Examples: M mfd

$$F : \mathrm{Open}(M) \xrightarrow{\text{covariant}} \mathrm{Vect}$$

(i) $F(U) = C_c^\infty(U) = \{ f : U \underset{\text{smooth}}{\to} \mathbb{R} \mid \underset{\text{compact}}{\mathrm{supp}(f)} \}$, $\mathrm{supp}(f) = \overline{\{x \in U \mid f(x) \neq 0\}}$

(ii) X mfd (or top. space) with basept $x_0 \in X$

$F(U) := C_c^\infty(U, X) = \{ f : U \underset{\text{smooth}}{\to} X \mid \mathrm{supp}(f) \text{ compact} \}$, $\mathrm{supp}(f) = \overline{\{x \in U \mid f(x) \neq x_0\}}$

(or $F(U) = \mathrm{map}_c(U, X)$
$\uparrow$ continuous maps with cp. support)

cov. $F : \mathrm{Open}(M) \longrightarrow \mathrm{Set}\ (\mathrm{Top})$

Q: Are these cosheafs?

recall: $F : \mathrm{Open}(M) \longrightarrow \mathcal{C}$ is a cosheaf if for any open cover $\{U_a\}_{a \in A}$ of $U \in \mathrm{Open}(M)$ the following holds:

$$U \xhookleftarrow[i_a]{} U_a \qquad U_a \cap U_b \xhookrightarrow[i^a_{ab}]{} U_a$$

$$(*) \quad F(U) \xleftarrow{\coprod (i_a)_*} \mathrm{coeq}\left(\coprod_{a \in A} F(U_a) \underset{\coprod j_b \circ (i^b_{ab})_*}{\overset{\coprod j_a \circ (i^a_{ab})_*}{\leftleftarrows}} \coprod_{a<b} F(U_a \cap U_b) \right)$$

is an isomorphism

Special case: U_1, U_2 disjoint open subsets, $U = U_1 \cup U_2$

$\Rightarrow \{U_1, U_2\}$ open cover of U,

Here $(*)$ simplifies to

$$F(U) \xleftarrow[\cong]{} F(U_1) \amalg F(U_2)$$

back to the examples:

(i) $C_c^\infty(U) = C_c^\infty(U_1 \amalg U_2) \cong C_c^\infty(U_1) \oplus C_c^\infty(U_2)$ ← yes, is the coproduct in Vect

$$f \longmapsto (f|_{U_1}, f|_{U_2})$$

$U = U_1 \amalg U_2$

(i.e. $\amalg$ means: $U_1 \cap U_2 = \emptyset$)

cosheaf property? for simplicity, let's check for cover $\{U_1, U_2\}$ of $U_1 \cup U_2 = U$.

$$F(U) \overset{?}{\underset{\cong}{\longleftarrow}} \underbrace{\mathrm{coeq}\Big(F(U_1) \oplus F(U_2) \;\substack{\xleftarrow{j_1 \circ i_{12}^1 = k_1} \\ \xleftarrow[j_2 \circ i_{12}^2 = k_2]{}}\; F(U_1 \cap U_2)\Big)}_{\mathrm{coker}\left(F(U_1) \oplus F(U_2) \xleftarrow{k_1 - k_2} F(U_1 \cap U_2)\right)}$$

$$\Updownarrow$$

exactness of

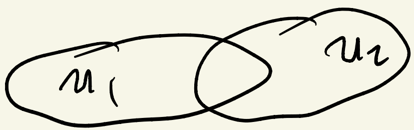
$$0 \leftarrow C_c^\infty(U) \longleftarrow C_c^\infty(U_1) \oplus C_c^\infty(U_2) \xleftarrow{k_1 - k_2} C_c^\infty(U_1 \cap U_2)$$

Injectivity here not required

surjective? Yes, using partitions of unity.

exactness here:

$$C_c^\infty(\mathcal{U}_1) \oplus C_c^\infty(\mathcal{U}_2) \ni (f_1, f_2) \longmapsto 0 = (i_1)_* f_1 + (i_2)_* f_2$$



$$\Rightarrow (i_1)_* f_1 = -(i_2)_* f_2$$

$$\Rightarrow \operatorname{supp}((i_1)_* f_1) = \operatorname{supp}((i_2)_* f_2)$$

$$\operatorname{supp}((i_1)_* f_1) = \operatorname{supp}(f_1) \subset \mathcal{U}_1, \qquad \operatorname{supp}((i_2)_* f_2) = \operatorname{supp}(f_2) \subset \mathcal{U}_2$$

$\Rightarrow \operatorname{supp}(f_i) \in \mathcal{U}_1 \cap \mathcal{U}_2$ and $f_1|_{\mathcal{U}_1 \cap \mathcal{U}_2} = -f_2|_{\mathcal{U}_1 \cap \mathcal{U}_2}$

$\Rightarrow (f_1, f_2)$ is $(k_1 - k_2)(f_1)$.

Hence $\mathcal{U} \longmapsto C_c^\infty(\mathcal{U})$ is a cosheaf.

Ex (ii): $F(\mathcal{U}) = \mathrm{map}_c(\mathcal{U}, X)$ cosheaf?

First test: $F(\mathcal{U}) \overset{?}{=} F(\mathcal{U}_1) \amalg F(\mathcal{U}_2)$ coproduct.

