EQUIVARIANT FACTORIZATION ALGEBRAS: AN ∞-OPERADIC APPROACH

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Abstract

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Factorization algebras are a mathematical tool for modeling the observables of field theories. In this dissertation, we consider two particular types of factorization algebras: G-equivariant factorization algebras on a model space M, where G is a group acting on M; and factorization algebras on a site of manifolds which locally look like M and with geometric structure encoded by the G-action. Our main result is that the $(\infty, 1)$-categories of these factorization algebras are equivalent. To show this, we formulate an alternative, categorical description of the locality (or descent) condition that factorization algebras satisfy, and show that this agrees with the original, more geometric descent condition. We then generalize the definition of factorization algebras to the ∞-operadic setting, and utilize higher algebraic techniques to prove the comparison result. One of the motivations for this new ∞-operadic perspective is the ability to use these general results in future work involving parameterized families of factorization algebras.

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CHAPTER 1

INTRODUCTION

1.1 Historical background

Quantum field theories, introduced in the 1920's, are used to great predictive success by physicists, but a precise mathematical description of them remains elusive. One mathematical approach to their description was proposed by Segal in the 1980's. Inspired by the path integral approach of physicists (which, however, has measure-theoretic issues which make it mathematically ill-defined), Segal abstracted important properties that the path integral satisfies and formulated these as a set of mathematical axioms that would define a field theory. In this approach, called the *functorial field theory* approach, a field theory is defined to be a functor from a bordism category (whose objects model the possible spacetimes on which the theory can be formulated) to a category of algebraic data (whose objects model the Hilbert spaces of observables of the field theory on the respective spacetimes). This functorial assignment must satisfy certain multiplicative and gluing conditions, which capture the locality of the field theory.

In the years since Segal formulated this axiomatic framework, these functorial field theories have sparked independent mathematical interest, and have been studied and developed in various directions: they are the object of study in the cobordism hypothesis, and there are various classification results relating them to classical topological objects and structures (for instance, the categories of topological field theories and E_n-algebras are equivalent).[9, 15]

If one wants these functorial field theories to encode the information of field theories relevant in physics, one would want to be able to incorporate geometric structure into this framework. In physics spacetime is usually taken not to be simply a topological space or even a smooth manifold; usually one is interested in a spacetime that includes more geometric structure, such as a metric or a conformal structure.

Not only is incorporating geometric structure interesting in terms of physical motivation; there are also interesting classification questions involving these geometric field theories. One particular classification of interest for this project comes from the Stolz-Teichner program. They consider what are called twisted-super-symmetric Euclidean functorial field theories; this involves equipping the objects and morphisms in the bordism category with a supersymmetric Euclidean structure, and further generalizing to what they call twisted field theories (see the survey paper [16] for more details). They have the following classification of these objects:

Theorem 1.1. *[16], [8] For X a smooth manifold, considering concondance classes of degree n-twisted supersymmetric (of dimension $0|1$ and $1|1$, respectively) Euclidean field theories, one has the natural group isomorphisms:*

$$0|1\text{-}EFT^n[X] \cong \begin{cases} H^{ev}_{dR}(X), & n \text{ even} \\ H^{odd}_{dR}(X), & n \text{ odd} \end{cases}$$

$$1|1\text{-}EFT^n[X] \cong KO^n(X)$$

Inspired by this, they conjecture:

Conjecture 1.2. *[16] There is an isomorphism $2|1\text{-}EFT^n[X] \cong TMF^n(X)$, compatible with the multiplicative structure.*

Introducing geometric structure into the functorial field theory framework is a subtle and nuanced matter. On the bordism side, issues arise with regard to the gluing data necessary for the composition of bordisms to respect the geometric structure. In the non-topological case, there are also issues involving the identity for this composition: there is no cylinder of length zero. Stolz-Teichner address these problems by equipping the objects and bordisms with collars, where the collars also carry the geometric structure in question, and gluing then involves matching up the geometric structure on the respective germs of collars in a compatible way–this involves carrying around a great deal of extra data.[16] On the algebraic side, there are also difficulties when one moves to non-topological functorial field theories: it is not clear what the appropriate choice of target category is in this case. The upshot is that it is difficult to construct physically meaningful, geometric (non-topological) examples in the twisted-functorial field theory perspective: even if one knew what axioms one wants such a field theory to satisfy, describing the categories involved and defining explicit functors and natural transformations is difficult. These difficulties make resolving the above conjecture especially challenging. One step towards further work in that direction would involve constructing some test case examples of $2|1$-EFT; given the above difficulties, however, this has proved a challenging task.[16]

Switching perspectives, another approach to mathematically modeling field theories is that of factorization algebras. These were were first introduced by Beilinson-Drinfeld in 2004 (under the name of *chiral algebras*, in the setting of algebraic curves).[2] They were further developed by Costello-Gwilliam in 2016 for the case of manifolds; it is this framework that we will use.[3, 4] Factorization algebras are also functors from a geometric category (subsets of space time) to an algebraic category (encoding the observables of the field theory living on that subset). This approach has the advantage of being more closely related to physical examples: for example, Costello-Gwilliam have showed how to construct explicit factorization algebras modeling perturbative quantum field theories.[3]

The factorization algebra perspective is related to the functorial field theory perspective: for instance, Scheimbauer has shown how to construct a fully extended topological field theory from a locally constant factorization algebra.[15] Additionally, and of special interest for the geometric, non-topological case, Dwyer-Stolz-Teichner have developed a construction that takes a factorization algebra (with a suitable geometric structure, $\mathcal{G}$, to be described in detail later) and creates from it a twisted (of the same geometric structure, $\mathcal{G}$) functorial field theory.

The current project is motivated by this relationship outlined by Dwyer-Stolz-Teichner. Given that $\mathcal{G}$-twisted field theories can be obtained from $\mathcal{G}$-factorization algebras, we would like to study the latter as a tool that can be used to help generate examples of physical interest of the former. Further, we will show that there is an alternative perspective on the $\mathcal{G}$-factorization algebras that only depends on the 'local' data of how the symmetry group acts on subsets of the spacetime manifold. Putting to use the fact that factorization algebras have a 'local-to-global' structure (coming from what is called the descent axiom), it suffices to describe this local model of the factorization algebra to obtain interesting 'global' examples of geometric factorization algebras. More precisely, we consider $\mathcal{G}$-structured manifolds which are obtained by gluing open subsets of a model manifold M together via the action of a group G on M. In such a case, one expects that the local-to-global axiom for a factorization algebra should allow one to construct a $\mathcal{G}$-factorization algebra from a G-equivariant factorization algebra on M. The goal of this work is to put this intuition on rigorous footing. Examples coming from equivariant factorization algebras (in the sense of Costello-Gwilliam) could then be transported into the functorial field theory framework (via the work of Dwyer-Stolz-Teichner) to obtain examples to test the classification conjectures of the Stolz-Teichner program.

1.2 Summary of contents

In chapter 2 we first review the usual notion of an equivariant factorization algebra on a manifold M, based on the framework of Costello-Gwilliam.[3, 4] We then introduce a generalized, geometric version of the category of factorization algebras, whose objects live on a site of manifolds equipped with a rigid geometric structure given by a group action on a model space. In chapter 3 we develop a categorical description of the descent axiom for factorization algebras, and show this is equivalent to the original, more geometrically motivated descent axiom in [3, 4]. Using this more categorical version of the descent axiom, in chapter 4, we translate the previous versions of factorization algebras into the ∞-operad or symmetric monoidal ∞-category framework developed by Lurie.[10] We review general background for the ∞-operad perspective in Appendix A. In our case, we obtain $(\infty, 1)$-categories of G-equivariant factorization algebras and generalized $\mathcal{G}$-factorization algebras. Higher algebra techniques relate these two $(\infty, 1)$-categories; we review the background for these techniques in Appendix B, Appendix C and Appendix D. In chapter 5 we show how imposing the descent condition gives an equivalence of the $(\infty, 1)$-categories:

Theorem 1.3. *There is an equivalence of $(\infty, 1)$-categories between the category of G-equivariant factorization algebras on M, Fac_M^G, and the category of $\mathcal{G}$-factorization algebras, $\mathcal{G}Fac$.*

In future work, we will generalize these results to the smooth setting, where we will consider smoothly G-equivarant factorization algebras on M, and smooth families of $\mathcal{G}$-factorization algebras. Doing this will utilize the flexibility of the ∞-operad framework as it will involve adapting the ∞-operad structure to one that incorporates the parameterizing category. Having formulatated the previous result in this suitably general framework, the hope is to naturally obtain a smooth version of the comparison result:

Conjecture 1.4. *The* $(\infty,1)$ *categories* $\mathcal{G}Fac^{fam}$ *and* $Fac_M^{G,fam}$ *are weakly equivalent.*

CHAPTER 2

BACKGROUND DEFINITIONS

In section 2.1, we recall the definition of factorization algebras as developed by Costello-Gwilliam [3, 4]. We then develop a generalized, geometric version of factorization algebras in section 2.2, which are of interest to us as relating to the twisted field theories of Dwyer-Stolz-Teichner, see [16].

2.1 G-equivariant factorization algebras on M

2.1.1 Factorization algebras on M

We start by reviewing the basic notion of a factorization algebra over a fixed manifold M, as presented in [3, 4]. This encodes the data of the observables of a given field theory over the manifold M.

In what follows, let M be a smooth n-manifold.

Definition 2.1. The multicategory $Open(M)$ consists of the following data:

- **objects:** open subsets $U \subset M$
- **multi-morphisms:** for open subsets $U_1, ..., U_n, V \subset M$

$$Open(M)(U_1, ..., U_n; V) := \begin{cases} i, \text{ inclusion map}, & \text{if } U_1, ..., U_n \subset V \text{ are disjoint} \\ \emptyset, & \text{else} \end{cases}$$

We view the symmetric monoidal category of cochain complexes, Ch, as a multicategory in the standard way, using the monoidal structure: for $C_1, ...C_n, D \in Ch$ define the multimorphism set to be $Ch(C_1, ..., C_n; D) := Hom(C_1 \otimes \cdots \otimes C_n; D)$.

Definition 2.2 ([3], Ch. 3, Defn 1.2.2)**.** Let M be an n-manifold. A (homotopy multiplicative) *pre-factorization algebra on M* is a multicategory functor

$$\mathcal{F} : Open(M) \to Ch.$$

Note that the multicategory structure gives what are called the *structure maps* of the factorization algebra:

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$$

for disjoint $U_1, ..., U_n \subseteq V$. These are associative with respect to inclusions; i.e. for disjoint $U_{i1}, ..., U_{ij} \subset V_i$ and disjoint $V_1, ..., V_k \subset W$, the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{F}(U_{11}) \otimes \cdots \otimes \mathcal{F}(U_{1j})) \otimes \cdots \otimes \mathcal{F}(U_{kl}) & & \\ \downarrow & \searrow & \\ \mathcal{F}(V_1) \otimes \cdots \otimes \mathcal{F}(V_k) & \longrightarrow & \mathcal{F}(W) \end{array}$$

This comes from forgetting the intermediate inclusions, as pictorially represented below:

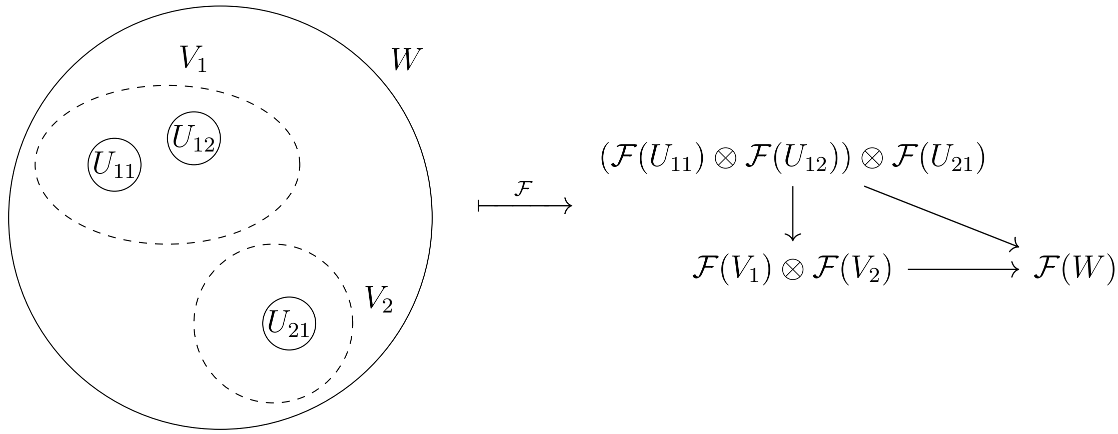


Figure 2.1. Example of associativity property of the structure maps.

A *factorization algebra* is a pre-factorization algebra which is a homotopy cosheaf with respect to a certain type of cover, called *Weiss covers*.

Definition 2.3 ([3], Ch. 6, Defn 1.2.1)**.** A *Weiss cover* $\mathfrak{U} = \{U_a\}$ of U is a cover of U such that for any finite set of points $S \subset U$, there exists a $U_a \in \mathfrak{U}$ such that $S \subset U_a$.

Example 2.4. Consider the interval $[b, c] \subset \mathbb{R}$. As an example of a Weiss cover of $[b, c]$, take $U_a := [b, c] \setminus \{x_a\}$, where $x_a \in [b, c]$.

Definition 2.5 ([3] Ch. 6, Defn 1.3.1)**.** A *factorization algebra on M* is a pre-factorization algebra on M, $\mathcal{F}$, which satisfies the following conditions:

(i) Multiplicative axiom: For disjoint $U_1, ..., U_n \subset M$, the structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \xrightarrow{\sim} \mathcal{F}(U_1 \sqcup \cdots \sqcup U_n)$$

is a weak equivalence.

(ii) Descent axiom: For any Weiss cover $\mathfrak{U} = \{U_a\}$ of an open subset $U \subseteq M$,

$$\check{C}(\mathfrak{U}; \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(U)$$

is a weak equivalence, where $\check{C}(-;-)$ denotes the Čech complex:

$$\check{C}(\mathfrak{U}; \mathcal{F}) := hocolim\Big(\bigoplus_{a_0} \mathcal{F}(U_{a_0}) \leftleftarrows \bigoplus_{a_0,a_1} \mathcal{F}(U_{a_0 a_1}) \cdots \Big)$$

where $U_{a_0 a_1} := U_{a_0} \cap U_{a_1}$.

Remark 2.6. To see the physical motivation for these axioms, it is helpful to consider the concrete case of one of the simplest examples of a classical field theory, scalar field theory (i.e. a field theory where the fields are functionals whose value is invariant when measured on different inertial frames related by a Lorentz transformation). Consider the scalar field theory where the fields on an open subset U are given by $\mathcal{E}(U) := C^\infty(U)$. The simplest field theory would be the one with action functional identically zero, $S \equiv 0$, so all of the fields satisfy the Euler-Lagrange equations: $EL(U) = C^\infty(U)$.

The classical observables on U should be thought of as (heuristically) being functions on this space of solutions to the Euler-Lagrange equations: we denote the classical observables by $Obs^{cl}(U) = C^\infty(EL(U)) =$ "$\Gamma(C^\infty(U))$". We still have to make sense of what we mean by the 'functions' on $C^\infty(U)$, "$\Gamma(C^\infty(U))$" (because $C^\infty(U)$ is in general an infinite dimensional vector space). We will discuss this issue more in the following remark; but for the moment, take the above as a heuristic set-up to motivate the factorization algebra axioms.

If we have $U \subset V$, then we would get a map from $EL(V) \to EL(U)$, by taking fields that satisfy the Euler-Lagrange equation on V, and restricting them to U. Then considering functions on these, $C^\infty(-)$ (a contravariant functor), one gets a map in the other direction: $Obs^{cl}(U) \to Obs^{cl}(V)$. This is the motivation for defining pre-factorization algebras to be *covariant* functors.

To see the physical motivation for the multiplicative axiom, consider two disjoint open subsets $U_1, U_2 \subset M$. Then solutions to the Euler-Lagrange equations on $U_1 \sqcup U_2$ will simply be pairs of solutions on each separate U_i. This gives:

$$\begin{aligned} C^\infty(EL(U_1 \sqcup U_2)) &= C^\infty(EL(U_1) \times EL(U_2)) \\ &\simeq C^\infty(EL(U_1)) \hat{\otimes} C^\infty(EL(U_2)) \end{aligned}$$

where $\hat{\otimes}$ denotes a suitable completion of the tensor product (e.g. the completed projective tensor product).

The descent axiom is motivated by the fact that field theories satisfy a 'locality' condition: knowing the observables on a collection of smaller open subsets of a space U, one should be able to 'glue' those together to acquire the observables on the whole of U. This gives general motivation for considering factorization algebras to be a type of cosheaf.

Remark 2.7. What is the motivation for defining factorization algebras as homotopy cosheaves with respect to Weiss covers specifically? Continuing to look at the example of the classical scalar field theory in the above remark, let's now zoom in to what we mean by "$\Gamma(C^\infty(U))$".

If V is a finite dimensional vector space, then

$$\begin{aligned} \Gamma(V) &:= \text{ polynomial functions on } V \\ &= \bigoplus_{k=0}^{\infty} \{\text{polynomial functions of degree } k\} \\ &= \bigoplus_{k=0}^{\infty} Sym^k(V^*) \\ &=: Sym(V^*) \end{aligned}$$

When we move to the case of $\Gamma(C^\infty(U))$, if we take as a first attempt at a definition:

$$\begin{aligned}\Gamma(C^\infty(U))\text{“}:=\text{”}\,Sym(C^\infty(U)^*)\\ = Sym(Hom^{\text{contin.}}(C^\infty(U),\mathbb{C})),\end{aligned}$$

then we get the symmetric algebra of the space of compactly supported distributions on U.

Knowing that we want to be able to generalize this description of the observables to the quantum level (and also to interacting field theories), we want to replace the space of compactly supported distributions by something ‘smaller’. More specifically, at the quantum level, in order to define the BV-Laplacian, one needs a degree $+1$ pairing on linear observables (given by multiplying the linear observables and then integrating over U). This multiplication does not make sense for distributions: one needs to replace the distributional linear observables with smooth linear observables (or ‘smeared observables’).

$$C_c^\infty(U) \hookrightarrow C^\infty(U)^*$$

This cosheaf gives homotopically equivalent cochain complexes to the cosheaf of compactly supported distributions, making it a suitable substitute, even at the classical level. See Chapter 4, Section 2 (specifically Lemma 2.1.1) of [3] for more details.

