**Lem:** $X_a$, $a \in A$ collection of sets.
Cartesian product $\prod_{a\in A} X_a$ has the following property

$$\begin{tikzcd}
\prod_{a\in A} X_a \arrow[dr, "p_a"] \arrow[drr, "p_b"] & & & \cdots \\
& X_a & X_b & \cdots \\
Z \arrow[uu, dashed, "\exists ! f"] \arrow[ur, "f_a"] \arrow[urr, "f_b"'] & & & \cdots
\end{tikzcd}$$

**Lem:** set map $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ ; $eq\left(X \overset{f}{\underset{g}{\rightrightarrows}} Y\right)$
property of equalizer:

$$\{x \mid f(x) = g(x)\}$$

$$
\begin{tikzcd}
\mathrm{eq}\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) \arrow[r, "i"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
& Z \arrow[ul, dashed, "\exists ! \phi"] \arrow[u, "\psi"'] &
\end{tikzcd}
$$

---

<u>Def</u>: Let $\mathcal{C}$ be a category.

$X_a$, $a \in A$, a family of objects.

A <u>categorical product</u> of $X_a$'s is some object $X$, together with morphisms $p_a$

$p_a : X \longrightarrow X_a$ s.t.

$$
\begin{tikzcd}
X \arrow[r, "p_a"] \arrow[rr, bend left, "p_b"] & X_a \quad \cdots & X_b \quad \cdots \\
Z \arrow[u, dashed, "\exists ! f"] \arrow[ur, "f_a"] \arrow[urr, "f_b"'] & &
\end{tikzcd}
$$

<u>note</u>: if it exists, $X$ is unique up to isomorphism.

notation for cat. prods: $\prod_{a \in A} X_a$

---

**Def:** given morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ the equalizer is char. by

$$\begin{tikzcd} eq(X \underset{g}{\overset{f}{\rightrightarrows}} Y) \arrow[r, "i"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ & Z \arrow[ul, dashed, "\exists!"] \arrow[u] & \end{tikzcd}$$

---

**Def:** let $X_a \in \text{ob}\,\mathcal{C}$, $a \in A$. The coproduct $\coprod_{a \in A} X_a$ is char. by univ. property:

$$
\begin{tikzcd}
\coprod_{a \in X} X_a \arrow[d, dashed, "\exists!\, f"'] & X_a \arrow[l, "i_a"'] \arrow[dl, "f_a"] & X_b \arrow[ll, "i_a"'] \arrow[dll, "f_b"] & \cdots \\
Z & & &
\end{tikzcd}
$$

<u>**Def:**</u> in cat $\mathcal{C}$ : $X \underset{g}{\overset{f}{\leftleftarrows}} Y$.

Then the <u>coequalizer</u> $\mathrm{coeq}(X \underset{g}{\overset{f}{\leftleftarrows}} Y)$ is determined by :

$$
\begin{tikzcd}
\mathrm{coeq}(X \underset{g}{\overset{f}{\leftleftarrows}} Y) \arrow[dr, dashed, "\exists!"'] & & X \arrow[ll] \arrow[dl] & Y \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] \\
& Z & &
\end{tikzcd}
$$

**Convention:** $\displaystyle\coprod_{a\in A} X_a \underset{A=\emptyset}{=} \emptyset$

$\emptyset$ ← empty set for $\mathcal{C} = \mathrm{Set}$

initial object in general

i.e. $\forall\, X \in \mathcal{C}$ $\exists!$ $\emptyset \to X$.