$\mathcal{U} = \mathcal{U}_1 \amalg \mathcal{U}_2$

$$\mathrm{map}_c(\mathcal{U}_1 \amalg \mathcal{U}_2, X) \xrightarrow{\cong} \mathrm{map}_c(\mathcal{U}_1, X) \times \mathrm{map}_c(\mathcal{U}_2, X)$$

$$f \longmapsto (f|_{\mathcal{U}_1}, f|_{\mathcal{U}_2})$$

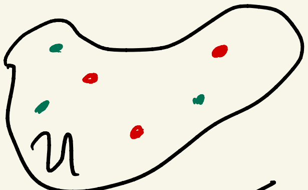
so $F(\mathcal{U}) = F(\mathcal{U}_1) \times F(\mathcal{U}_2)$

not the coproduct!
(in the cat. of sets or top. spaces)

so F is <u>not</u> a cosheaf!

Ex: Configuration spaces

$$F(\mathcal{U}) := \mathrm{Conf}_k(\mathcal{U}) = \left\{ \begin{matrix} \text{configuration of } k \\ \text{points in } \mathcal{U} \end{matrix} \right\}$$



$\in \mathrm{Conf}_3(\mathcal{U})$

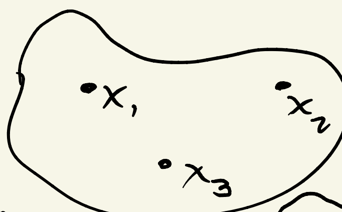
$\in$

two configurations

labeled configuration space

$$\widetilde{\mathrm{Conf}}_k(\mathcal{U}) = \left\{ (x_1, \dots, x_k) \in \underbrace{\mathcal{U} \times \dots \times \mathcal{U}}_{k} \text{ s.t. } x_i \neq x_j \text{ for } i \neq j \right\}$$

equipped with subspace top. $\subseteq \mathcal{U} \times \dots \times \mathcal{U}$



$\in \widetilde{\mathrm{Conf}}_3(\mathcal{U})$

quotient topology $\rightarrow$ $\mathrm{Conf}_k(\mathcal{U}) := \widetilde{\mathrm{Conf}}_k(\mathcal{U}) / \Sigma_k$ (action of Σ_k is given by permuting labels $1, \dots, k$)

$$\mathrm{Conf}(\mathcal{U}) := \coprod_{k=0,1,2,\dots} \mathrm{Conf}_k(\mathcal{U})$$

$\mathcal{U} \hookrightarrow V \implies$ induced map $\mathrm{Conf}(\mathcal{U}) \to \mathrm{Conf}(V)$

$\implies$ functor $\mathrm{Open}(M) \longrightarrow \mathrm{Top}$

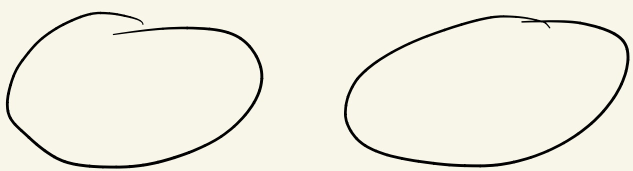
$\mathcal{U} \longmapsto \mathrm{Conf}(\mathcal{U})$

Q: Cosheaf?

$$\mathrm{Conf}_k(\mathcal{U}_1 \sqcup \mathcal{U}_2) = \coprod_{k=k_1+k_2} \mathrm{Conf}_{k_1}(\mathcal{U}_1) \times \mathrm{Conf}_{k_2}(\mathcal{U}_2)$$

$$\Downarrow$$

$$\mathrm{Conf}(\mathcal{U}_1 \sqcup \mathcal{U}_2) = \mathrm{Conf}(\mathcal{U}_1) \times \mathrm{Conf}(\mathcal{U}_2)$$



$\mathcal{U}_1$ $\mathcal{U}_2$

Upshot: this is not a cosheaf.

digression on classical field theories

Definition of classical FT

class. FT consists of data (on mfd M)

- $M \underset{\text{open}}{\supset} U \longmapsto \mathcal{F}(U) =$ "space of fields in U"
- an action functional $S : \mathcal{F}(U) \longrightarrow \mathbb{R}$ smooth
- classical fields are critical points of S;

$\{\text{class. field}\} = \mathrm{Crit}(S) = \left\{ \begin{array}{l} \text{solutions of} \\ \text{differential} \\ \text{equations} \\ \text{called} \\ \text{Euler-Lagrange} \\ \text{equations} \end{array} \right\}$

$EL(U) \stackrel{\cdot}{=}$

Example of class. FT

fix Riem. metric of M & a Riem. "target mfd" X

- $\mathcal{F}(U) = C^\infty(U, X)$

e.g.: $M = \mathbb{R} \Rightarrow \mathcal{F}(U) = \left\{ \begin{array}{l} \text{paths in } X \\ \text{param. by} \\ U \end{array} \right\}$

- $\phi : U \to X$ field

$S(\phi) = \underline{\text{energy of } \phi} := \int_U \|d\phi_p\|^2$

$(d\phi)_p \in \underbrace{\mathrm{Hom}(T_pM, T_{\phi(p)}X)}_{\text{has induced inner product}}$

$p \in U$

inner product spaces

$\|d\phi_p\|^2$