Taking $C_c^\infty(U)$ as the suitable substitute for the space of distributions, we define:

$$\begin{aligned}\Gamma(C^\infty(U)) &:= Sym(C_c^\infty(U))\\ &= \bigoplus_{k=0}^{\infty} Sym^k(C_c^\infty(U)).\end{aligned}$$

Recall that $Sym^k(C_c^\infty(U)) = (C_c^\infty(U)^{\otimes k})_{\Sigma_k}$, the coinvariants of the tensor algebra under the symmetric group, where $(\otimes k)$ denotes a suitable completed tensor product (depending on the choice of target category; for example, the completed projective tensor product or the completed bornological tensor product). Coinvariants can be identified as a colimit:

$$Sym^k(C_c^\infty(U)) = (C_c^\infty(U))_{\Sigma_k}^{\otimes k} = colim_{*//\Sigma_k}\Big(* //\Sigma_k \xrightarrow{(C_c^\infty(U))^{\otimes k}} Vect\Big)$$

We will show that $C_c^\infty(-)^{\otimes k}$ is a cosheaf with respect to Weiss covers. Then we will use the above description of $Sym^k(C_c^\infty(U))$ to show that $Sym^k(C_c^\infty(-))$ is also a cosheaf with respect to Weiss covers.

To do this, we will first show that $C^\infty(-)^{\otimes k}$ is a *sheaf* with respect to Weiss covers; sheaves are contravariant, and taking functions with compact support is covariant; taking functions with compact support, $C_c^\infty(-)^{\otimes k}$, we will get a *cosheaf* with respect to Weiss covers.

Take a cover $\mathfrak{U} = \{U_a \hookrightarrow U\}_{a\in A}$ of U, remaining agnostic about what type of cover (ordinary or Weiss) this is for the moment. For $C^\infty(-)^{\otimes k}$ to be a sheaf with respect to this cover, we would need:

$$C^\infty(U)^{\otimes k} \xrightarrow{\simeq ?} eq\Big(\bigoplus_{a_0} C^\infty(U_a)^{\otimes k} \rightrightarrows \bigoplus_{a_0,a_1} C^\infty(U_{a_0a_1})^{\otimes k}\Big)$$

What conditions on the cover $\mathfrak{U}$ give this desired isomorphism in the sheaf condition?

Using the fact that we're dealing with the completed tensor product, note that

$$\begin{aligned} C^\infty(U_a)^{\otimes k} &= C^\infty(U_a \times \cdots \times U_a), \\ C^\infty(U_{a_0a_1})^{\otimes k} &= C^\infty(U_{a_0a_1} \times \cdots \times U_{a_0a_1}) = C^\infty((U_{a_0} \times \cdots \times U_{a_0}) \cap (U_{a_1} \times \cdots \times U_{a_1})). \end{aligned}$$

We can translate this problem into a question about the sheaf of smooth functions, $C^\infty(-)$, which we know is a sheaf with respect to ordinary covers: i.e. for $\{U_a \times \cdots \times$

$U_a\}_{a\in A}$ an ordinary cover for $U\times\cdots\times U$, then we have the identification:

$$C^\infty(U\times\cdots\times U)\xrightarrow{\simeq} eq\Big(\bigoplus_{a_0} C^\infty(U_{a_0}\times\cdots\times U_{a_0})\rightrightarrows\bigoplus_{a_0,a_1} C^\infty((U_{a_0}\times\cdots\times U_{a_0})\cap(U_{a_1}\times\cdots\times U_{a_1}))\Big)$$

What does it mean for $\{U_a\times\cdots\times U_a\}_{a\in A}$ to be an ordinary cover of $U\times\cdots\times U$? Given any point $x=(x_1,...,x_k)\in(U\times\cdots\times U)$, then x must land in some element of the cover; i.e. $x_1,...,x_k\in U_a$ for some U_a in $\mathfrak{U}$. This is precisely what it would mean for $\mathfrak{U}$ to be a Weiss cover for U.

So $C^\infty(-)^{\otimes k}$ is a sheaf with respect to Weiss covers. We claim that $C_c^\infty(-)^{\otimes k}$ is a *cosheaf* with respect to Weiss covers. For more details behind this, see [3] (Ch. 6, Thm 5.2.1; Appendix A, section 4.4; Appendix B section 7.2). Now we want to use this to show that $Obs^{cl}(-)$ is in turn a cosheaf with respect to Weiss covers.

$$\begin{aligned}
Sym^k(C_c^\infty(U)) &= colim_{*//\Sigma_k}\Big((C_c^\infty(U))^{\otimes k}\Big)\\
&\simeq colim_{*//\Sigma_k}\Big(colim_{\cdot\leftleftarrows\cdot}\Big(\bigoplus_{a_0} C_c^\infty(U_{a_0})^{\otimes k}\leftleftarrows\bigoplus_{a_0,a_1} C_c^\infty(U_{a_0a_1})^{\otimes k}\Big)\Big)\\
&\simeq colim_{\cdot\leftleftarrows\cdot}\Big(colim_{*//\Sigma_k}\Big(\bigoplus_{a_0} C_c^\infty(U_{a_0})^{\otimes k}\leftleftarrows\bigoplus_{a_0,a_1} C_c^\infty(U_{a_0a_1})^{\otimes k}\Big)\Big)\\
&\simeq colim_{\cdot\leftleftarrows\cdot}\Big(\bigoplus_{a_0}(C_c^\infty(U_{a_0}))^{\otimes k}_{\Sigma_k}\leftleftarrows\bigoplus_{a_0,a_1}(C_c^\infty(U_{a_0a_1}))^{\otimes k}_{\Sigma_k}\Big)\\
&= colim_{\cdot\leftleftarrows\cdot}\Big(\bigoplus_{a_0} Sym^k(C_c^\infty(U_{a_0}))\leftleftarrows\bigoplus_{a_0,a_1} Sym^k(C_c^\infty(U_{a_0a_1}))\Big)
\end{aligned}$$

Thus we get that it is precisely with respect to Weiss covers that $Sym^k(C_c^\infty(-))$, our model for the observables in a classical scalar field theory, satisfies the descent condition; thus we take Weiss covers to be the appropriate notion of cover in the definition of the descent axiom for factorization algebras.

In more abstract language, this motivating example is a case of the following more general motivation: for any topological space X, one can consider the Ran space $Ran(X)$ of all non-empty finite subsets of X. One can put an interesting topology on this space $Ran(X)$; see section 5.5.1 of [10] for more details. It turns out that ordinary covers of $Ran(X)$ for this topology come from Weiss covers on the original space X.

2.1.2 Some examples

Example 2.8. Factorization algebra on $\mathbb{R}$: Let A be an associative algebra. We can construct a factorization algebra on $\mathbb{R}$ as follows. Define

$$\mathcal{F}((i,j)) := A$$

for any interval $(i,j) \subset \mathbb{R}$.

Given two open, disjoint subsets $(i',j'),(i'',j'') \subset (i,j)$ (in other words, a triple of intervals with $i < i' < j' < i'' < j'' < j$, as illustrated below), the structure map coming from the inclusion $(i',j') \sqcup (i'',j'') \hookrightarrow (i,j)$ is defined by the multiplication map $A \otimes A \xrightarrow{\mu} A$.

This is an example of a *locally constant factorization algebra.*

Definition 2.9. A *locally constant factorization algebra* is a factorization algebra $\mathcal{F}$ such that for any $U \subset U'$ where U is a deformation retract of U', $\mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(U')$ is a weak equivalence.

It turns out that, conversely, a locally constant factorization algebra on $\mathbb{R}$ determines an E_1-object in the target category of the factorization algebra (i.e. a weakly associative algebra). In fact, this generalizes to higher dimensions, as shown in the following classification result.

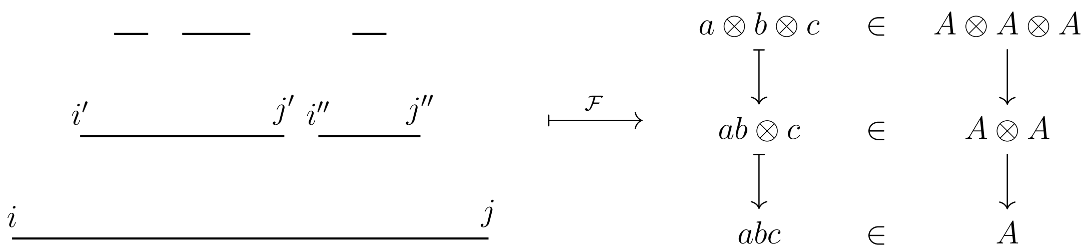


Figure 2.2. Example of inclusions for a locally constant factorization algebra on $\mathbb{R}$.

Theorem 2.10 ([10], Thm 5.4.5.9)**.** *E_n-algebras are equivalent (as $(\infty, 1)$-categories) to locally constant factorization algebras on $\mathbb{R}^n$.*

Example 2.11. Factorization algebra on S^1: Again, let A be an associative algebra. Identifying the circle as $S^1 = [0,1]/_{0\sim 1}$, define a factorization algebra $\mathcal{F}$ as follows. For any interval $(b, c) \subset S^1$, take

$$\mathcal{F}((b,c)) := A.$$

To evaluate $\mathcal{F}$ on the whole circle, we use the descent axiom. Take the Weiss cover on S^1 given by

$$\mathfrak{U} = \{U_a := [0,1] \setminus \{x_a\} | x_a \in [0,1)\}.$$

Then

$$\begin{aligned}
\mathcal{F}(S^1) &\xleftarrow{\sim} hocolim\Big(\bigoplus_{a_0} \mathcal{F}(S^1 \setminus \{x_{a_0}\}) \leftleftarrows \bigoplus_{a_0,a_1} \mathcal{F}(S^1 \setminus \{x_{a_0}, x_{a_1}\}) \cdots\Big) \\
&\simeq hocolim\Big(A \leftleftarrows A \otimes A \cdots\Big) \\
&\simeq Tot\Big(A \leftleftarrows A \otimes A \cdots\Big)
\end{aligned}$$

This is the Hochschild chain complex associated to A. (See [5], Cor 5.)

Example 2.12. Costello-Gwilliam [3, 4] show how to construct factorization algebras from perturbative QFT; topological examples, such as perturbative Chern-Simons theory, give locally constant factorization algebras.

2.1.3 G-equivariance

Field theories that arise in physics often involve symmetries of the physical system. One of the ways to implement this in terms of the observables of the physical system is to look at *equivariant factorization algebras*, where the spacetime manifold M is equipped with an action by a group G, and this G-action is required to be compatible with the factorization algebra structure as described in the following definition.

Definition 2.13 ([3], Defn 7.1.1; [5], Defn 18)**.** Let G be a group acting on M: $G \times M \to M$. A *G-equivariant factorization algebra on M* is a factorization algebra

$$\mathcal{F} : Open(M) \to Ch$$

together with quasi-isomorphisms

$$\sigma_g^U : F(U) \to F(gU)$$

for every $g \in G$ and every $U \subseteq M$, which satisfy the following conditions:

(i) $\sigma_1 = id$

(ii) $\sigma_{gh}^U = \sigma_g^{hU} \circ \sigma_h^U : \mathcal{F}(U) \to \mathcal{F}(ghU)$

(iii) the σ_i are compatible with the structure maps of the factorization algebra, i.e. squares of the following form commute:

$$\begin{array}{ccc} \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) & \xrightarrow{\simeq} & \mathcal{F}(gU_1) \otimes \cdots \otimes \mathcal{F}(gU_n) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\simeq} & \mathcal{F}(gV) \end{array}$$

Alternatively, one can think of G-equivariant factorization algebras as being factorization algebras over a multicategory $Open^G(M)$ defined as follows:

Definition 2.14 ([3], Defn 7.2.1)**.** Let M be a manifold with a group G acting on it. Then $Open^G(M)$ is the multicategory (i.e. colored operad) with objects the opens in M and multimorphism sets given by

$$Open^G(M)(U_1, ..., U_n|V) = \{(g_1, ..., g_n) \in G^n | \forall i, g_i U_i \subset V; \forall i \neq j, g_i U_i \cap g_j U_j = \emptyset\}.$$

Definition 2.15. A *G-equivariant factorization algebra on M* is a multicategory functor

$$\mathcal{F} : Open^G(M) \to Ch$$

which satisfies the following conditions:

(i) Multiplicative axiom: For disjoint $U_1, ..., U_n \subset M$, the structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \xrightarrow{\sim} \mathcal{F}(U_1 \sqcup \cdots \sqcup U_n)$$

is a weak equivalence.

(ii) Descent axiom: For any Weiss cover $\mathfrak{U} = \{U_a\}$ of an open subset $U \subseteq M$,

$$hocolim\Big(\bigoplus_{a_0} \mathcal{F}(U_{a_0}) \leftleftarrows \bigoplus_{a_0, a_1} \mathcal{F}(U_{a_0 a_1}) \cdots \Big) \xrightarrow{\sim} \mathcal{F}(U)$$

is a weak equivalence.

Lemma 7.2.2 of [3] shows that 2.13 and 2.15 are equivalent. We will use the notion of G-equivariant factorization algebras as maps out of the multicategory $Open^G(M)$ for our work.

Example 2.16 ([5], Propn 21)**.** If G is a discrete group and acts properly and discontinuously on M, then G-equivariant factorization algebras on M are equivalent to factorization algebras on M/G.

Example 2.17 ([5], Propn 22)**.** Let $q : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be the universal cover of S^1. Locally constant factorization algebras on S^1 are equivalent to $\mathbb{Z}$-equivariant locally constant factorization algebras on $\mathbb{R}$. If $\mathcal{F}$ is a locally constant factorization algebra on S^1, $q^*(\mathcal{F})$ is the corresponding $\mathbb{Z}$-equivariant locally constant factorization algebra on $\mathbb{R}$.

Recall that by 2.10 locally constant factorization algebras on $\mathbb{R}$ are equivalent to E_1-algebras. Also, the $\mathbb{Z}$-equivariance gives us a map:

$$\sigma_1 : 1^*(q^*(\mathcal{F})) \xrightarrow{\simeq} q^*(\mathcal{F}).$$

Using this and the canonical equivalence coming from the locally constant property of $\mathcal{F}$, we get the following self-equivalence:

$$mon : q^*\mathcal{F} \simeq 1^*(q^*\mathcal{F}) \xrightarrow{\sigma_1} q^*\mathcal{F}.$$

This is called the 'monodromy' of $\mathcal{F}$.

Then [5], Cor 4, gives us

$$Fac^{l.c.}_{S^1} \simeq Aut(E_1 - Alg)$$

where the right side is the ∞-category of E_1-algebras equipped with self-equivalences.

Using a similar argument, $Fac^{l.c.}_{S^1 \times S^1}$ is equivalent to the category of E_2-algebras equipped with commuting monodromies (i.e. self-equivalences).

Example 2.18 ([7], Ch. 6)**.** The $\beta\gamma$-system of mass m gives the following factorization algebra:

$$Obs^q(U) := \Big(Sym(\Omega_c^{1,*}(U)\,[1] \oplus \Omega_c^{0,*}(U)\,[1])\,[\hbar]\,, Q + \hbar\Delta \Big),$$

where U is an open of $\mathbb{C}$, $Q = \bar{\partial} - md\bar{z}$, $\bar{\partial}$ is the Dolbeault differential and Δ is the BV Laplacian. This is an example of a $\mathbb{R}^2$-equivariant factorization algebra.

Example 2.19 ([3], Ch. 4, Propn 3.0.3)**.** If $\mathcal{F}$ is a locally constant smoothly translation invariant factorization algebra on $\mathbb{R}$ valued in vector spaces which corresponds to the observables of a free scalar field theory on $\mathbb{R}$ with mass m, then the associative algebra given by $\mathcal{F}((0,1)) = A$ is the Weyl algebra. The smooth translation action gives a smooth map $\mathbb{R} \to Aut(A)$; differentiating this map and evaluating on the basis element gives the Hamiltonian of the field theory.

As an additional connection to physically motivated examples, note that Costello-Gwilliam show in general how unital, S^1-equivariant, holomorphically translation invariant pre-factorization algebras on $\mathbb{C}$, valued in differentiable vector spaces (with some additional assumptions), give vertex algebras ([3], Thm 2.2.1).

2.2 $\mathcal{G}$-factorization algebras

Inspired by the fact that factorization algebras are built with a local-to-global structure (the descent axiom), rather than simply looking at factorization algebras on open subsets of a fixed manifold, one could look at factorization algebras that live on a site of manifolds equipped with a geometric structure that can also be described locally. In this dissertation, we will be looking at the particular case where a rigid geometric structure on the manifolds in question is given by the action of a group G on a model space M: we want to consider manifolds which locally look like M and whose transition functions are given by the G-action. We denote this pair $(M, G) =: \mathcal{G}$; before defining this new version of $\mathcal{G}$-factorization algebras, we define the site of manifolds with $\mathcal{G}$-structure which these factorization algebras will live over. This definition comes from the $\mathcal{G}$-bordism categories in [16].

2.2.1 $\mathcal{G}$Man

Let M^n be a manifold and G a group with a left action on M: $G \times M \to M$.

Definition 2.20. The category $\mathcal{G}$Man consists of:

- **objects:** manifolds X^n with $\mathcal{G} := (G, M)$ structure, i.e. a maximal atlas of charts:

$$\{X \supset U_i \xrightarrow{\phi_i} V_i \subset M\}_{i \in I}$$

  which are diffeomorphisms, where the U_i cover X; as well as the data of a collection $\{g_{ij} \in G\}$ which determine the transition functions:

$$\begin{array}{ccc} & U_i \cap U_j & \\ {}^{\phi_i}\swarrow & & \searrow^{\phi_j} \\ V_i & \xrightarrow{g_{ij}} & V_j \end{array}$$

  These are required to satisfy a cocycle condition:

$$g_{jk} \circ g_{ij} = g_{ik}.$$

- **morphisms:** smooth embeddings $f : X \hookrightarrow Y$, and a collection $\{f_{i,i'} \in G\}$ for every pair of charts $(U_i, \phi_i), (U_i', \phi_i')$ in the $\mathcal{G}$ structure for X and Y, respectively, where $f(U_i) \subseteq U_{i'}$. These are required to make diagrams of the following form commute:

$$\begin{array}{ccc} U_i & \xrightarrow{f} & U_i' \\ {\scriptstyle\phi_i}\downarrow & & \downarrow{\scriptstyle\phi_i'} \\ M & \xrightarrow[f_{i,i'}]{} & M \end{array}$$

- **composition:** composing embeddings; given two embeddings $f : X \hookrightarrow Y$, $g : Y \hookrightarrow Z$, then $g \circ f$ gives an embedding $X \hookrightarrow Z$.
- **monoidal structure:** disjoint union; for $X, Y \in \mathcal{G}$Man, $X \sqcup Y$ is a $\mathcal{G}$-manifold with the $\mathcal{G}$-atlas given by taking the union of the atlases for X and Y and taking the maximal atlas this belongs to. (Note that for U_i an element of the atlas on X and V_j an element of the atlas on Y, the transition function for $U_i \cap V_j$ will be trivial because $U_i \cap V_j = \emptyset$.)

Remark 2.21. Note that the elements $\{f_{i,i'} \in G\}$ are extra data included in a morphism from X to Y in $\mathcal{G}Man$.