$\displaystyle\prod_{a\in A} X_a \underset{A=\emptyset}{=} *$

$*$ ← single point for $\mathcal{C} = \mathrm{set}$

terminal ob. in general,

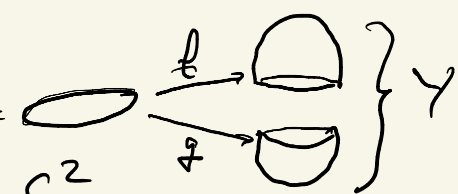
i.e. $\forall\, X \in \mathcal{C}$ $\exists!$ $X \longrightarrow *$

---

**Ex:** $\mathcal{C} = \mathrm{Top}$ (cat. of top. spaces + continuous maps)

$$\mathrm{coeq}\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) = Y \big/ f(x) \sim g(x)$$

**Ex:** $X = S^1 =$




$\mathrm{coeq}(X \rightrightarrows Y) \cong S^2$

$f$ = inclusion of $\partial$-circle in top disc

$g$ = " bottom "

| category | product | coproduct | initial / | terminal ob |
|---|---|---|---|---|
| Set | Cartesian product | disjoint union | $\emptyset$ | $*$ |
| Top | " | " | $\emptyset$ | $*$ |
| Vect | $\prod_{a\in A} V_a$ | $\bigoplus_{a\in A} V_a \subset \prod_{a\in V} V_a$ <br> $\parallel$ <br> $\{(v_a)_{a\in A} \mid v_a \in V_a$ only finitely many $v_a \neq 0\}$ | $\{0\}$ | $\{0\}$ |
| Open(M) | $\prod_{a\in A} U_a = \bigcap_{a\in A} U_a$ | $\coprod_{a\in A} U_a = \bigcup_a U_a$ | $\emptyset$ | $M$ |

provided this intersection is open;
so finite products always exist, but infinite product may not exist.

---

Ex: $F : \mathrm{Open}(M) \longrightarrow \mathrm{Vect}$

$$F(U) := C_c^\infty(U) = \{ f : U \to \mathbb{R} \,/\, \text{smooth}, \ \mathrm{supp}(f) \text{ compact} \}$$

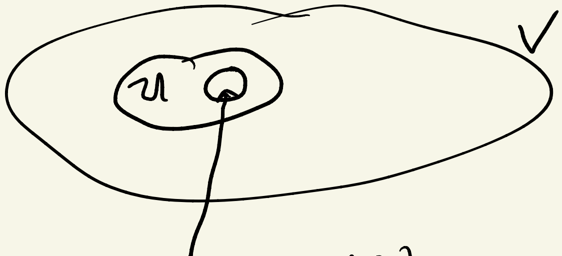
$$\mathrm{supp}(f) = \text{closure}\left(\{ x \in U \mid f(x) \neq 0 \}\right)$$

$i_U^V : U \hookrightarrow V$

"extend by 0"

$$F(i_U^V) = (i_U^V)_* : \underset{f}{C_c^\infty(U)} \longrightarrow C_c^\infty(V) \quad \text{covariant.}$$

$$g = \left((i_U^V)_* f\right)(x) = \begin{cases} f(x) & x \in U \\ 0 & x \notin U \end{cases}$$




is smooth since

$S = \mathrm{supp}(f)$ compact $\Rightarrow V = U \cup V \setminus S$

$\Rightarrow S \underset{\text{closed}}{\subset} V$

$g|_U$ smooth, $g|_{V\setminus S} \equiv 0$

$\Rightarrow g$ smooth

Q: is $F$ a cosheaf?

**Def:** (covariant) functor $F : \mathrm{Open}(X) \to \mathcal{C}$

$\{U_a\}$ cover of $V$

$$F(V) \xleftarrow{\coprod (i_a)_*} \mathrm{coeq}\left( \coprod_{a\in A} F(U_a) \underset{\coprod j_b\circ(i^b_{ab})_*}{\overset{\coprod j_a\circ(i^a_{ab})_*}{\leftleftarrows}} \coprod_{a<b} F(U_a\cap U_b)\right)$$

$F$ is a <u>cosheaf</u> if $\coprod (i_a)_*$ is an isomorphism.

$i_a : U_a \hookrightarrow V$

$i^a_{ab} : U_a \cap U_b \hookrightarrow U_a$