$U=(a,b)$ $\mathcal{F}(U)=\{\text{path } (a,b)\to X\}$
action functional S	$S : \mathcal{F}(U) \to \mathbb{R}$	$\phi : U \to X$ $S(\phi) = \text{energy of } \phi$ $= \int_U \lVert d\phi_p \rVert^2 \, dvol(p)$ $p \in U$	$S(\phi) = \int_a^b \lVert \dot\phi(t) \rVert^2 \, dt$
space of classical fields	$Crit(S) = \{\text{critical pts of } S\}$ $= \{\phi \in \mathcal{F}(U) \mid \text{satisfy Euler-Langrange eqs}\}$ $= EL(U)$ $U \mapsto EL(U)$ sheaf	$EL(U) = \{\phi : U \to X \text{ harmonic}\}$	$EL(U) = \{\phi : (a,b) \to X \text{ geodesic}\}$ $= \{\phi \mid \nabla_t \dot\phi(t) = 0\}$ $EL(U) \xrightarrow{\cong} TM$ $\phi \longmapsto (\phi(c), \dot\phi(c))$ $c \in (a,b)$

vector space of classical observables $\mathrm{Obs}(\mathcal{U})$ = all quantities measurable in spacetime region $\mathcal{U} \subset M$

$\mathrm{Obs}(\mathcal{U})$
$\|$
suitable space of functions on $EL(\mathcal{U})$
$\|$
$\mathcal{O}(EL(\mathcal{U}))$

$\mathrm{Obs}(\mathcal{U}) = \mathcal{O}(TM)$

e.g. $= C^\infty(TM)$

note: e.g. position or velocity functions.

$$\begin{tikzcd} M \supset \mathcal{U} \arrow[r, mapsto, "\uparrow \text{ contravariant}"'] \arrow[rr, bend right, mapsto, "\text{covariant}"'] & EL(\mathcal{U}) \arrow[r, mapsto, "\uparrow \text{ contravariant}"'] & \mathcal{O}(EL(\mathcal{U})) \end{tikzcd}$$

$\mathcal{O}(EL(\mathcal{U})) = \mathrm{Obs}(\mathcal{U})$

<u>Q</u>: Is $\mathcal{U} \longmapsto \mathrm{Obs}(\mathcal{U})$ a cosheaf?

$$\mathrm{Obs}(\mathcal{U}_1 \sqcup \mathcal{U}_2) = \mathcal{O}(EL(\mathcal{U}_1 \sqcup \mathcal{U}_2)) \simeq \mathcal{O}(EL(\mathcal{U}_1) \times EL(\mathcal{U}_2))$$

how is that expressable in terms of

$\mathcal{O}(EL(\mathcal{U}_1))$ & $\mathcal{O}(EL(\mathcal{U}_2))$.

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disgression on functions on products

$\mathcal{O}(X) = \{$ suitable functions $f : X \to k \}$

$k = \mathbb{R}, \mathbb{C}$

product of functions

$$\begin{tikzcd}
(f,g) \arrow[r, mapsto] & (p_1^* f)\cdot(p_2^* g) \\
\mathcal{O}(X) \times \mathcal{O}(Y) \arrow[r, "\text{bilinear}"] \arrow[d] & \mathcal{O}(X\times Y) \\
\mathcal{O}(X) \otimes \mathcal{O}(Y) \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

Is Φ an isomorphism?

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(1) polynomial functions $\qquad P(\mathbb{R}) = \mathbb{R}[x]$

$P(V) = \{ f : V \to k \}$ $\qquad P(\mathbb{R}^n) = \mathbb{R}[x_1, \dots, x_n]$

vector space $\qquad$ polynomial function

$$\mathcal{P}(\mathbb{R}^m \times \mathbb{R}^n) = \mathcal{P}(\mathbb{R}^{m+n}) = \mathbb{R}[x_1, \dots, x_{m+n}]$$

$$= \mathbb{R}[x_1, \dots, x_m] \otimes \mathbb{R}[x_{m+1}, \dots, x_{m+n}]$$

$$= \mathcal{P}(\mathbb{R}^m) \otimes \mathcal{P}(\mathbb{R}^n)$$

basis free argument:

$$\mathcal{P}(V) := \bigoplus_{k=0}^{\infty} P_k(V) = \mathrm{Sym}(V^\vee) := \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(V^\vee)$$

homog. polynomials of degree k

$$P_0(V) := k$$

$$P_1(V) := \mathrm{Hom}(V, k) = V^\vee \quad \text{dual vector space}$$

$$P_k(V) := \left(\underbrace{V^\vee \otimes \cdots\cdots \otimes V^\vee}_{k}\right) \Big/ S_k =: \mathrm{Sym}^k(V^\vee)$$

symmetric group acting by permuting factors.

The construction $W \longmapsto Sym(W)$ is <u>exponential</u>,

i.e, $Sym(W \oplus U) = Sym(W) \otimes Sym(U)$

$$\Longrightarrow \quad P(V \oplus W) = Sym(V^\vee \oplus W^\vee) = Sym(V^\vee) \otimes Sym(W^\vee)$$

$$\shortparallel$$

$$P(V) \otimes P(W).$$

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(2) finite measure space X

Hilbert space $L^2 X = \{ f : X \to \mathbb{R} \text{ measurable} \mid \int_X f(x)^2 d\mu(x) < \infty \}$

with inner product $\langle f, g \rangle := \int_X f(x) g(x) d\mu(x)$

$$L^2 X \otimes L^2 Y \xrightarrow{\ \Phi\ } L^2(X \times Y)$$

$$f_i \otimes g_i \longmapsto (p_1^* f_i) \cdot (p_2^* g_i)$$

$$\langle \Phi(f_1\otimes g_1), \Phi(f_2\otimes g_2)\rangle = \int_{X\times Y} \underbrace{f_1(x)g_1(y)}_{\Phi(f_1\otimes g_1)} \; \underbrace{f_2(x)g_2(y)}_{\Phi(f_2\otimes g_2)} \, d\mu(x,y)$$

$$= \int_{X\times Y} f_1(x)f_2(x)\cdot g_1(y)g_2(y)\,d\mu(x,y)$$

$$= \left(\int_X f_1(x)f(x_2)\,d\mu(x)\right)\cdot\left(\int_Y g_1(y)g_2(y)\,d\mu(y)\right)$$

$$= \langle f_1, f_2\rangle_{L^2X} \cdot \langle g_1, g_2\rangle_{L^2Y}$$

Hence Φ is an isometry, provided we define $\langle f_1\otimes g_1, f_2\otimes g_2\rangle := \langle f_1, f_2\rangle_{L^2X}\cdot\langle g_1, g_2\rangle_{L^2Y}$

$\Longrightarrow$ Φ is injective, its image is dense

$$\Longrightarrow \text{isom. } \underbrace{L^2X \otimes_H L^2Y}_{\substack{\| \\ \text{completion of } L^2X \otimes L^2Y \\ \text{w.r.t. inner product defined above}}} \xrightarrow{\cong} L^2(X \times Y)$$