In particular, note that for U, V connected open subsets of M, the morphisms $U \hookrightarrow V$ considered as objects in $\mathcal{G}Man$ are different from the morphisms $U \hookrightarrow V$ when considered as objects of $Open(M)$. Viewed as objects in $\mathcal{G}Man$, the morphisms are of the following form:

Lemma 2.22. *For $U, V \in Open^G(M) \subset \mathcal{G}Man$, a morphism $f \in \mathcal{G}Man(U, V)$ can be factored in the following form:*

$$f : U \xrightarrow{g} gU \overset{i}{\hookrightarrow} V$$

for some $g \in G$.

Proof. Consider the identity charts $U \xrightarrow{\phi_U = id} U$, $V \xrightarrow{\phi_V = id} V$ in the maximal atlases for U and V, respectively. Then the fact that f is $\mathcal{G}$-structure preserving means that we have:

(i) a smooth embedding $f : U \hookrightarrow V$

(ii) because $f(U) \subseteq V$, we have an element $f_{UV} \in G$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \overset{f}{\hookrightarrow} & V \\ {\scriptstyle \phi_U = id}\downarrow & & \downarrow{\scriptstyle \phi_V = id} \\ M & \underset{f_{UV}}{\longrightarrow} & M \end{array}$$

This diagram gives that $f_{UV}(U) \subset V$; take f_{UV} as the element of G in the above factorization. $\square$

Some examples of $\mathcal{G}$-structures of particular interest to us are the following:

Example 2.23. Euclidean structure: Take $M = \mathbb{R}^n, G = \mathbb{R}^n \rtimes O(n)$ (where $\mathbb{R}^n$ corresponds to the translations and $O(n)$ corresponds to the dilations). In this case, a $\mathcal{G}$-structure on X means that X has a flat Riemannian metric (i.e. a Euclidean structure); it is induced by pulling back the standard Riemannian metric on the open subsets $U_i \subset \mathbb{R}^n$ via the chart maps, ϕ_i.

Example 2.24. Euclidean spin structure: Take $M = \mathbb{R}^n, G = \mathbb{R} \rtimes Spin(n)$, proceed as above.

Example 2.25. Conformal Euclidean structure: Take $M = \mathbb{R}^n, G = \mathbb{R}^n \rtimes (SO(n) \times \mathbb{R}_+)$, where $\mathbb{R}_+$ acts by dilations of $\mathbb{R}^n$.

2.2.2 $\mathcal{G}$-factorization algebras

We now want to define a generalized version of factorization algebras which, instead of living over the opens of a fixed manifold M, will live over the category $\mathcal{G}Man$. We will call these $\mathcal{G}$-factorization algebras.

Analogous to the factorization algebras on M, $\mathcal{G}$-factorization algebras will be cosheaves with respect to Weiss covers. In the case of $\mathcal{G}Man$, a Weiss cover is as follows:

Definition 2.26. A Weiss cover of $X \in \mathcal{G}$ Man is a collection of morphisms in $\mathcal{G}$Man, $\mathfrak{U} = \{U_a \overset{f_a}{\hookrightarrow} X\}_{a \in A}$, such that for any finite set of points $S \subseteq X$, there is some $U_a \overset{f_a}{\hookrightarrow} X$ in $\mathfrak{U}$ such that $S \subseteq f_a(U_a)$.

Definition 2.27. A $\mathcal{G}$*-factorization algebra* is a (lax) symmetric monoidal functor

$$\hat{\mathcal{F}} : \mathcal{G}Man \to Ch$$

which satisfies:

(i) Multiplicative axiom: For any $X_1, ..., X_n \in \mathcal{G}$Man, the map from the lax symmetric monoidal structure

$$\hat{\mathcal{F}}(X_1) \otimes \cdots \otimes \hat{\mathcal{F}}(X_n) \xrightarrow{\sim} \hat{\mathcal{F}}(X_1 \sqcup \cdots \sqcup X_n)$$

is a weak equivalence.

(ii) Descent axiom: For any Weiss cover $\mathfrak{U} = \{U_a \to X\}$ of $X \in \mathcal{G}Man$,

$$hocolim\Big(\bigoplus_{a_0} \hat{\mathcal{F}}(U_{a_0}) \leftleftarrows \bigoplus_{a_0,a_1} \hat{\mathcal{F}}(U_{a_0} \times_X U_{a_1}) \cdots \Big) \xrightarrow{\sim} \hat{\mathcal{F}}(X)$$

is a weak equivalence.

CHAPTER 3

NEW VERSION OF THE DESCENT AXIOM

Our eventual goal is to develop an ∞-categorical generalization of factorization algebras, which will be well suited to talk about families (or parameterized versions) of factorization algebras. Before doing so, it is helpful to first relate the descent axiom for the factorization algebras in chapter 2, which was very geometric, to a more categorical version of the descent axiom. We develop this categorical version of the descent axiom in this section, and show it is equivalent to the previous descent axioms in definitions 2.5, 2.15 and 2.27. The generalization to ∞-categorical versions of factorization algebras will be done in chapter 4; the categorical version of the descent axiom from this section will generalize particularly nicely in that context.

To motivate the new categorical descent axiom, note what role descent philosophically plays. For a factorization algebra $\mathcal{F}$ to satisfy descent means that $\mathcal{F}$ satisfies a locality condition: evaluating $\mathcal{F}$ on a manifold X is equivalent to evaluating $\mathcal{F}$ on a collection of smaller subsets of X and then gluing that data together. In other words, we pull back $\mathcal{F}$ to a cover of X, i.e. restrict $\mathcal{F}$ to that cover; and then 'push it forward' via gluing to say what $\mathcal{F}$ gives on the whole X, i.e. take a left Kan extension related to the cover. If this restriction-left Kan extension composite agrees with what $\mathcal{F}$ originally gave on X, then $\mathcal{F}$ satisfies descent.

In order to formally describe this categorical process of restriction-left Kan extension, we will want to deal with a precise categorical formulation of Weiss covers: this is given by the notion of a Grothendieck topology on a category. In section 3.1 we review the axioms of a Grothendieck topology and look at what this means for our categories and covers of interest. Then in section 3.2 we give a categorical version of the descent axiom, and show that it agrees with the previous versions of the descent axiom in chapter 2.

3.1 Background on Grothendieck topologies

Let $\mathcal{C}$ be a small category.

Definition 3.1 ([12], Section I.4)**.** A *sieve* $\mathcal{S}$ on an object $c \in \mathcal{C}$ is a full subcategory $\mathcal{S} \subset \mathcal{C}/c$ which is closed under precomposition with morphisms in $\mathcal{C}$; i.e. if $f : c' \to c$ is in $\mathcal{S}$ and $g : d \to c'$ is a morphism in $\mathcal{C}$, then $f \circ g$ is in $\mathcal{S}$.

Given a morphism $h : d \to c$ in $\mathcal{C}$ and a sieve $\mathcal{S}$ on c, the *pullback sieve* $h^*\mathcal{S}$ (on d) is defined to be the full subcategory of $\mathcal{C}/d$ spanned by morphisms $g : d' \to d$ such that $h \circ g$ is in $\mathcal{S}$.

Definition 3.2 ([12], III.2, Defn 1)**.** A *Grothendieck topology* on a category $\mathcal{C}$ is a function J which assigns to each object $c \in \mathcal{C}$ a collection $J(c)$ of sieves on c, called *covering sieves*, which satisfy the following conditions:

(i) (Trivial sieve covers:) For all $c \in \mathcal{C}$, $\mathcal{C}/c$ is in $J(c)$.

(ii) (Stability axiom:) If $f : c \to d$ is a morphism in $\mathcal{C}$ and $\mathcal{S} \in J(d)$, then $f^*\mathcal{S} \in J(c)$.

(iii) (Transitivity axiom:) Let $c \in \mathcal{C}$, $\mathcal{S}$ be in $J(c)$ and $\mathcal{S}'$ be an arbitrary sieve on c. If for every $f : d \to c$ in $\mathcal{S}$, the pullback $f^*\mathcal{S}'$ is in $J(d)$, then $\mathcal{S}'$ is in $J(c)$.

A *site* is a pair $(\mathcal{C}, J)$ of a small category $\mathcal{C}$ with a Grothendieck topology J on $\mathcal{C}$.

Remark 3.3. Note that for the sources of our factorization algebras, we are dealing with something slightly more complicated than a category: $Open(M), Open^G(M)$ are multicategories and $\mathcal{G}Man$ is a symmetric monoidal category. For these specific cases, we take a Grothendieck topology on a multicategory (resp. symmetric monoidal category) to be a Grothendieck topology on its underlying category (i.e. we only allow multimorphisms which have a single object as the source, so are ordinary morphisms). This is not necessarily a general notion of a Grothendieck topology on any multicategory, but it works for the specific multicategories $Open(M), Open^G(M)$ and $\mathcal{G}Man$.

This notion of a Grothendieck topology works for these specific examples of multicategories $\mathcal{B} = Open(M), Open^G(M), \mathcal{G}Man$ because they all share a special property: if $f \in Mul_{\mathcal{B}}(b_1, ..., b_n; b)$, then the source of f can be thought of as another object of $\mathcal{B}$ itself (for $Open(M)$, $U_1 \sqcup \cdots \sqcup U_n \hookrightarrow U$; for $Open^G$, $g_1U_1 \sqcup \cdots \sqcup g_nU_n \hookrightarrow U$; for $\mathcal{G}Man$, $X_1 \sqcup \cdots \sqcup X_n \hookrightarrow X$). This allows us to make sense of pullback sieves in these multicategories, which is needed for the stability and transitivity axioms of a Grothendieck topology. However, for an arbitrary multicategory, the multimorphisms don't necessarily have such a property.

For our multicategories of interest $\mathcal{B} = Open(M), Open^G(M), \mathcal{G}Man$, we need to consider a saturated version of the Weiss covers, in order to obtain a collection of sieves:

Definition 3.4. Let $\mathcal{B} = Open(M), Open^G(M)$ or $\mathcal{G}Man$. Let $\mathfrak{U} = \{u_a \to b\}$ be a Weiss cover of $b \in \mathcal{B}$. Define $\overline{\mathfrak{U}}$ to be the full subcategory of $\mathcal{B}/b$ consisting of objects which factor through an element of $\mathfrak{U}$, i.e.:

$$\begin{array}{ccc} v & & \\ \downarrow & \searrow & \\ u_a & \longrightarrow & b \end{array}$$

for some $(u_a \to b) \in \mathfrak{U}$. We refer to $\overline{\mathfrak{U}}$ as a *saturated Weiss cover* of b.

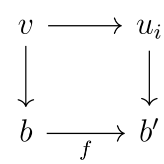
Note that there is a canonical forgetful functor $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{B}$, which forgets the map to b; when we refer to a saturated cover of an object $b \in \mathcal{B}$ we will often include this canonical functor.

Lemma 3.5. *Let $\mathcal{B} = Open(M), Open^G(M)$ or $\mathcal{G}Man$. The collection of all saturated Weiss covers defines a Grothendieck topology on $\mathcal{B}$, called the* Weiss topology.

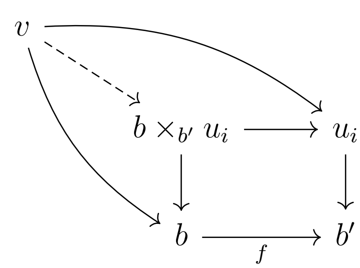
Proof. Note that by definition, the saturated Weiss covers are sieves (closed under precomposition).

(i) For all $b \in \mathcal{B}$, the identity map $b \to b$ gives a Weiss cover. Taking the saturated version of this gives the overcategory $\mathcal{B}/b$ itself, which is thus a covering sieve.

(ii) If $f : b \to b'$ is a 1-ary morphism in $\mathcal{B}$ and $\overline{\mathfrak{U}'}$ is a saturated Weiss cover of b', then $f^*\overline{\mathfrak{U}'}$ consists of objects $v \to b$ of the following form:



which factor through some element u_i of the Weiss cover $\mathfrak{U}'$ of b'. Taking the pullback gives an element of a Weiss cover of b; by the universal property of the pullback, v factors through this element:



Varying over all elements u_i of $\mathfrak{U}'$, the pullbacks $b \times_{b'} u_i$ provide a Weiss cover of b. Thus $f^*\overline{\mathfrak{U}'}$ is a covering sieve of b.

(iii) Let $b \in \mathcal{B}$ and $\overline{\mathfrak{U}}$ be a saturated Weiss cover of b. Consider any other sieve $\mathcal{S}'$ on b, such that for any $f : b' \to b$ in $\overline{\mathfrak{U}}$, $f^*\mathcal{S}'$ is a saturated Weiss cover for b'. In particular, for $X \subseteq b$ a finite set of points in b, then because $\overline{\mathfrak{U}}$ is a saturated Weiss cover of b, there exists some $u_i \xrightarrow{f} b$ such that $X \subseteq f(u_i)$. The preimage $f^{-1}(X) \subseteq u_i$ will be a finite set of points. Because $f^*\mathcal{S}'$ is a saturated Weiss cover for u_i, there exists a $v_j \xrightarrow{g} u_i$ in $f^*\mathcal{S}'$ such that the image contains $f^{-1}(X)$. By definition, $(v_j \xrightarrow{f\circ g} b) \in \mathcal{S}'$, and $X \subseteq f \circ g(v_j)$; thus $\mathcal{S}'$ is a saturated Weiss cover of b.

□

In our cases of interest ($Open(M), Open^G(M), \mathcal{G}Man$, with their respective Weiss topologies), the saturated Weiss covers $\overline{\mathfrak{U}}$ satisfy an even stronger property:

Condition 3.6. Let $\overline{\mathfrak{U}}$ be a saturated Weiss cover of $b \in \mathcal{B}$. If $(v \xrightarrow{\phi} b)$ is an element of $\overline{\mathfrak{U}}$, then for any other map ψ, $(v \xrightarrow{\psi} b)$ is also in $\overline{\mathfrak{U}}$.

We will refer to this as the *translation property* of $\overline{\mathfrak{U}}$, as it comes, in our specific examples, by considering possible translations of v by the G-action, and requiring that these still sit inside some element of the Weiss cover.

3.2 Categorical descent axiom

Having this construction of a Grothendieck topology associated to Weiss covers, we can now describe a categorical version of the descent axiom as a composite of restriction-left Kan extension; we show this is equivalenct to the Čech complex.

Proposition 3.7. *Let $\mathcal{B} = Open(M), Open^G(M), \mathcal{G}Man$. Let $\mathfrak{U} = \{u_a \to b\}$ be a Weiss cover of $b \in \mathcal{B}$, and $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{B}$ its associated saturated Weiss cover (as in 3.4). Define $\mathcal{U}$ to be the full subcategory of $\mathcal{B}$ consisting of objects in the image of $\bar{j}$; denote the inclusion functor by $j : \mathcal{U} \to \mathcal{B}$. Then*

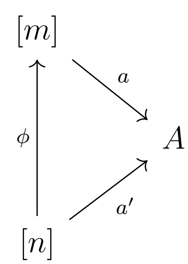
$$\operatorname{hocolim}\Big(\bigoplus_{a_0} \mathcal{F}(u_{a_0}) \leftleftarrows \bigoplus_{a_0,a_1} \mathcal{F}(u_{a_0} \times_b u_{a_1}) \cdots \Big) \simeq j_! j^* \mathcal{F}(b).$$

We will break down the proof of 3.7 into several lemmas: First, in 3.9 we reindex the Čech colimit using a new category which arranges the data in a more convenient combinatorial way; we define this new category in 3.8. In 3.12 we relate this to a colimit involving the saturated Weiss cover, using the Grothendieck topology formalism introduced in the previous section. Finally, in 3.13 we relate this to the restriction-left Kan extension composite. This equivalence of colimits allows us to reformulate the original definitions of the factorization algebras given in chapter 2 in terms of a more categorical descent axiom.

Take $\mathcal{B} = Open(M), Open^G(M)$ or $\mathcal{G}Man$. We start by defining a category which we will use to re-index the Čech colimit.

Definition 3.8. Given a set A, define the category $P_{fin}(A)$ to consist of:

- **objects:** $a : [m] \to A$ for $[m] \in \Delta^{op}$
- **morphisms:** a morphism from $[m] \xrightarrow{a} A$ to $[n] \xrightarrow{a'} A$ consists of a map in Δ^{op} from $[m]$ to $[n]$ (i.e. an order-preserving map $\phi : [n] \to [m]$), which makes the following diagram commute:



When A is the indexing set corresponding to a cover $\mathfrak{U} = \{u_a\}_{a \in A}$ of $b \in \mathcal{B}$, there is a natural functor:

$$\alpha : P_{fin}(A) \to \mathcal{B}$$

which sends an object $a : [m] \to A$ to $\alpha(a) := u_{a(0)} \times_b \cdots \times_b u_{a(m)}$. A morphism $\phi : a \to a'$ is sent to an inclusion map:

$$u_{a(0)} \times_b \cdots \times_b u_{a(m)} \hookrightarrow u_{a'(0)} \times_b \cdots \times_b u_{a'(n)}.$$

We now want to re-index the colimit for the Čech complex in 2.5 in terms of the category $P_{fin}(A)$.

Lemma 3.9. *Let $\mathfrak{U} = \{u_a \to b\}_{a\in A}$ be a Weiss cover of $b \in \mathcal{B}$. There is a weak equivalence:*

$$\mathrm{hocolim}\left(\bigoplus_{a_0} \mathcal{F}(u_{a_0}) \leftleftarrows \bigoplus_{a_0,a_1} \mathcal{F}(u_{a_0} \times_b u_{a_1}) \cdots\right) \simeq \mathrm{hocolim}\left(P_{fin}(A) \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right)$$

Proof. Denote the composition of the functors on the right by

$$G := \mathcal{F} \circ \alpha : P_{fin}(A) \to Ch.$$

Then consider the diagram:

$$\begin{array}{ccccc} P_{fin}(A) & \xrightarrow{\beta} & \Delta^{op} & \xrightarrow{T} & * \\ {\scriptstyle G}\downarrow & \swarrow{\scriptstyle \beta_! G} & & & \\ Ch & & & & \end{array}$$

where $\beta : P_{fin}(A) \to \Delta^{op}$ denotes the forgetful functor, which forgets the map into A; and T denotes the terminal functor.

Note that the functorial property of the left Kan extension gives:

$$\begin{aligned} \mathrm{hocolim}_{\Delta^{op}}(\beta_! G) &\simeq T_!(\beta_! G) \\ &= (T \circ \beta)_!(G) \\ &\simeq \mathrm{hocolim}_{P_{fin}(A)}(G). \end{aligned}$$

We want to show that $\mathrm{hocolim}_{\Delta^{op}}(\beta_! G)$ corresponds to the Čech complex. Note that the Čech complex is also a colimit indexed by the simplicial category Δ^{op}; the n-simplices correspond to taking a direct sum of the factorization algebra evaluated on all possible intersections of n-tuples of elements of the cover $\mathfrak{U}$. We will show that the $\beta_! G$ recovers this direct sum.

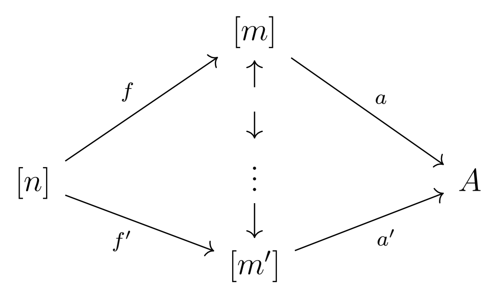
By definition of the left Kan extension, $\beta_! G$ acts on elements $[n] \in \Delta^{op}$ as follows:

$$(\beta_! G)([n]) = \operatorname{hocolim}\left((P_{fin}(A)/ [n]) \xrightarrow{\text{forget}} P_{fin}(A) \xrightarrow{G} Ch \right)$$

The objects in $P_{fin}(A)/ [n]$ consist of pairs (a, f) where a is an object of $P_{fin}(A)$ and f is a map in Δ^{op}:

$$[n] \xrightarrow{f} [m] \xrightarrow{a} A.$$

Each connected component of $P_{fin}(A)/ [n]$ contains a terminal object: if $(a, f), (a', f')$ are in the same connected component, there is a zig-zag of maps:



such that the whole diagram commutes. In particular, for any $(a, f), (a', f')$ in the same connected component, $a \circ f = a' \circ f'$. The terminal object for the connected component containing (a, f) can then be taken to be:

$$[n] \xrightarrow{id} [n] \xrightarrow{a \circ f} A$$

This composite corresponds to a particular n-tuple of elements in A. Because the homotopy colimit breaks into the direct sum of the homotopy colimit for each connected component of the indexing category, and each connected component of $P_{fin}(A)/ [n]$ has a terminal object, one obtains the following:

$$
\begin{aligned}
(\beta_! G)([n]) &= \operatorname{hocolim}\left((P_{fin}(A)/[n]) \xrightarrow{\text{forget}} P_{fin}(A) \xrightarrow{G} Ch \right) \\
&\simeq \bigoplus_{conn.comp.} G(a \circ f) \\
&\simeq \bigoplus_{a_0,\ldots,a_n} \mathcal{F}(u_{a_0} \times_b \cdots \times_b u_{a_n})
\end{aligned}
$$

Thus,

$$
\operatorname{hocolim}\left(\bigoplus_{a_0} \mathcal{F}(u_{a_0}) \leftleftarrows \bigoplus_{a_0,a_1} \mathcal{F}(u_{a_0} \times_b u_{a_1}) \cdots \right) \simeq \operatorname{hocolim}\left(P_{fin}(A) \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\mathcal{F}} Ch \right)
$$

□

We now want to relate the colimit involving a Weiss cover $\mathfrak{U}$ to the colimit involving its associated saturated Weiss cover $\overline{\mathfrak{U}}$. We will use the $P_{fin}(-)$ category to organize and relate the different information.

Before doing so, we recall some definitions and results that will be of use.

Definition 3.10 ([14], Defn 8.5.1)**.** A functor $K : \mathcal{C} \to \mathcal{D}$ is *homotopy final* if for all $d \in \mathcal{D}$ the simplicial set $N(d/K)$ is contractible.

The reason why homotopy final functors will be useful for us has to do with the following result comparing different homotopy colimits.

Proposition 3.11 ([14], Thm 8.5.6)**.** *Let $F : \mathcal{D} \to \mathcal{M}$ be any functor into a simplicial model category. If $K : \mathcal{C} \to \mathcal{D}$ is homotopy final, then*

$$
\operatorname{hocolim}_{\mathcal{C}}(F \circ K) \xrightarrow{\sim} \operatorname{hocolim}_{\mathcal{D}}(F)
$$

is a weak equivalence.

Note that we are dealing with a simplicial model category $\mathcal{M} = Ch$ in all of the situations we consider.

Using these tools, we now relate the colimit indexed by $P_{fin}(A)$ to a colimit indexed by the saturated Weiss cover.

Lemma 3.12. *Let $\mathfrak{U} = \{u_a\}_{a\in A}$ be a Weiss cover of $b \in \mathcal{B}$. Consider the saturated version of this cover, $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{B}$. as in 3.4. Then there is a weak equivalence:*

$$\operatorname{hocolim}\left(P_{fin}(A) \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right) \simeq \operatorname{hocolim}\left(\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right)$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} P_{fin}(A) & \xrightarrow{K} & \overline{\mathfrak{U}} \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \bar{j}} \\ & & \mathcal{B} \\ & & \downarrow{\scriptstyle \mathcal{F}} \\ & & Ch \end{array}$$

where K sends $([m] \xrightarrow{a} A) \in P_{fin}(A)$ to $(u_{a(0)} \times_b \cdots \times_b u_{a(m)} \to b) \in \overline{\mathfrak{U}}$ (note that by construction this factors through an element of $\mathfrak{U}$, namely any of the $u_{a(i)}$'s).

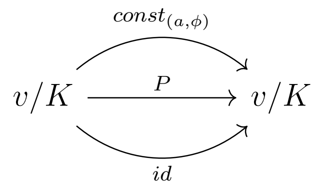
We claim that K is homotopy final; by the commutativity of the above diagram, this gives the desired equivalence of homotopy colimits.

Take an arbitrary object $(v \to b) \in \overline{\mathfrak{U}}$; note that this means the maps factors through $v \to u_i \to b$ for some $(u_i \to b)$ of $\mathfrak{U}$. Consider the slice category v/K. The objects of this category consist of $([m] \xrightarrow{a} A; \phi)$ where $([m] \xrightarrow{a} A) \in P_{fin}(A)$ and ϕ is a map in $\mathcal{B}/b$ making the following diagram commute:

$$\begin{array}{ccc} v & \longrightarrow & b \\ \downarrow{\scriptstyle \phi} & \nearrow & \\ u_{a(0)} \times_b \cdots \times_b u_{a(m)} & & \end{array}$$

Because of the definition of v as an object that factors through some element of the cover $\mathfrak{U}$, the category v/K is always non-empty.

Take an arbitrary object $(a, \phi) \in v/K$. Consider the following diagram:



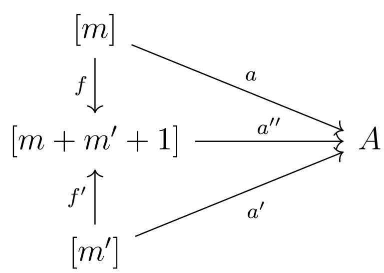
where $const_{(a,\phi)}$ is the constant functor which sends every object to (a, ϕ), and P is the functor which sends (a', ϕ') to $(a'', \phi'') := P(a', \phi')$ where

$$[m + m' + 1] \xrightarrow{a''} A$$

$$a''(i) := \begin{cases} a(i), \text{ if } 0 \leq i \leq m, \\ a'(i - m - 1), \text{ if } m + 1 \leq i \leq m + m' + 1 \end{cases} .$$

and $\phi'' : v \to u_{a''(0)} \times_b \cdots \times_b u_{a''(m+m'+1)}$ is defined by the universal property of the pullback, using the maps ϕ and ϕ'.

There are natural transformations $P \Rightarrow const_{(a,\phi)}, P \Rightarrow id$, consisting of maps f, f' for objects $(a', \phi') \in v/K$:



defined as $f(i) := i, f'(i) := i + m + 1$.

Upon taking the nerve, these natural transformations define a zig-zag of homotopies from the identity to a constant map, giving the contractibility of $N(v/K)$.

Thus K is a homotopy final functor. This gives:

$$\begin{aligned} \text{hocolim}\left(P_{fin}(A) \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right) &= \text{hocolim}\left(P_{fin}(A) \xrightarrow{K} \overline{\mathfrak{U}} \xrightarrow{\bar{\jmath}} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right) \\ &\simeq \text{hocolim}\left(\overline{\mathfrak{U}} \xrightarrow{\bar{\jmath}} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right). \end{aligned}$$

$\square$

Finally, we relate the colimit on the right above to the colimit given by the left Kan extension.

Lemma 3.13. *Let* $\overline{\mathfrak{U}} \xrightarrow{\overline{j}} \mathcal{B}$ *be a saturated Weiss cover of* $b \in \mathcal{B}$ *which satisfies the translation property (3.6). Let* $\mathcal{U}$ *be the full subcategory of* $\mathcal{B}$ *consisting of objects in the image of* $\overline{j}$*; i.e.* $\mathcal{U} := \overline{j}(\overline{\mathfrak{U}})$*. Denote the inclusion functor by* $j : \mathcal{U} \to \mathcal{B}$*.*

Then there is a weak equivalence:

$$\operatorname{hocolim}\left(\overline{\mathfrak{U}} \xrightarrow{\overline{j}} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right) \simeq j_! j^* \mathcal{F}(b).$$

Proof. Note that the definition of the left Kan extension gives:

$$j_! j^* \mathcal{F}(b) \simeq \operatorname{hocolim}\left(\mathcal{U}/b \to \mathcal{U} \xrightarrow{j} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right).$$

The categories $\mathcal{U}/b$ and $\overline{\mathfrak{U}}$ are isomorphic. This follows from the translation property (see 3.6). The composite $\mathcal{U}/b \to \mathcal{U} \xrightarrow{j} \mathcal{B}$ is equal to the functor $\overline{\mathfrak{U}} \xrightarrow{\overline{j}} \mathcal{B}$. Thus,

$$\begin{aligned} j_! j^* \mathcal{F}(b) &\simeq \operatorname{hocolim}\left(\mathcal{U}/b \to \mathcal{U} \xrightarrow{j} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right) \\ &= \operatorname{hocolim}\left(\overline{\mathfrak{U}} \xrightarrow{\overline{j}} \mathcal{B} \xrightarrow{\mathcal{F}} Ch\right). \end{aligned}$$

$\square$

Proof of 3.7. The equivalence follows from 3.9, 3.12, and 3.13. $\square$

We can thus give alternative, equivalent definitions to 2.5, 2.15, 2.27, utilizing this more categorical version of the descent axiom:

Definition 3.14. A *factorization algebra on M* is a multicategory functor

$$\mathcal{F} : Open(M) \to Ch,$$

which satisfies the following conditions:

(i) Multiplicative axiom: For disjoint $U_1, ..., U_n \subset M$, the structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \xrightarrow{\sim} \mathcal{F}(U_1 \sqcup \cdots \sqcup U_n)$$

is a weak equivalence.

(ii) Descent axiom: For any saturated Weiss cover $\overline{\mathfrak{U}} \xrightarrow{\overline{j}} Open(M)$ of $U \in Open(M)$ and associated $\mathcal{U} := \overline{j}(\overline{\mathfrak{U}}) \xrightarrow{j} Open(M)$,

$$j_! j^* \mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(U)$$

is a weak equivalence.

Definition 3.15. A *G-equivariant factorization algebra on M* is a multicategory functor

$$\mathcal{F} : Open^G(M) \to Ch,$$

which satisfies the following conditions:

(i) Multiplicative axiom: For disjoint $U_1, ..., U_n \subset M$, the structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \xrightarrow{\sim} \mathcal{F}(U_1 \sqcup \cdots \sqcup U_n)$$

is a weak equivalence.

(ii) Descent axiom: For any saturated Weiss cover $\overline{\mathfrak{U}} \xrightarrow{\overline{j}} Open^G(M)$ of $U \in Open^G(M)$

which satisfies the translation property, and associated $\mathcal{U} := \overline{j}(\overline{\mathfrak{U}}) \xrightarrow{j} Open^G(M)$,

$$j_! j^* \mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(U)$$

is a weak equivalence.

Definition 3.16. A *$\mathcal{G}$-factorization algebra* is a lax symmetric monoidal functor

$$\mathcal{F} : \mathcal{G}Man \to Ch,$$

which satisfies the following conditions:

(i) Multiplicative axiom: For $X_1, ..., X_n \in \mathcal{G}Man$, the structure map

$$\mathcal{F}(X_1) \otimes \cdots \otimes \mathcal{F}(X_n) \xrightarrow{\sim} \mathcal{F}(X_1 \sqcup \cdots \sqcup X_n)$$

is a weak equivalence.

(ii) Descent axiom: For any saturated Weiss cover $\overline{\mathfrak{U}} \xrightarrow{\overline{j}} \mathcal{G}Man$ of $X \in \mathcal{G}Man$ which satisfies the translation property, and associated $\mathcal{U} := \overline{j}(\overline{\mathfrak{U}}) \xrightarrow{j} \mathcal{G}Man$,

$$j_! j^* \mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(X)$$

is a weak equivalence.

CHAPTER 4

DEFINITION OF ∞-VERSION OF FACTORIZATION ALGEBRAS

In this section we generalize the previous definitions to the world of $(\infty, 1)$-operads and symmetric monoidal $(\infty, 1)$-categories. There are several reasons for this move:

(a) Factorization algebras use a weak notion of equivalence throughout (quasi-isomorphisms of chain complexes show up in both the descent axiom and the multiplicative axiom); $(\infty, 1)$-categories are the natural place to talk about things up to this weaker notion of equivalence.

(b) The $(\infty, 1)$-framework is a natural place to develop the smooth or parameterized version of factorization algebras, which is ultimately the result we're interested in, in order to talk about smooth families of field theories.

(c) The higher algebraic techniques for dealing with $(\infty, 1)$-categories in the quasi-category perspective in particular have been well developed in [9] and [10], and this perspective of $(\infty, 1)$-categories is currently used in the literature on factorization homology (see [1] for example). This makes the quasi-categorical perspective for a higher categorical version of factorization algebras a convenient one to take.

The goal of this section is to (i) define a new $(\infty, 1)$-version of factorization algebras, and (ii) show that this is generalization is compatible with the original definition of factorization algebras as given by [3, 4], outlined in chapter 2. In chapter 5 we will use techniques of higher algebra to prove an equivalence of two $(\infty, 1)$-categories of these new $(\infty, 1)$-factorization algebras. The motivation for proving this result at this level of generality is for ease of application to future work on the parameterized version of this result, involving smooth families of factorization algebras.

In what follows, when we mention ∞-operads or ∞-categories, we really mean the $(\infty, 1)$-versions of those; we will work with the particular model of quasi-categories throughout. We summarize the background on ∞-operads from [6, 10] needed for this project in Appendix A for ease of reference, and refer the reader there for more motivation and details behind the following paragraphs.

Given a symmetric monoidal category $(\mathcal{C}, \otimes)$ (in the sense of [11]), one can equivalently view this as an associated category $\mathcal{C}^{\otimes}$ (see A.2) equipped with a functor to pointed finite sets, $\mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ which is an op-fibration and satisfies a Segal-like condition (allowing one to decompose the fiber over arbitrary objects of Fin_* in terms of a product of the fiber over the finite set consisting of a single element; see A.6 and A.7 for the precise details of these conditions).

Applying the nerve to this functor gives a map of quasi-categories: $N\mathcal{C}^{\otimes} \to N\mathsf{Fin}_*$. One can then define a symmetric monoidal ∞-category to be a quasi-category with a map to $N\mathsf{Fin}_*$ which satisfies the quasi-categorical analog of the properties in the previous paragraph (see A.14). By construction, $N\mathcal{C}^{\otimes} \to N\mathsf{Fin}_*$ is an example of a symmetric monoidal ∞-category.

Multicategories (i.e. colored operads) are a generalization or weakening of symmetric monoidal categories. Roughly, one can see this by noting that multi-morphisms are biased as to whether a collection of objects are in the source or the target of the morphism; if the former, one is allowed to 'combine' them (i.e. there are multimorphisms from multiple source objects), but not in the case of the latter. Symmetric monoidal categories 'correct' this bias: one can combine objects in general, regardless of whether they are in the source or target of the desired multi-morphism set. Just as there were two views of the symmetric monoidal category $\mathcal{C}$ above, an analogous situation holds in the case of a multicategory $\mathcal{O}$: one can equivalently think of this as a category equipped with a functor, $\mathcal{O}^{\otimes} \to \mathsf{Fin}_*$, which satisfies certain properties, weakened versions of the properties above (specifically, the condition of being an op-fibration is relaxed).

Again, one can then define ∞-operads to be the quasi-categorical version of this notion (see A.12). In particular, taking the nerve gives an ∞-operad, $N\mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$. More details of these constructions and definitions are spelled out in Appendix A.

We can thus translate the operads and symmetric monoidal categories of interest to us for factorization algebras ($Open(M), Open^G(M), \mathcal{G}Man, Ch$ of chapter 2) into the appropriate ∞-versions. In section 4.1 we give a generalized definition of ∞-factorization algebras using this ∞-operadic perspective. In subsection 4.1.1 and subsection 4.1.2, we show that this ∞-version of factorization algebras generalizes the definitions factorization algebras in chapter 3. One key feature is that the categorical formulation of the descent axiom in chapter 3 generalizes well to this more abstract, ∞-operadic context. Another subtlety to note involves how we translate the category Ch into the ∞-category setting; the category Ch is not just a symmetric monoidal category, but a dg-category. As such, there is a special notion of an ∞-category associated to it (by applying what is called the dg-nerve). We will not use the full higher categorical structure available here in this project, as we are primarily concerned with looking at the examples of G-equivariant factorization algebras and $\mathcal{G}$-factorization algebras. However, we mention the structure that is available and point to possibilities of where it could be used in future work in section 4.2.

4.1 ∞-version of factorization algebras

Since factorization algebras satisfy a 'local-to-global' property (the descent axiom), we need a notion of an open covering for the case of $(\infty, 1)$-categories. We review the background definitions for this ∞-analog of a site here; see [9] (section

6.2.2) or [17] for more details.

Definition 4.1 ([9], Defn 6.2.2.1)**.** Let $\mathcal{C}$ be an ∞-category. A *sieve on* $\mathcal{C}$ is a full-subcategory $\mathcal{S} \subset \mathcal{C}$ which is closed under precomposition with morphisms in $\mathcal{C}$, i.e. such that if $f : c \to d$ is a morphism of $\mathcal{C}$ and $d \in \mathcal{S}$, then $c \in \mathcal{S}$. We refer to this property as the *saturation condition* of a sieve.

If $c \in \mathcal{C}$, then a *sieve on* c is a sieve on the slice ∞-category $\mathcal{C}/c$.

Given a morphism $f : c \to d$ in $\mathcal{C}$ and a sieve $\mathcal{S}$ on d, the *pullback sieve* $f^*\mathcal{S}$ on c is the full-subcategory of $\mathcal{C}/c$ spanned by morphisms which, after post-composition with f, are equivalent to a morphism in $\mathcal{S}$.

Analogously to the case of Grothendieck topologies on ordinary categories, in the ∞-categorical world one has:

Definition 4.2. A *Grothendieck topology* on an ∞-category $\mathcal{C}$ consists of an assignment to every object $c \in \mathcal{C}$ of a collection of sieves on c, called *covering sieves*, which satisfy the following conditions:

(i) (Trivial sieve covers:) If $c \in \mathcal{C}$, then $\mathcal{C}/c \subset \mathcal{C}/c$ is a covering sieve on c.

(ii) (Stability axiom:) If $f : c \to d$ is a morphism of $\mathcal{C}$ and $\mathcal{S}$ is a covering sieve on d, then $f^*\mathcal{S}$ is a covering sieve on c.

(iii) (Transitivity axiom:) Let $c \in \mathcal{C}$, $\mathcal{S}$ be a covering sieve on c and $\mathcal{S}'$ be an arbitrary sieve on c. If for each $f : d \to c$ in $\mathcal{S}$ the pullback $f^*\mathcal{S}'$ is a covering sieve on d, then $\mathcal{S}'$ is a covering sieve on c.

For our new formulation of the descent axiom, we will want the following construction from a Grothendieck topology on an ∞-category $\mathcal{C}$.

Notation 4.3. Let $\mathcal{C}$ be an ∞-category equipped with a Grothendieck topology. Let $c \in \mathcal{C}$ and $\mathcal{S}$ be a particular covering sieve on c. Note that there is a canonical forgetful functor $\mathcal{S} \xrightarrow{\bar{j}} \mathcal{C}$, which forgets the map to c. We will often refer to this functor when we speak of a cover of c.

The specific cases of ∞-categories that we're interested in for factorization algebras at the moment come from a multicategory (or symmetric monoidal category) $\mathcal{B}$, equipped with a Grothendieck topology; we then take the associated category $\mathcal{B}^{\otimes}$ (see A.2 for the construction of this associated category); finally we apply the nerve to this category to obtain an ∞-category. In our cases of interest, the Grothendieck topologies on $\mathcal{B}$ induce a Grothendieck topology on the category $\mathcal{B}^{\otimes}$ (see 4.5). We can then utilize the following observation:

Remark 4.4. If $\mathcal{B}^{\otimes}$ is a category with a Grothendieck topology, a Grothendieck topology on $N(\mathcal{B}^{\otimes})$ reduces to the original notion of Grothendieck topology on the category $\mathcal{B}^{\otimes}$. Unpacking this, for $b \in \mathcal{B}^{\otimes}$ (i.e. a 0-simplex in $N(\mathcal{B}^{\otimes})$), the slice ∞-category $N(\mathcal{B}^{\otimes})/b$ will consist of composable tuples of morphisms in $\mathcal{B}^{\otimes}$ with the final target b. The sieve property (closed under pre-composition by morphisms in $\mathcal{B}^{\otimes}$) makes sieves on $\mathcal{B}^{\otimes}/b$ and sieves on $N(\mathcal{B}^{\otimes})/b$ equivalent. Thus having a collection of covering sieves on $N(\mathcal{B}^{\otimes})$ is equivalent to having a collection of covering sieves on $\mathcal{B}^{\otimes}$. In fact, in the case of a general ∞-category $\mathcal{C}$, the set of Grothendieck topologies on $\mathcal{C}$ is in natural bijection with the set of Grothendieck topologies on the homotopy category $h\mathcal{C}$. See Remark 6.2.2.3 of [9].

Construction 4.5. For $\mathcal{B} = Open(M), Open^G(M), \mathcal{G}Man$, we obtain a Grothendieck topology on $\mathcal{B}^{\otimes}$ by taking, for any object $(\langle m \rangle; b_1, ..., b_m) \in \mathcal{B}^{\otimes}$, the collection of covering sieves to be given by all m-tuples of saturated Weiss covers $\overline{\mathfrak{U}}_1, ..., \overline{\mathfrak{U}}_m$ for each $b_i \in \mathcal{B}, 1 \leq i \leq m$.

More explicitly, for a particular m-tuple of saturated Weiss covers $\overline{\mathfrak{U}}_1, ..., \overline{\mathfrak{U}}_m$ of $b_1, ..., b_m \in \mathcal{B}$, take the sieve that contains all objects

$$(\langle n \rangle; U_1, ..., U_n) \xrightarrow{f} (\langle m \rangle; b_1, ..., b_m)$$

consisting of a map $\alpha : \langle n \rangle \to \langle m \rangle$ in Fin_*; and such that each $f_j \in Mul_{\mathcal{B}}(\{U_i\}_{i\in\alpha^{-1}(j)}; b_j)$ is in $\overline{\mathfrak{U}}_j$.

Varying over all m-tuples of saturated Weiss covers of $b_1, ..., b_m$ gives the collection of covering sieves for $\mathcal{B}^{\otimes}$. It inherits the properties of a covering sieve from the covering sieves of the objects in $\mathcal{B}$.

Thus in our cases of interest (taking $Open(M), Open^G(M), \mathcal{G}Man$ as source categories), we get induced Grothendieck topologies on the ∞-categories $N(Open(M)^{\otimes})$, $N(Open^G(M)^{\otimes})$, $N(\mathcal{G}Man^{\otimes})$. We can extend the previous definition of factorization algebras to more general ∞-operads, where the source is a general ∞-operad equipped with a Grothendieck topology, as follows.

Definition 4.6. Let $\mathcal{O}^{\otimes} \to N(\mathsf{Fin}_*)$ be an ∞-operad with a Grothendieck topology on the underlying ∞-category of $\mathcal{O}^{\otimes}$, and such that pullbacks exist in $\mathcal{O}^{\otimes}$. A *factorization algebra on* $\mathcal{O}^{\otimes}$ is a map of ∞-operads:

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{\quad \mathcal{F} \quad} & N(Ch^{\otimes}) \\ & \searrow \quad \swarrow & \\ & N(\mathsf{Fin}_*) & \end{array}$$

which satisfies:

(i) Multiplicative axiom: For every coCartesian morphism $\tilde{\alpha}$ in $\mathcal{O}^{\otimes}$, $\mathcal{F}(\tilde{\alpha})$ is a coCartesian morphism in $N(Ch^{\otimes})$. (See A.10 for the definition of a coCartesian morphism.)

(ii) Descent axiom: For every object $U \in \mathcal{O}^{\otimes}$ and any covering sieve on U, $\mathcal{S} \xrightarrow{\bar{j}} \mathcal{O}^{\otimes}$, denote the image of $\bar{j}$ by $\mathcal{V} := \bar{j}(\mathcal{S})$ and the inclusion functor by $j : \mathcal{V} \to \mathcal{O}^{\otimes}$. Then there is a weak equivalence:

$$j_! j^* \mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(U).$$

We denote the $(\infty, 1)$-category of factorization algebras on $\mathcal{O}^{\otimes}$ as $\mathrm{Fun}^{\otimes, desc}(\mathcal{O}^{\otimes}, N(Ch^{\otimes}))$.

Notation 4.7. If $\mathcal{O}^{\otimes} = N(Open(M)^{\otimes})$, we call these 'factorization algebras on M' and denote the $(\infty,1)$-category $\mathrm{Fun}^{\otimes,desc}(N(Open(M)^{\otimes}), N(Ch^{\otimes}))$ by Fac_M.

Notation 4.8. If $\mathcal{O}^{\otimes} = N(Open^G(M)^{\otimes})$, we call these '$G$-equivariant factorization algebras on M' and denote the $(\infty,1)$-category $\mathrm{Fun}^{\otimes,desc}(N(Open^G(M)^{\otimes}), N(Ch^{\otimes}))$ by Fac^G_M.

Notation 4.9. If $\mathcal{O}^{\otimes} = N(\mathcal{G}Man^{\otimes})$, we call these '$\mathcal{G}$-factorization algebras' and denote the $(\infty,1)$-category $\mathrm{Fun}^{\otimes,desc}(N(\mathcal{G}Man^{\otimes}), N(Ch^{\otimes}))$ by $\mathcal{G}Fac$.

We claim that 4.6 is a good generalization of 2.5 (or equivalently, 3.14; similarly for 3.15 and 3.16) in that, given a factorization algebra $\mathcal{F}$ in the latter sense, if one applies the $N((-)^{\otimes})$ construction to obtain a map of ∞-operads, then $\mathcal{F}$ satisfies the descent axiom of 3.14 (3.15, 3.16) if and only if $N(\mathcal{F}^{\otimes})$ satisfies the descent axiom of 4.6, and likewise for the multiplicative axioms. We prove these two claims in the following sections.

4.1.1 ∞-version of the descent axiom

Having the reformulation of the descent axiom in 3.14 (3.15, 3.16, respectively), and the proof that this new descent axiom is equivalent to the one of 2.5 (2.15, 2.27, respectively), makes the compatibility between the ∞-version of the descent axiom and the original descent axiom follow formally, from the property of a Grothendieck topology on an ∞-category $N(\mathcal{B}^{\otimes})$.

Remark 4.10. More explicitly, we use the following facts about the Grothendieck topology on $N(\mathcal{B}^{\otimes})$. Let $\mathcal{B}$ be a multicategory with a Grothendieck topology and $b \in \mathcal{B}$. Consider any covering sieve $\overline{\mathfrak{U}} \xrightarrow{\overline{j}} \mathcal{B}$ of b. In the result that follows, we will use the image of $\overline{j}$ along with the inclusion functor; denote these: $\mathcal{V} := \overline{j}(\overline{\mathfrak{U}}) \xhookrightarrow{j} \mathcal{B}$. We can think of b as an object $(\langle 1 \rangle; b) \in N(\mathcal{B}^{\otimes})$. Because $N(\mathcal{B}^{\otimes})$ is the nerve of an ordinary category, the Grothendieck topology on $N(\mathcal{B}^{\otimes})$ is determined by the

Grothendieck topology on $\mathcal{B}^{\otimes}$ (see 4.4). So we can think of $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} N(\mathcal{B}^{\otimes})$ as also giving a covering sieve of $(\langle 1\rangle; b)$. Abusing notation slightly, we will denote the image of this functor also by $\mathcal{V}$, and denote the inclusion functor $\mathcal{V} \overset{j}{\hookrightarrow} N(\mathcal{B}^{\otimes})$; this abuse of notation is justified by the fact that the higher simplices (of $\mathcal{V}$ thought of in $N(\mathcal{B}^{\otimes})$), which correspond to composable strings of morphisms, are already included (in $\mathcal{V}$ thought of in $\mathcal{B}$) by the saturation property of covering sieves.

Proposition 4.11. *Let $\mathcal{B} = Open(M), Open^G(M)$ or $\mathcal{G}Man$. Let $\mathcal{F} : \mathcal{B} \to Ch$ be a map of multicategories (i.e. colored operads). Take any $b \in \mathcal{B}$ and any saturated Weiss cover $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{B}$ of b. Let $\mathcal{V} := \bar{j}(\overline{\mathfrak{U}}) \overset{j}{\hookrightarrow} \mathcal{B}$ be the inclusion functor.*

Then the following are equivalent:

(1) The map

$$j_! j^* \mathcal{F}(b) \xrightarrow{\sim} \mathcal{F}(b)$$

is a weak equivalence.

(2) Let $\hat{\mathcal{F}} := N(\mathcal{F}^{\otimes})$. Think of b as the object $(\langle 1\rangle; b) \in N(\mathcal{B}^{\otimes})$, with saturated Weiss cover corresponding to $\overline{\mathfrak{U}}$. Following 4.10 we denote the inclusion functor $\mathcal{V} \overset{j}{\hookrightarrow} N(\mathcal{B}^{\otimes})$. The map

$$j_! j^* \hat{\mathcal{F}}((\langle 1\rangle; b)) \xrightarrow{\sim} \hat{\mathcal{F}}((\langle 1\rangle; b))$$

is a weak equivalence.

For the pre-factorization algebras $\mathcal{F}, N(\mathcal{F}^{\otimes})$ to satisfy the *descent axiom*, they must satisfy descent for all objects in $\mathcal{B}$, $N(\mathcal{B}^{\otimes})$, respectively, with respect to any Weiss cover. This recovers the descent axioms for 3.14 (equivalently, 2.5), 3.15, 3.16 and 4.6.

Proof. Note that it suffices to consider objects of the form $(\langle 1\rangle; b) \in N(\mathcal{B}^{\otimes})$ because the fiber over $\langle m\rangle$, $N(\mathcal{B}^{\otimes})_{\langle m\rangle}$ is equivalent to an m-fold product of the fiber over $\langle 1\rangle$: $N(\mathcal{B}^{\otimes})_{\langle m\rangle} \simeq (N(\mathcal{B}^{\otimes})_{\langle 1\rangle})^m$ (see A.7).

By definition of $\hat{\mathcal{F}}$, $\hat{\mathcal{F}}(\langle 1\rangle; b) = (\langle 1\rangle; F(b))$. The morphism in $N(Ch^{\otimes})$ in (2) consists of the following morphism in $Ch^{\otimes}$:

$$\begin{array}{ccc} (\langle 1\rangle; j_! j^* \mathcal{F}(b)) & \xrightarrow{f} & (\langle 1\rangle; \mathcal{F}(b)) \\ \downarrow & & \downarrow \\ \langle 1\rangle & \xrightarrow{id} & \langle 1\rangle \end{array}$$

which in turn consists of a morphism in Ch:

$$f_1 : j_! j^* \mathcal{F}(b) \to \mathcal{F}(b).$$

The morphism in $N(Ch^{\otimes})$ is a weak equivalence if the morphism in Ch is a weak equivalence, giving the desired result. (See section 4.2 for more details about the weak equivalences in $N(Ch^{\otimes})$.) □

4.1.2 ∞-version of the multiplicative axiom

We now want to show that the multiplicative axioms of 2.5 (3.15, 3.16, respectively) and 4.6 are equivalent. We will work with the multicategories $Open(M), Open^G(M)$ separately from $\mathcal{G}Man$, as the latter is a symmetric monoidal category and so has structure not available in the other two multicategories.

Proposition 4.12. *Let $\mathcal{B} = Open(M), Open^G(M)$. Let $\mathcal{F} : \mathcal{B} \to Ch$ be a map of multicategories (i.e. colored operads). The following are equivalent:*

(1) For disjoint $U_1, ..., U_n$ in $\mathcal{B}$, the structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \xrightarrow{\sim} \mathcal{F}(U_1 \sqcup \cdots \sqcup U_n)$$

is a weak equivalence.

(2) The map $N(\mathcal{F}^{\otimes}) : N(\mathcal{B}^{\otimes}) \to N(Ch^{\otimes})$ of ∞-operads preserves coCartesian morphisms.

Proposition 4.13. *Let $\mathcal{B} = \mathcal{G}Man$ and $\mathcal{F} : \mathcal{G}Man \to Ch$ be a lax symmetric monoidal functor. The following are equivalent:*

(1) For $X_1, ..., X_n \in \mathcal{B}$, the structure map

$$\mathcal{F}(X_1) \otimes \cdots \otimes \mathcal{F}(X_n) \xrightarrow{\sim} \mathcal{F}(X_1 \sqcup \cdots \sqcup X_n)$$

is a weak equivalence. (I.e. $\mathcal{F}$ is a symmetric monoidal functor.)

(2) The map $N(\mathcal{F}^{\otimes}) : N(\mathcal{G}Man^{\otimes}) \to N(Ch^{\otimes})$ preserves coCartesian morphisms.

Proof of 4.13. The equivalence follows by definition: in the case where $\mathcal{O}^{\otimes}$ is a symmetric monoidal ∞-category, the (∞-)multiplicative axiom is equivalent to requiring that $N(\mathcal{F}^{\otimes})$ is actually a symmetric monoidal ∞-functor (see A.24). Thus for the case of $\mathcal{B} = \mathcal{G}Man$, the definition of the multiplicative axiom of 2.27, i.e. the requirement that $\mathcal{F} : \mathcal{G}Man \to Ch$ is a symmetric monoidal functor, directly generalizes upon applying the nerve construction to the requirement that $N(\mathcal{F}^{\otimes}) : N(\mathcal{G}Man^{\otimes}) \to N(Ch^{\otimes})$ be a symmetric monoidal ∞-functor. □

The proof of 4.12 will make use of the following lemmas, which provide a characterization of some of the coCartesian morphisms in $\mathcal{B}$ and Ch:

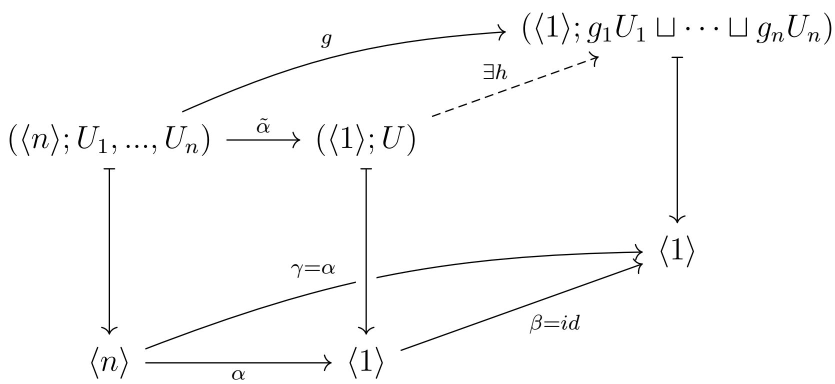
Lemma 4.14. *Let $\mathcal{B} = Open(M), Open^G(M)$. Consider $N(\mathcal{B}^{\otimes})$. Let $\alpha : \langle n \rangle \to \langle 1 \rangle$ be the unique active morphism in $N(\mathsf{Fin}_*)$ and $\tilde{\alpha}$ a morphism in $N(\mathcal{B}^{\otimes})$ over α, as shown below:*

$$\begin{array}{ccc} (\langle n \rangle; U_1, ..., U_n) & \xrightarrow{\tilde{\alpha}} & (\langle 1 \rangle; U) \\ \big\downarrow & & \big\downarrow \\ \langle n \rangle & \xrightarrow{\alpha} & \langle 1 \rangle \end{array}$$

Then $\tilde{\alpha}$ is coCartesian if and only if $\tilde{\alpha}_1 \in Mul_{\mathcal{B}}(U_1, ..., U_n; U)$ is an isomorphism (i.e. $U_1 \sqcup \cdots \sqcup U_n \cong U$ for $\mathcal{B} = Open(M)$; or there exist $g_1, \cdots g_n \in G$ so that $g_1U_1 \sqcup \cdots \sqcup g_nU_n \cong U$ for $\mathcal{B} = Open^G(M)$).

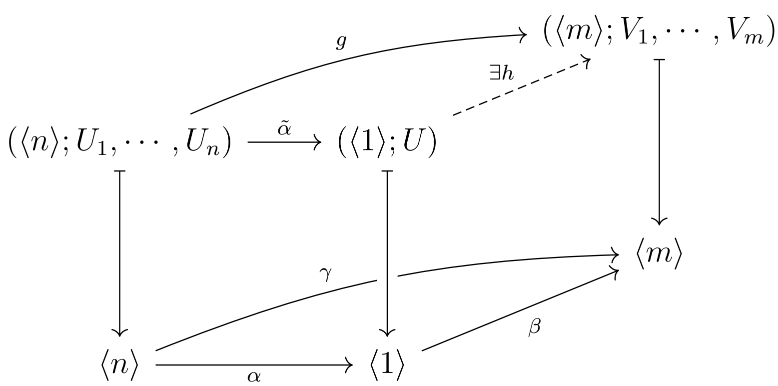
Proof. Assume $\tilde{\alpha}$ is coCartesian. $\tilde{\alpha}$ consists of a morphism $\tilde{\alpha}_1 \in Mul_{\mathcal{B}}(U_1, ..., U_n; U)$; i.e. of elements $g_i \in G$ such that $g_1U_1 \sqcup \cdots \sqcup g_nU_n \hookrightarrow U$ (where if $\mathcal{B} = Open(M)$, then $g_i = id_G$ for all i).

Consider the maps g (where g consists of $g_1U_1 \sqcup \cdots \sqcup g_nU_n \overset{id}{\hookrightarrow} g_1U_1 \sqcup \cdots \sqcup g_nU_n$), $\gamma := \alpha, \beta := id$ in the diagram below:



Because $\tilde{\alpha}$ is coCartesian, there is a contractible space of maps h filling the diagram. The maps h_1 give an inverse to $\tilde{\alpha}_1$. Thus $g_1U_1 \sqcup \cdots \sqcup g_nU_n \cong U$.

Conversely, assume that $\tilde{\alpha}_1$ is an isomorphism (i.e. $g_1U_1 \sqcup \cdots \sqcup g_nU_n \cong U$). Then for arbitrary g, γ, β making the diagram below commute:



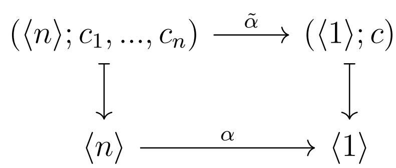
there is a contractible space of h's filling the diagram above, given (up to weak equivalence) by:

- $\beta : \langle 1 \rangle \to \langle m \rangle$, mapping $\{1\} \mapsto \{\gamma(1) = \cdots = \gamma(n)\}$
- $h_i := \begin{cases} U \xrightarrow{g_{\beta(1)} \circ \tilde{\alpha}_1^{-1}} V_{\beta(1)}, \text{ if } i = \beta(1) \neq * \\ \mathbb{1} \to V_i, \text{ else} \end{cases}$

Thus, $\tilde{\alpha}$ is coCartesian.

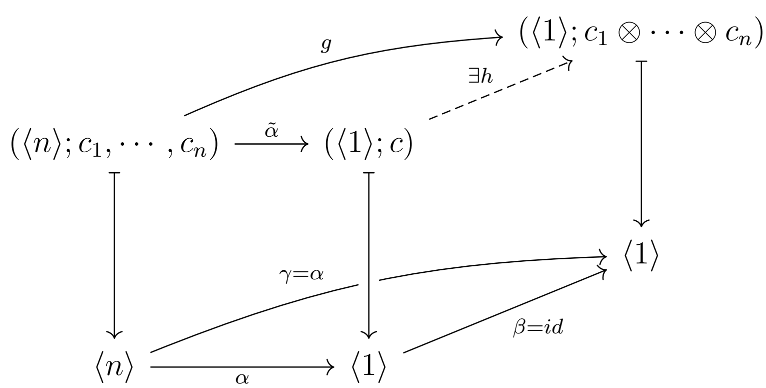
□

Lemma 4.15. *Let $\mathcal{C}$ be a symmetric monoidal category with a notion of weak equivalence (for example, $\mathcal{C} = Ch$). Consider $N(\mathcal{C}^{\otimes})$. Let $\alpha : \langle n \rangle \to \langle 1 \rangle$ be the unique active morphism and $\tilde{\alpha}$ a morphism in $N(\mathcal{C}^{\otimes})$ over α, as shown below:*



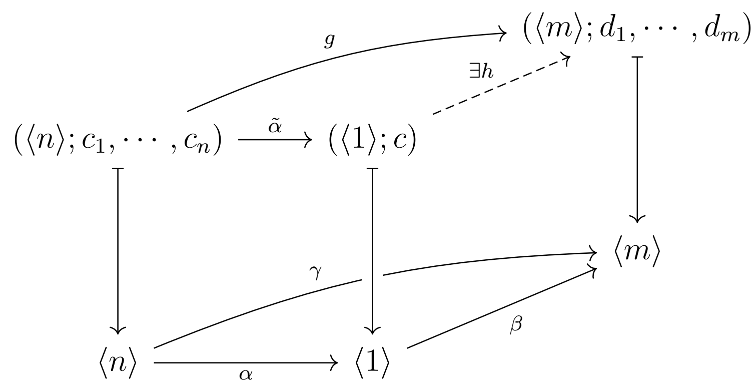
Then $\tilde{\alpha}$ is coCartesian if and only if $\tilde{\alpha}_1 : c_1 \otimes \cdots \otimes c_n \to c$ is a weak equivalence.

Proof. Assume $\tilde{\alpha}$ is coCartesian. Consider the maps g (where $g_1 := id$), $\gamma := \alpha, \beta := id$ in the diagram below:



Because $\tilde{\alpha}$ is coCartesian, the space of h's filling the diagram is contractible. The maps h_1 give weak inverses to $\tilde{\alpha}_1$. Thus $c_1 \otimes \cdots \otimes c_n \simeq c$.

Conversely, assume that $\tilde{\alpha}_1 : c_1 \otimes \cdots \otimes c_n \to c$ is a weak equivalence. Then for arbitrary g, γ, β making the diagram below commute:



there is a contractible space of h's filling the diagram above, given (up to weak equivalence) by:

- $\beta : \langle 1\rangle \to \langle m\rangle$, mapping $\{1\} \mapsto \{\gamma(1) = \cdots = \gamma(n)\}$
- $h_i := \begin{cases} c \xleftarrow{\sim} (c_1 \otimes \cdots \otimes c_n) \xrightarrow{g_{\beta(1)}} d_{\beta(1)}, & \text{if } i = \beta(1) \neq * \\ \mathbb{1} \to d_i, & \text{else} \end{cases}$

Thus, $\tilde{\alpha}$ is coCartesian. □

Proof of 4.12. Let $\hat{\mathcal{F}} := N(\mathcal{F}^{\otimes})$. First, assume $\mathcal{F}$ satisfies (1).

Consider a coCartesian morphism $\tilde{\alpha}$ in $N(\mathcal{B}^{\otimes})$:

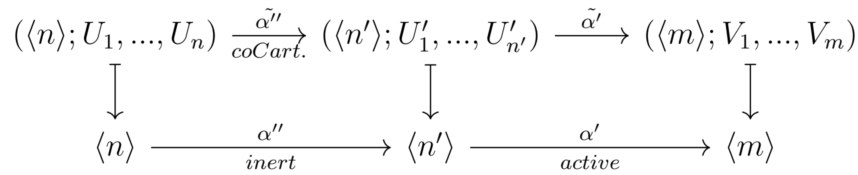
$$\begin{array}{ccc} (\langle n\rangle; U_1, ..., U_n) & \xrightarrow{\tilde{\alpha}} & (\langle m\rangle; V_1, ..., V_m) \\ \downarrow & & \downarrow \\ \langle n\rangle & \xrightarrow{\alpha} & \langle m\rangle \end{array}$$

α factors uniquely (up to unique isomorphism) as an inert morphism (α'') followed by an active morphism (α') (see A.8, B.1 for definitions):

$$\langle n\rangle \xrightarrow{\alpha''} \langle n'\rangle \xrightarrow{\alpha'} \langle m\rangle,$$

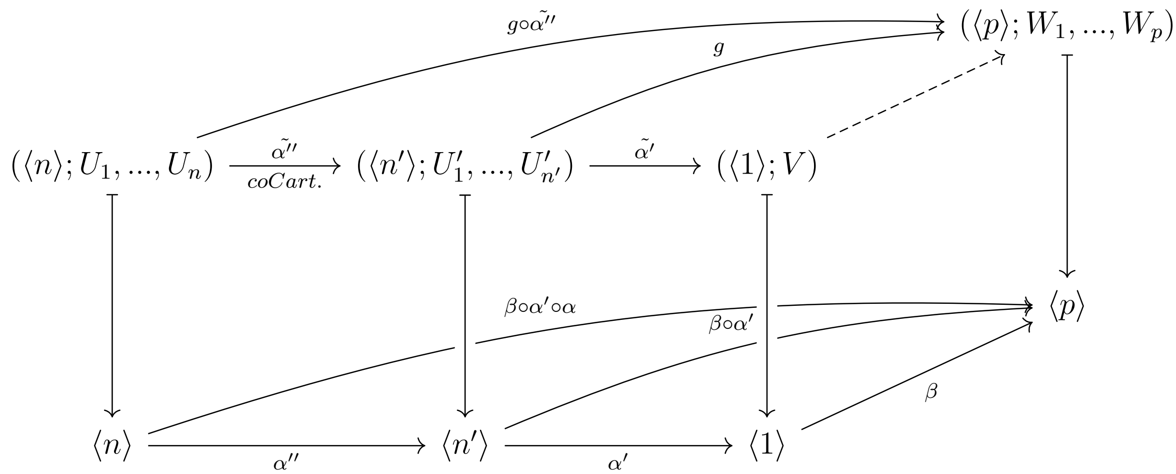
where α'' sends all the elements in $\langle n\rangle$ to the basepoint which α sends to the basepoint, and is injective on all the other elements; and α' only sends the basepoint to the basepoint, and does what α does for the other, non-trivial maps (see [9], Rmk 2.1.2.2).

This factorization system for the morphisms of Fin_* induces a factorization system for the morphisms of any ∞-operad (see [9], Propn 2.1.2.4). In our situation, $\tilde{\alpha}$ factors as follows:



For ease of argument, we assume $m = 1$; for more general active morphisms, consider a union of maps in the following argument. We know that $\tilde{\alpha''}$ is coCartesian by the lifting property in the definition of ∞-operads. We claim that $\tilde{\alpha'}$ is also coCartesian.

To see this, consider any g, β as in the diagram below:



Because $\tilde{\alpha}$ itself is coCartesian, considering maps $g \circ \tilde{\alpha''}, \beta$, which by construction make the diagram commute, there is a contractible space of morphisms $(\langle 1\rangle; V) \to (\langle p\rangle; W_1, ..., W_p)$ which fill the diagram; these also serve as the necessary fillers to show that $\tilde{\alpha'}$ is coCartesian.

By 4.14, this implies that $(\tilde{\alpha'})_1 \in Mul_{\mathcal{B}}(U'_1, ..., U'_n; V)$ is an isomorphism; i.e. there exist $g_i \in G$ such that $\tilde{\alpha'}_1 : g_1 U'_1 \sqcup \cdots \sqcup g_{n'} U'_{n'} \cong V$.

Because $\hat{\mathcal{F}}$ is a map of ∞-operads, it preserves inert coCartesian morphisms; thus $\hat{\mathcal{F}}(\tilde{\alpha''})$ is coCartesian. Note that $\hat{\mathcal{F}}(\tilde{\alpha'})$ consists of the following map:

$$\begin{array}{ccc} \mathcal{F}(U_1') \otimes \cdots \otimes \mathcal{F}(U_{n'}') & \xrightarrow{\hat{\mathcal{F}}(\tilde{\alpha'})_1} & \mathcal{F}(V) \\ \downarrow \simeq & \nearrow \sim & \\ \mathcal{F}(g_1 U_1') \otimes \cdots \otimes \mathcal{F}(g_n U_n') & & \end{array}$$

where the vertical map comes from the natural isomorphisms that are part of the data of a G-equivariant factorization algebra; and the diagonal map comes from the fact that $\mathcal{F}$ satisfies (1). (In the case of $\mathcal{B} = Open(M)$, the vertical map is an equality, as $g_i = id_G$ for all i.) Thus, $\hat{\mathcal{F}}(\tilde{\alpha'})_1$ is a weak equivalence. By 4.15 $\hat{\mathcal{F}}(\tilde{\alpha'})$ is coCartesian. The composition of coCartesian morphisms is coCartesian, so $\hat{\mathcal{F}}(\tilde{\alpha})$ is coCartesian. Thus $\hat{\mathcal{F}}$ satisfies (2).

Now suppose that $\hat{\mathcal{F}}$ satisfies (2). Let $U_1, ..., U_n$ in $\mathcal{B}$ be disjoint. Consider the following coCarteisan morphism in $N(\mathcal{B}^{\otimes})$, where $\alpha : \langle n \rangle \to \langle 1 \rangle$ is the unique active morphism:

$$\begin{array}{ccc} (\langle n \rangle; U_1, ..., U_n) & \xrightarrow{f} & (\langle 1 \rangle; U_1 \sqcup \cdots \sqcup U_n) \\ \downarrow & & \downarrow \\ \langle n \rangle & \xrightarrow{\alpha} & \langle 1 \rangle \end{array}$$

where f consists of:

- $\alpha : \langle n \rangle \to \langle 1 \rangle$
- $f_1 := U_1 \sqcup \cdots \sqcup U_n \overset{id}{\hookrightarrow} U_1 \sqcup \cdots \sqcup U_n$

By 4.14, f is coCartesian. Because $\hat{\mathcal{F}}$ satisfies (2), this implies that

$$\hat{\mathcal{F}}(f) : (\langle n \rangle; \mathcal{F}(U_1), \cdots, \mathcal{F}(U_n)) \to (\langle 1 \rangle; \mathcal{F}(U_1 \sqcup \cdots \sqcup U_n))$$

is also a coCartesian morphism. By 4.15, $\hat{\mathcal{F}}(f)_1$ is a weak equivalence, i.e. the structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(U_1 \sqcup \cdots \sqcup U_n)$$

is a weak equivalence. Thus $\mathcal{F}$ satisfies (1).

□

4.2 Subtleties with chain complexes and weak equivalences

There are subtleties involved in what we mean by the nerve of the category Ch, coming from the fact that this category has several different types of structure: it is a symmetric monoidal category, it has a notion of weak equivalence (quasi-isomorphisms), and it is a dg-category (enriched over chain complexes). When we ask for the ∞-category associated to Ch, we want to take these structures into account. We have already discussed what it means to look at the ∞-analog of a symmetric monoidal category; we construct an associated category $Ch^{\otimes}$ with a special functor to Fin_* and then take the nerve of that combination of data (see section A.2 for more details). In this section we survey aspects of the dg-categorical structure, and note to what extent this plays a role in the current project.

We start with some generalities. Let $\mathcal{D}$ be a dg-category. There are two ordinary categories we can associate to $\mathcal{D}$:

(1) the *underlying category*, $\mathcal{D}_0$; whose objects are the objects of $\mathcal{D}$; and whose morphisms are given by chain maps, $Hom_{\mathcal{D}_0}(X,Y) := \{f \in Map_{\mathcal{D}}(X,Y)_0 | df = 0\}$. (See [9] Rmk 1.3.1.4.)

(2) the *homotopy category*, $h\mathcal{D}$; whose objects are the objects of $\mathcal{D}$; and whose morphisms are given by $Hom_{h\mathcal{D}}(X,Y) := H_0(Map_{\mathcal{D}}(X,Y)_*)$. (See [9], Rmk 1.3.1.5.)

There are various (equivalent) ways to associate a quasi-category to the dg-category $\mathcal{D}$. Lurie gives a particularly nice description a quasi-category associated to $\mathcal{D}$ via what he calls the *dg nerve* ([9], Construction 1.3.1.6). Define $N_{dg}(\mathcal{D})$ to be the simplicial set with n-simplices the set of ordered pairs:

$$N_{dg}(\mathcal{D})_n = \{(\{X_i\}_{0\leq i\leq n}, \{f_I\})\}$$

where

- X_i are objects of $\mathcal{D}$, for all $0 \leq i \leq n$
- for every subset $I = \{i_- < i_m < \cdots < i_1 < i_+\} \subseteq [n], m \geq 0,\ f_I \in Map_{\mathcal{D}}(X_{i_-}, X_{i_+})_m$ such that

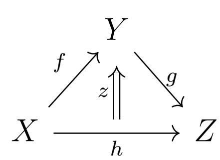
$$df_I = \sum_{1\leq j\leq m} (-1)^j \big(f_{(I-\{i_j\})} - f_{\{i_j<\cdots<i_+\}} \circ f_{\{i_-<\cdots<i_j\}}\big)$$

Unpacking this for the first few n-simplices (see [9], Ex 1.3.1.8):

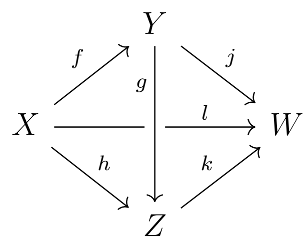
- $N_{dg}(\mathcal{D})_0$: objects of $\mathcal{D}$
- $N_{dg}(\mathcal{D})_1$: $X, Y \in \mathcal{D}$ plus $f \in Map_{\mathcal{D}}(X,Y)_0$ such that $df = 0$

$$X \xrightarrow{\ f\ } Y$$

- $N_{dg}(\mathcal{D})_2$: $X, Y, Z \in \mathcal{D}$; $f \in Map_{\mathcal{D}}(X,Y)_0, g \in Map_{\mathcal{D}}(Y,Z)_0, h \in Map_{\mathcal{D}}(X,Z)_0$ such that $df = dg = dh = 0$; $z \in Map_{\mathcal{D}}(X,Z)_1$ such that $dz = (g \circ f) - h$

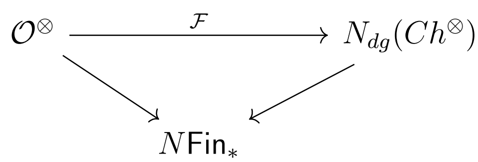


- $N_{dg}(\mathcal{D})_3$: $X, Y, Z, W \in \mathcal{D}$ corresponding to vertices of a 3-simplex; 6 elements corresponding to the edges, in degree 0 of the various morphism-complexes; 4 elements corresponding to the faces, in degree 1 of the various morphism-complexes; 1 element in $Map_{\mathcal{D}}(X,W)_2$ which corresponds to the inside of the 3-simplex, witnessing the compatibility of the different faces



Remark 4.16. (See [9], Rmk 1.3.1.9) Consider the nerve of the underlying (ordinary) category associated to $\mathcal{D}$, $N(\mathcal{D}_0)$. The simplicial set $N(\mathcal{D}_0)$ is isomorphic to the simplicial subset of $N_{dg}(\mathcal{D})$ with trivial higher simplices information ($f_I = 0$ whenever I has more than 2-elements; this means we only really see things up to (composable strings of) 1-simplices). More precisely, the map $N(\mathcal{D}_0) \to N_{dg}(\mathcal{D})$ is bijective on n-simplices for $n \leq 1$.

In general it would be interesting to think about what an ∞-factorization algebra would be as a map of ∞-operads



satisfying the multiplicative and descent axioms. We could then use the full data of the higher simplices in the quasi-category $N_{dg}(\mathcal{D})$ for the descent axiom: we could require that the edge in the descent axiom be a chain homotopy equivalence (which would be a quasi-isomorphism, but not conversely).

However, for the specific examples of ∞-factorization algebras which we consider in this project, where the source ∞-categories are $N(\mathcal{B}^{\otimes}) = N(Open(M)^{\otimes})$, $N(Open^G(M)^{\otimes})$, $N(\mathcal{G}Man^{\otimes})$ (i.e. are the nerves of ordinary categories), our factorization algebras do not involve this higher categorical structure. Because of the simplicity of the source categories (the higher simplices are strings of composable morphisms), the image of the functor $\mathcal{F}$ can only have trivial higher simplicial data; i.e. $\mathcal{F}$ factors through a more basic simplicial subset of $N_{dg}(Ch^{\otimes})$, namely $N(Ch_0^{\otimes})$:

$$\begin{array}{ccccc} N(\mathcal{B}^{\otimes}) & \xrightarrow{N(\mathcal{F}^{\otimes})} & N(Ch_0^{\otimes}) & \longrightarrow & N_{dg}(Ch^{\otimes}) \\ & \searrow & \downarrow & \swarrow & \\ & & N\mathsf{Fin}_* & & \end{array}$$

In this dissertation, we will take $N(Ch^{\otimes})$ to mean this subset $N(Ch_0^{\otimes})$; it is a symmetric monoidal ∞-category, thus making sense of $N(\mathcal{F}^{\otimes})$ as a map of ∞-operads (or a symmetric monoidal ∞-functor in the case of $\mathcal{B} = \mathcal{G}Man$). We call an edge in $N(Ch_0^{\otimes})$ (i.e. a chain map) a weak equivalence if it is a quasi-isomorphism.

The questions about how this fits into a more honest ∞-categorical framework (where one does use the higher simplicial data in $N_{dg}(Ch^{\otimes})$ in a non-trivial way) are left for future work. In particular, to make sense of the diagram above, we would like to address the following questions:

1. Is $N_{dg}(Ch^{\otimes}) \to N\mathsf{Fin}_*$ a symmetric monoidal ∞-category?

2. Is the map $N(Ch_0^{\otimes}) \to N_{dg}(Ch^{\otimes})$ a symmetric monoidal ∞-functor? This should follow immediately: if an edge is coCartesian in $N(Ch_0^{\otimes})$, it would also be coCartesian in $N_{dg}(Ch^{\otimes})$ by taking the same contractible space of lifts. In fact, the ∞-category $N_{dg}(Ch^{\otimes})$ has more ways of 'contracting' edges (it has non-trivial 2-simplices). This would imply that the composite in the diagram above satisfies the multiplicative axiom.

3. For the descent axiom, in the case of $N_{dg}(Ch^{\otimes})$ the requirement that an edge be a weak equivalence could be weakened from the version of requiring it to be a quasi-isomorphism: we could instead ask that the edge be a chain homotopy equivalence. This would use the higher simplicial data available in $N_{dg}(Ch^{\otimes})$ in a non-trivial way.

We leave these details about the higher categorical structure available for general ∞-factorization algebras and the implications this structure would have for future work. In the next section, chapter 5, we focus on showing that the ∞-categories Fac_M^G and $\mathcal{G}Fac$ are equivalent; we use higher algebra techniques to show this, with a view towards being able to translate these results into more general cases (choosing different source ∞-categories for the factorization algebras, in particular parameterized or family versions of the ones currently discussed, and possibly using the higher structure available in $N_{dg}(Ch^{\otimes})$).

Remark 4.17. From now on we will only be interested in the ∞-version of factorization algebras and will drop the ∞-notation, leaving it implicit throughout. I.e. when we write $Open^G(M), \mathcal{G}Man, Ch$ or Fin_*, we are actually referring to $N(Open^G(M)^{\otimes}), N(\mathcal{G}Man^{\otimes}), N(Ch^{\otimes}), N(\mathsf{Fin}_*)$ but drop the nerve notation for simplicity.

CHAPTER 5

COMPARISON RESULT: G-EQUIVARIANT FACTORIZATION ALGEBRAS ON M ARE EQUIVALENT TO $\mathcal{G}$-FACTORIZATION ALGEBRAS

Inspired by the fact that factorization algebras satisfy a local-to-global property (the descent axiom), we now want to relate factorization algebras on M, equipped with a structure given by a group G acting on M (i.e. G-equivariant factorization algebras), to factorization algebras on more general manifolds which locally look like M with a geometric structure given by the G-action (i.e. $\mathcal{G}$-factorization algebras). More precisely, we show that the $(\infty, 1)$-categories Fac_M^G and $\mathcal{G}Fac$ are equivalent. While we look specifically at these two particular flavors of factorization algebras in this project, many of the results utilize general features of ∞-operads, and so could readily be generalized to other flavors of factorization algebras. In future work, we will investigate a generalization of the current result to the case of parameterized or smooth family versions of the factorization algebras considered here. This is of interest as related to the families of twisted field theories in the work of Dwyer-Stolz-Teichner.

In this section we first recall results that hold from higher algebra techniques developed in [1, 10]; these techniques are summarized in more detail in Appendix B and Appendix D. These constructions give us maps between functor categories involving the desired ∞-operads for our factorization algebras. We then show in section 5.1 and section 5.2 how imposing the axioms of factorization algebras, as defined in 4.6, yields an equivalence of $(\infty, 1)$-categories. For a review of background material about equivalences and adjunctions of ∞-categories, using the model of quasi-categories, see

Appendix C.

Consider the ∞-operads $Open^G(M), \mathcal{G}Man$ and Ch as described above. The monoidal envelope construction, which can be thought of as a monoidal 'exterior completion,' constructs a symmetric monoidal ∞-category from an ∞-operad; see Appendix B for more details. We apply this construction to the ∞-operad $Open^G(M)$: the associated symmetric monoidal ∞-category is denoted $Env(Open^G(M))$. For any ∞-operad $\mathcal{O}^{\otimes}$, there is a natural functor into its monoidal envelope: $\mathcal{O}^{\otimes} \hookrightarrow Env(\mathcal{O}^{\otimes})$. This induces a map between the respective functor categories out of these ∞-operads. When these functor categories are into another symmetric monoidal ∞-category (for example, Ch in the diagram below), this map is a weak equivalence.

In our case, we have an inclusion of symmetric monoidal ∞-categories

$$i : Env(Open^G(M)) \to \mathcal{G}Man.$$

It is the inclusion of the full subcategory consisting of all finite disjoint unions of opens in M into the category of general $\mathcal{G}$-manifolds. This functor i satisfies conditions which, together with properties of the ∞-category Ch, make the restriction and operadic left Kan extension adjoint functors. This is a particular application of a result in [1]; see Appendix D for more details.

We summarize these results from the literature in the diagram below:

$$\mathrm{Fun}^{inert}(Open^G(M), Ch) \xleftarrow[\sim]{(1)} \mathrm{Fun}^{\otimes}(Env(Open^G(M)), Ch) \underset{\longleftarrow}{\overset{(2)}{\underset{\perp}{\longrightarrow}}} \mathrm{Fun}^{\otimes}(\mathcal{G}Man, Ch)$$

It is known that:

(1) is an equivalence of $(\infty, 1)$-categories by the universal property of the monoidal envelope (see [10], Propn 2.2.4.9; further details are summarized in Appendix B), and

(2) is an adjunction (see [1], Lemma 2.16; further details about how this result applies to our particular ∞-categories are given in Appendix D).

As is, these results simply apply to functor categories (of ∞-operads and symmetric monoidal ∞-categories respectively). To raise these to a comparison of factorization algebra categories we need to restrict to functors which satisfy the factorization algebra axioms. Imposing these axioms on all functor categories involved gives G-equivariant factorization algebras on M (Fac_M^G) on the left and $\mathcal{G}$-factorization algebras ($\mathcal{G}Fac$) on the right of the diagram above. In section 5.1 we show that imposing the axioms preserves the equivalence in (1). In section 5.2 we show that imposing the axioms lifts the adjunction in (2) to an equivalence of $(\infty, 1)$-categories.

This gives the following main result:

Theorem 5.1. *Let M be a manifold and G be a group acting on M. There is an $(\infty, 1)$-categorical equivalence between the G-equivariant factorization algebras on M and $\mathcal{G}$-factorization algebras:*

$$Fac_M^G \simeq \mathcal{G}Fac.$$

5.1 Equivalence involving the monoidal envelope

Because $Open^G(M)$ is an ∞-operad and Ch is a symmetric monoidal ∞-category, by B.4 we have an equivalence of ∞-categories:

$$\mathrm{Fun}^{\otimes}(Env(Open^G(M)), Ch) \xrightarrow{\sim} \mathrm{Fun}^{inert}(Open^G(M), Ch)$$

where the left side denotes symmetric monoidal ∞-functors and the right side denotes ∞-operad maps.

We impose the additional axioms for factorization algebras on both sides and claim that the equivalence of ∞-categories is preserved:

Proposition 5.2. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad equipped with a Grothendieck topology on its underlying ∞-category. Let $Env(\mathcal{O}^{\otimes})$ be its monoidal envelope (i.e. its associated symmetric monoidal ∞-category) equipped with a Grothendieck topology compatible with that on $\mathcal{O}^{\otimes}$ in the following sense: if $i : \mathcal{O}^{\otimes} \to Env(\mathcal{O}^{\otimes})$ denotes the inclusion functor, $X \in \mathcal{O}^{\otimes}$ and $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{O}^{\otimes}$ is a covering sieve of X, then $\overline{\mathfrak{U}} \xrightarrow{i \circ \bar{j}} Env(\mathcal{O}^{\otimes})$ is a covering sieve of $iX \in Env(\mathcal{O}^{\otimes})$.*

Then there is an equivalence of ∞-categories:

$$\mathrm{Fun}^{\otimes, desc}(Env(\mathcal{O}^{\otimes}), Ch) \xrightarrow{\sim} \mathrm{Fun}^{\otimes, desc}(\mathcal{O}^{\otimes}, Ch).$$

In particular, for $\mathcal{O}^{\otimes} = Open^{G}(M)$ there is an equivalences of ∞-categories involving G-equivariant factorization algebras on M:

$$\mathrm{Fun}^{\otimes, desc}(Env(Open^{G}(M)), Ch) \simeq \mathrm{Fun}^{\otimes, desc}(Open^{G}(M), Ch) =: Fac_{M}^{G}.$$

Proof. As B.4 gives an equivalence of the functor categories, it remains to show that this equivalence is preserved after imposing the multiplicative and descent axioms on both sides of the equivalence.

(i) Multiplicative axiom: The map above sends $F \in \mathrm{Fun}^{\otimes}(Env(\mathcal{O}^{\otimes}), Ch)$ to $i^{*}F \in \mathrm{Fun}^{inert}(\mathcal{O}^{\otimes}, Ch)$. Note that if $f \in \mathcal{O}^{\otimes}$ is coCartesian, then $i^{*}F(f)$ is also coCartesian, as illustrated below, as F preserves coCartesian morphisms:

$$\begin{array}{ccc} (\langle m\rangle; U_1, ..., U_m) & \xrightarrow[(coCart.)]{f} & (\langle n\rangle; V_1, ..., V_n) \\ \downarrow & & \downarrow \\ \langle m\rangle & \xrightarrow[\alpha]{} & \langle n\rangle \end{array} \qquad \overset{i}{\mapsto}$$

$$
\begin{array}{ccc}
(\langle m\rangle; U_1, ..., U_m) & \xrightarrow[(coCart.)]{f} & (\langle n\rangle; V_1, ..., V_n) \\
\downarrow & & \downarrow \\
\langle m\rangle & \xrightarrow[\alpha]{} & \langle n\rangle \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\
\langle m\rangle & \xrightarrow[\alpha]{} & \langle n\rangle
\end{array}
\qquad \overset{F}{\longmapsto}
$$

$$
\begin{array}{ccc}
(\langle m\rangle; F(U_1), ...) & \xrightarrow[(coCart.)]{F(f)} & (\langle n\rangle; F(V_1), ...) \\
\downarrow & & \downarrow \\
\langle m\rangle & \xrightarrow[\alpha]{} & \langle n\rangle
\end{array}
$$

This means that $(F \circ i) \in \mathrm{Fun}^{\otimes}(\mathcal{O}^{\otimes}, Ch) \subset \mathrm{Fun}^{inert}(\mathcal{O}^{\otimes}, Ch)$ satisfies the multiplicative axiom. Thus the functor in B.4 restricts to a fully faithful and essentially surjective functor onto $\mathrm{Fun}^{\otimes}(\mathcal{O}^{\otimes}, Ch)$, the category of functors satisfying the multiplicative axiom.

(ii) Descent axiom: Assume $F \in \mathrm{Fun}^{\otimes}(Env(\mathcal{O}^{\otimes}), Ch)$ satisfies the descent axiom. We want to show that $i^*F \in \mathrm{Fun}^{inert}(\mathcal{O}^{\otimes}, Ch)$ also satisfies the descent axiom. Take an arbitrary $X \in \mathcal{O}^{\otimes}$ and any covering sieve $\mathfrak{U} \xrightarrow{\bar{j}} \mathcal{O}^{\otimes}$ of X; consider the inclusion functor $\mathcal{U} := \bar{j}(\mathfrak{U}) \overset{j}{\hookrightarrow} \mathcal{O}^{\otimes}$. Showing that i^*F satisfies descent with respect to X and $\mathfrak{U}$ amounts to showing that:

$$j_! j^*(i^*F)(X) \xrightarrow{\simeq ?} i^*F(X).$$

Consider $iX \in Env(\mathcal{O}^{\otimes})$. By assumption, $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{O}^{\otimes} \xrightarrow{i} Env(\mathcal{O}^{\otimes})$ is a covering sieve of iX. Consider $\mathcal{U} \xrightarrow{j} \mathcal{O}^{\otimes} \xrightarrow{i} Env(\mathcal{O}^{\otimes})$. Since F satisfies the descent axiom,

$$(i \circ j)_!(i \circ j)^* F(iX) \xrightarrow{\simeq} F(iX).$$

Also note that $i : \mathcal{O}^{\otimes} \to Env(\mathcal{O}^{\otimes})$ is a full and faithful functor; thus $i^* i_! \simeq \mathbb{1}$, see D.6.

Thus we have:

$$\begin{aligned} j_! j^*(i^* F)(X) &\simeq (i^* i_!) j_! j^* i^* F(X) \\ &= (i_! j_! j^* i^* F)(iX) \\ &\simeq F(iX) \\ &= i^* F(X). \end{aligned}$$

Varying over all X and all covering sieves $\mathfrak{U}$ gives the desired descent axiom for $i^* F$.

□

5.2 Adjoint equivalence involving the operadic Lan

Note that there is an inclusion functor of a full subcategory $i : Env(Open^G(M)) \hookrightarrow \mathcal{G}Man$. This induces an adjunction:

$$\mathrm{Fun}^{\otimes}(Env(Open^G(M)), Ch) \underset{i^*}{\overset{i_!}{\rightleftarrows}} \mathrm{Fun}^{\otimes}(\mathcal{G}Man, Ch) \qquad (i_! \dashv i^*)$$

which we get for formal reasons involving the operadic left Kan extension (see Appendix D, specifically D.8; [1], Lemma 2.16).

In the particular case of $Env(Open^G(M))$ and $\mathcal{G}Man$, there is a special compatibility of the respective Grothendieck topologies: for any $X \in \mathcal{G}Man$, there is a covering sieve for X consisting of objects in $Env(Open^G(M))$; i.e. there is a Weiss covering of X built from open subsets of M. Imposing the descent axiom on both sides lifts the adjunction to an equivalence of categories:

$$\mathrm{Fun}^{\otimes,desc}(Env(Open^G(M)), Ch) \simeq \mathrm{Fun}^{\otimes,desc}(\mathcal{G}Man, Ch)$$

This is a special case of a more general result; we prove the more general result here, with a view towards being able to apply this to the case of smooth, family versions of the respective ∞-categories.

Proposition 5.3. *Let $\mathcal{A}, \mathcal{B}$ be symmetric monoidal ∞-categories; let $i : \mathcal{A} \hookrightarrow \mathcal{B}$ be a symmetric monoidal functor satisfying the conditions of D.7 ([1], Lemma 2.16). Let $\mathcal{A}, \mathcal{B}$ be equipped with Grothendieck topologies which are compatible in the following sense: if $a \in \mathcal{A}$ and $\overline{\mathfrak{W}} \xrightarrow{\bar{k}} \mathcal{A}$ is a covering sieve of a, then $\overline{\mathfrak{W}} \xrightarrow{\bar{k}} \mathcal{A} \xrightarrow{i} \mathcal{B}$ is a covering sieve of $ia \in \mathcal{B}$; and for every $b \in \mathcal{B}$ there is a covering sieve on b consisting of objects in $\mathcal{A}$.*

Then there is an equivalence of ∞-categories:

$$\mathrm{Fun}^{\otimes, desc}(\mathcal{A}, Ch) \simeq \mathrm{Fun}^{\otimes, desc}(\mathcal{B}, Ch)$$

witnessed by the adjoint functors $i^*, i_!$.

In particular, for $\mathcal{A} = Env(Open^G(M)), \mathcal{B} = \mathcal{G}Man$, there is an equivalence of ∞-categories:

$$\mathrm{Fun}^{\otimes, desc}(Env(Open^G(M)), Ch) \simeq \mathcal{G}Fac.$$

We will prove this proposition in the following steps: We first need to show that when we restrict the functor catgories by imposing the descent axiom, the functors $i^*, i_!$ still have the appropriate source and target categories; we do this in 5.4. We then need to show that these functors are (a) still adjoint functors, and (b) witness the categorical equivalence; we will do these steps together by showing the functors give an adjoint equivalence in 5.5. See Appendix C for background on adjoint equivalences in the setting of ∞-categories.

Lemma 5.4. *Let $i : \mathcal{A} \to \mathcal{B}$ as above. When restricting to functors that satisfy the descent axiom, the functors:*

$$\mathrm{Fun}^{\otimes,desc}(\mathcal{A},Ch) \underset{i^*}{\overset{i_!}{\rightleftarrows}} \mathrm{Fun}^{\otimes,desc}(\mathcal{B},Ch)$$

have the desired sources and targets.

Proof. We first show that i^* preserves the descent axiom. Assume that $G \in \mathrm{Fun}^{\otimes}(\mathcal{B}, Ch)$ satisfies descent; i.e. for any object $b \in \mathcal{B}$ and for any covering sieve $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{B}$ of b, denote the inclusion functor by $\mathcal{U} := \bar{j}(\overline{\mathfrak{U}}) \xrightarrow{j} \mathcal{B}$. Then

$$j_! j^* G(b) \simeq G(b).$$

Take an arbitrary $a \in \mathcal{A}$ and any covering sieve $\overline{\mathfrak{W}} \xrightarrow{\overline{J}} \mathcal{A}$ of a. Note that $\overline{\mathfrak{W}} \xrightarrow{\overline{J}} \mathcal{A} \xrightarrow{i} \mathcal{B}$ is then a covering sieve of $i(a) \in \mathcal{B}$. Denote the inclusion functor $\mathcal{W} := \overline{J}(\overline{\mathfrak{W}}) \overset{J}{\hookrightarrow} \mathcal{A} \xrightarrow{i} \mathcal{B}$. The fact that G satisfies descent in particular means that:

$$(i \circ J)_!(i \circ J)^* G(ia) \simeq G(ia).$$

Then

$$\begin{aligned} i^*G(a) &= G(ia) \\ &\simeq (i \circ J)_!(i \circ J)^* G(ia) \\ &= i_! J_! J^* i^* G(ia) \\ &\simeq \mathrm{hocolim}\left(\mathcal{A}/_{ia} \to \mathcal{A} \xrightarrow{J_! J^* i^* G} Ch\right) \\ &\simeq J_! J^* i^* G(a) \end{aligned}$$

Note that the fifth line follows because i is a full functor; the terminal object in the category $\mathcal{A}/_{ia}$ is $(a, ia \xrightarrow{id} ia)$; the colimit can then be evaluated by applying the composite functor to the terminal object.

Thus i^* preserves the descent condition.

We now show that $i_!$ preserves the descent axiom. Assume that $F \in \mathrm{Fun}^{\otimes}(\mathcal{A}, Ch)$ satisfies descent; i.e. for any object $a \in \mathcal{A}$ and for any covering sieve $\overline{\mathfrak{W}} \xrightarrow{\overline{J}} \mathcal{A}$ of a, with inclusion functor denoted $\mathcal{W} := \overline{J}(\overline{\mathfrak{W}}) \xrightarrow{J} \mathcal{A}$, then

$$J_! J^* F(a) \simeq F(a).$$

Take an arbitrary $b \in \mathcal{B}$ and a covering sieve $\overline{\mathfrak{U}} \xrightarrow{\overline{j}} \mathcal{B}$ of b. Denote the corresponding inclusion functor $\mathcal{U} := \overline{j}(\overline{\mathfrak{U}}) \overset{j}{\hookrightarrow} \mathcal{B}$. Let $\overline{\mathfrak{V}}$ be the full subcategory of $\mathcal{A}/(\mathcal{A}/b)$ consisting of objects of the form:

$$\begin{array}{ccc} a' & \longrightarrow & a \\ \downarrow & & \downarrow \\ u & \longrightarrow & b \end{array}$$

where $u \to b$ is an object of $\overline{\mathfrak{U}}$, $a' \to u$ is an element of the covering sieve on u consisting of elements in $\mathcal{A}$, and $a \to b$ is an object of $\mathcal{A}/b$.

We will denote such an object by (a', a, u), suppressing the maps for ease of notation. Note that by the stability axiom of Grothendieck topologies, if we fix an object $(a \xrightarrow{f} b) \in \mathcal{A}/b$, the collection of all (a', a, u) with this a fixed gives the pullback covering sieve $f^*\overline{\mathfrak{U}}$ on a.

Let $\mathcal{V}$ denote the image of $\overline{\mathfrak{V}}$ in $\mathcal{A}$ (i.e. the objects $a' \in \mathcal{A}$). Consider the commutative diagram:

$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{J} & \mathcal{A} & \xrightarrow{F} & Ch \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle i} & \nearrow{\scriptstyle i_! F} & \\ \mathcal{U} & \xrightarrow[j]{} & \mathcal{B} & & \end{array}$$

where J and I are inclusion functors. (Note that $a' \in \mathcal{V}$ gives an object of $\mathcal{U}$ by the saturation property of $\overline{\mathfrak{U}}$.)

For any $a \in \mathcal{A}/b$, the descent axiom for F gives:

$$F(a) \simeq J_! J^* F(a).$$

Also note that $I_! J^* F \simeq j^* i_! F : \mathcal{U} \to Ch$; this can be seen by comparing the respective colimits as follows: Take $u \in \mathcal{U}$. The definition of the left Kan extension gives:

$$I_! J^* F(u) = \text{hocolim}\left(\mathcal{V}/u \to \mathcal{V} \xrightarrow{J} \mathcal{A} \xrightarrow{F} Ch\right)$$

$$\begin{aligned} j^* i_! F(u) &= i_! F(j(u)) \\ &= i_! F(u) \\ &\simeq \text{hocolim}\left(\mathcal{A}/u \to \mathcal{A} \xrightarrow{F} Ch\right) \end{aligned}$$

Note that $\mathcal{A}/u \simeq \mathcal{V}/u$ and the following diagram commutes trivially:

$$\begin{array}{ccc} \mathcal{V}/u & \xrightarrow{forget} & \mathcal{V} \\ \simeq\downarrow & & \downarrow J \\ \mathcal{A}/u & \xrightarrow[forget]{} & \mathcal{A} \end{array}$$

Putting these results together, one gets:

$$
\begin{aligned}
i_! F(b) &\simeq \operatorname{hocolim}\left(\mathcal{A}/b \to \mathcal{A} \xrightarrow{\mathcal{F}} Ch\right) \\
&\simeq \operatorname{hocolim}_{a \in \mathcal{A}/b}\left(F(a)\right) \\
&\simeq \operatorname{hocolim}_{a \in \mathcal{A}/b}\left(J_! J^* F(a)\right) \\
&\simeq i_! J_! J^* F(b) \\
&= j_! I_! J^* F(b) \\
&\simeq j_! j^* i_! F(b)
\end{aligned}
$$

Thus $i_!$ preserves the descent condition.

□

We now show that $i^*, i_!$ give an adjoint equivalence between the ∞-categories of factorization algebras.

Lemma 5.5. *Let $i : \mathcal{A} \to \mathcal{B}$ as in 5.4. Then*

$$
\operatorname{Fun}^{\otimes, desc}(\mathcal{A}, Ch) \underset{i^*}{\overset{i_!}{\rightleftarrows}} \operatorname{Fun}^{\otimes, desc}(\mathcal{B}, Ch) \quad (\simeq)
$$

gives an adjoint equivalence of ∞-categories.

Proof. For background about adjoint equivalences of ∞-categories see Appendix C. In summary, we will show that there exists a unit transformation $u : \mathbb{1}_{\operatorname{Fun}^{\otimes, desc}(\mathcal{A}, Ch)} \to i^* i_!$ (i.e. $i_!, i^*$ are adjoint functors), and that the desired compositions are naturally isomorphic to the respective identity functors:

$$
\mathbb{1}_{\operatorname{Fun}^{\otimes, desc}(\mathcal{A}, Ch)} \cong i^* i_! \qquad i_! i^* \cong \mathbb{1}_{\operatorname{Fun}^{\otimes, desc}(\mathcal{B}, Ch)}.
$$

The first of these equivalences will be given by the unit transformation.

Since $i : \mathcal{A} \to \mathcal{B}$ is an inclusion of a full subcategory, there is a particularly nice choice for the unit transformation, namely the identity, (see D.5). This is a natural isomorphism, giving the desired equivalence:

$$\mathbb{1}_{\mathrm{Fun}^{\otimes,desc}(\mathcal{A},Ch)} \cong i^* i_!.$$

For the other equivalence, take an arbitrary $F \in \mathrm{Fun}^{\otimes,desc}(\mathcal{B}, Ch)$. We want to show, for any $b \in \mathcal{B}$,

$$i_! i^* F(b) \simeq^? F(b).$$

Let $\overline{\mathfrak{U}} \xrightarrow{\bar{j}} \mathcal{B}$ be an arbitrary covering sieve of b, with associated category $\mathcal{U} := \bar{j}(\overline{\mathfrak{U}}) \overset{j}{\hookrightarrow} \mathcal{B}$. Consider the following diagram, involving the same $\mathcal{V}, I, J$ as in the proof of 5.4

$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{J} & \mathcal{A} & & \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle i} & \searrow^{i^*F} & \\ \mathcal{U} & \xrightarrow[j]{} & \mathcal{B} & \xrightarrow[F]{} & Ch \end{array}$$

Note that the stability axiom of Grothendieck topologies implies that $\mathcal{V} \xrightarrow{J} \mathcal{A}$ gives a cover for all $a \in \mathcal{A}/b$; thus $i^*F|_{\mathcal{A}/b} \simeq J_! J^* i^* F|_{\mathcal{A}/b}$.

Also note that $\mathcal{V} \xrightarrow{J} \mathcal{A} \xrightarrow{i} \mathcal{B}$ gives a cover of b, by definition of $\mathcal{V}$ and the transitivity axiom of Grothendieck topologies.

Then

$$\begin{aligned} i_! i^* F(b) &\simeq i_! J_! J^* i^* F(b) \\ &= (i \circ J)_! (i \circ J)^* F(b) \\ &\simeq F(b) \end{aligned}$$

which gives the desired equivalence. □

Proof of 5.3. Putting 5.4 and 5.5 together gives the proof of the desired equivalence.

□

APPENDIX A

∞-OPERAD BACKGROUND

In this section we review the ∞-operad and symmetric monoidal ∞-category background used in this project. This material is taken from [10] and [6]; there is no claim of originality. It is included as an introduction to this background material and for the sake of completeness.

A.1 Motivation from (non-∞) symmetric monoidal categories and operads

There are different ways to think of symmetric monoidal categories. The standard approach (à la MacLane) defines a symmetric monoidal category as a category, $\mathcal{C}$, equipped with a bifunctor, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and a unit object, $\mathbb{1}_{\mathcal{C}}$, for the monoidal structure $\otimes$; these are required to satisfy certain coherence data, witnessed by natural isomorphisms (the associator and unitors); see [11]. For 1-categories this is a good perspective; but if one is interested in generalizing this to a notion of symmetric monoidal higher categories, the coherence data quickly become prohibitively complicated.

An alternative approach that avoids this difficulty is to formulate a symmetric monoidal category instead as a category equipped with a special functor to the category of pointed finite sets, Fin_*; this functor is required to satisfy certain lifting conditions. In contrast to the first approach, this outlook is convenient for generalizations: for $(\infty, 1)$-categories one looks at a functor between quasi-categories which satisfies appropriate generalizations of the lifting properties, thus yielding a natural (and more workable) notion of a symmetric monoidal higher category. The historical motivation for this viewpoint is due to Segal: he outlined how one could view a symmetric monoidal category as a pseudo-functor from $\mathsf{Fin}_* \to Cat$ plus a certain condition; or, equivalently, as a cofibration $\mathcal{C} \to \mathsf{Fin}_*$. This led to the more general notion of an operad as a category that is 'partially cofibered' over Fin_*.

We outline this functor-plus-properties perspective for ordinary multicategories (i.e. colored operads) and symmetric monoidal categories in this section. We do this to motivate the definition of symmetric monoidal ∞-categories and ∞-operads which will follow in section A.2. Just as ordinary symmetric monoidal categories can be viewed as a special case of an ordinary operad, an analogous situation holds in the world of ∞-categories: a symmetric monoidal ∞-category will be a special case of an ∞-operad. After reviewing these definitions, we include descriptions of two constructions with ∞-operads which are used in this project in Appendix B and Appendix D. In Appendix C we review the notions of adjunctions and equivalences between ∞-operads.

Let Fin_* denote the category whose objects are pointed finite sets, denoted $\langle m \rangle = \{*, 1, ..., m\}$, where $m \geq 0$; and whose morphisms are basepoint preserving maps of sets. Let $\langle m \rangle^\circ$ denote the set $\langle m \rangle$ without the basepoint.

Remark A.1. Note that any finite set is isomorphic to one of this kind, so it suffices to consider this category. See Remark 2.0.0.3 of [10]; we follow this notation.

We define some particular maps in Fin_* that will be of use: for $1 \leq i \leq n$ define $\rho^i : \langle n \rangle \to \langle 1 \rangle$ to be the map given by

$$\rho^i(j) = \begin{cases} 1, & \text{if } i = j \\ *, & \text{else} \end{cases}.$$

Construction A.2. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category, as defined in [11]. Then one can construct a new category from it, denoted $\mathcal{C}^{\otimes}$, consisting of:

- **objects:** finite (possibly empty) sequences of objects in $\mathcal{C}$; denote these by $(\langle m \rangle; c_1, ..., c_m)$, where the object $\langle m \rangle \in \mathsf{Fin}_*$ is the indexing set for the objects of $\mathcal{C}$

- **morphisms:** a morphism from $(\langle m \rangle; c_1, ..., c_m) \to (\langle n \rangle; c'_1, ..., c'_n)$ consists of:

  - a morphism in Fin_*, $\alpha : \langle m \rangle \to \langle n \rangle$

  - a collection of morphisms in $\mathcal{C}$:

$$\left\{ f_i : \bigotimes_{j \in \alpha^{-1}(i)} c_j \to c'_i \right\}_{1 \leq i \leq n}$$

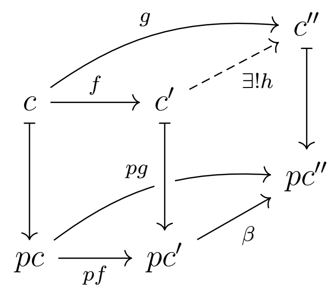
- **composition:** from composition in Fin_* and $\mathcal{C}$

Remark A.3. Note that the domain of f_i above is not a well-defined object (there are choices involved both in how the objects in the tensor product are ordered, and how one brackets the resulting tensor product). However, due to MacLane's coherence theorem all such choices of objects are isomorphic, making the *morphism* set with this domain well-defined.

There is a natural forgetful functor $p : \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$. When $\mathcal{C}^{\otimes}$ comes from a symmetric monoidal category $\mathcal{C}$ as in the construction above, this functor p satisfies some additional properties, which we now unpack. We first recall some terminology.

Notation A.4. Let $\mathcal{C}^{\otimes}_{\langle m \rangle}$ denote the fiber of p over $\langle m \rangle \in \mathsf{Fin}_*$.

Definition A.5. Let $p : \mathcal{C} \to \mathcal{D}$ be a functor. An arrow $f : c \to c'$ in $\mathcal{C}$ is *p-coCartesian* if for any arrow $g : c \to c''$ in $\mathcal{C}$ and any arrow $\beta : pc' \to pc''$ in $\mathcal{D}$ with $\beta \circ pf = pg$, there exists a unique arrow $h : c' \to c''$ with $ph = \beta$ and $h \circ f = g$, as shown below:



Equivalently, $f : c \to c'$ is p-coCartesian if the following is a pullback diagram:

$$\begin{array}{ccc} hom_{\mathcal{C}}(c', c'') & \xrightarrow{f^*} & hom_{\mathcal{C}}(c, c'') \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ hom_{\mathcal{D}}(pc', pc'') & \xrightarrow[(pf)^*]{} & hom_{\mathcal{D}}(pc, pc'') \end{array}$$

Definition A.6. A functor $p : \mathcal{C} \to \mathcal{D}$ is a *(Grothendieck) op-fibration* if for all $c \in \mathcal{C}$ and any morphism $\alpha : pc \to d'$ in $\mathcal{D}$, there exists a p-coCartesian arrow $f : c \to c'$, such that $pf = \alpha$ (i.e. there exists a coCartesian lift of α, f).

Remark A.7. In the construction above, where $\mathcal{C}^{\otimes}$ comes from a symmetric monoidal category $\mathcal{C}$, the forgetful functor $p : \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ enjoys two features of note:

(i) The functor p is an op-fibration.

(ii) (Segal condition) There is an equivalence $\mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C}$; more generally, the functors $\{\rho_i : \langle n \rangle \to \langle 1 \rangle\}_{1 \le i \le n}$ induce equivalences $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{C})^n$ for all $n \ge 0$.

The symmetric monoidal structure on $\mathcal{C}$ is determined, up to equivalence, by $\mathcal{C}^{\otimes}$ and the functor p. It turns out that the reverse is true as well: having a functor $p : \mathcal{D} \to \mathsf{Fin}_*$ which satisfies the properties (i) and (ii) above gives $\mathcal{D}_{\langle 1 \rangle}$ the structure of a symmetric monoidal category. See pages 167-168 of [10] or Section 4.1 of [6] for more details. In other words, a symmetric monoidal category is equivalent to a functor $\mathcal{D} \to \mathsf{Fin}_*$ satisfying (i) and (ii) above.

There are different advantages of these two views of symmetric monoidal categories. One of the key advantages of the latter is that the coherence conditions of a symmetric monoidal category, such as the commutativity of MacLane's pentagon diagram, are hidden in the combinatorial data of Fin_*. Especially when one wants to consider a higher categorical version of monoidal categories, where the coherence conditions become exceedingly more complicated, the functor-plus-property viewpoint becomes much more convenient. Following [10], we thus use this viewpoint to define symmetric monoidal ∞-categories.

Analogous to the case with ordinary categories, where multicategories (or colored operads) are a generalization of symmetric monoidal categories, relaxing the lifting conditions on the fibration p gives the more general notion of an ∞-operad. More precisely, we will be asking for coCartesian lifts for particular types of maps which are called *inert*, instead of for all maps. In the following section we first review this more general theory, before addressing the particular case of symmetric monoidal ∞-categories. We then discuss various notions of maps between ∞-operads, building up to the notion of symmetric monoidal ∞-functors.

A.2 ∞-operads and symmetric monoidal ∞-categories

For general ∞-operads, we will want lifts with respect to a particular class of maps in $N(\mathsf{Fin}_*)$.

Definition A.8 ([10], Defn 2.1.1.8)**.** A morphism $f : \langle m \rangle \to \langle n \rangle$ in Fin_* is *inert* if for every element $i \in \langle n \rangle^\circ$, the preimage $f^{-1}(i)$ has exactly one element.

A coCartesian fibration of ∞-categories is defined in an analogous way to the Grothendieck op-fibrations above, using the ∞-categorical version of a coCartesian lift.

Definition A.9 ([6], Defn 1.37)**.** Let $p : X \to S$ be a morphism of simplicial sets. p is an *inner fibration* if it has the right lifting property with respect to all maps $\Lambda^n_k \to \Delta^n$ for $0 < k < n$.

Definition A.10 ([6], Defn 4.12)**.** Let $p : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. $f : c_1 \to c_2$ in $\mathcal{C}$ is a *p-coCartesian arrow* if

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{pc_1/}} \mathcal{D}_{pf/}$$

is an acyclic Kan fibration.

Definition A.11 ([6], Defn 4.13)**.** Let $p : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories p is a *p-coCartesian fibration* if

1. p is an inner fibration.

2. for all $c_1 \in \mathcal{C}$ and for all morphisms $\alpha : p(c_1) = d_1 \to d_2$ in $\mathcal{D}$, there exists a p-coCartesian lift $f : c_1 \to c_2$ of α.

Both ∞-operads and symmetric monoidal ∞-categories will be defined as ∞-categories with functors to $N(\mathsf{Fin}_*)$, which have coCartesian lifts with respect to certain morphisms, as well as certain other properties.

Definition A.12 ([10], Defn 2.1.1.10)**.** An ∞-operad is a functor $p : \mathcal{O}^{\otimes} \to N(\mathsf{Fin}_*)$ between ∞-categories satisfying:

(i) For every inert morphism $f : \langle m \rangle \to \langle n \rangle$ in $N(\mathsf{Fin}_*)$ and every object $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, there exists a p-coCartesian lift $\bar{f} : C \to C'$ in $\mathcal{O}^{\otimes}$ of f. In particular, f induces a functor $f_! : \mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle n \rangle}$.

(ii) For $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}, C' \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and $f : \langle m \rangle \to \langle n \rangle$, let $Map^{f}_{\mathcal{O}^{\otimes}}(C, C')$ be the union of those connected components of $Map_{\mathcal{O}^{\otimes}}(C, C')$ which lie over f. Choose p-coCartesian morphisms $C' \to C'_i$ lying over the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. Then the induced map

$$Map^{f}_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{1 \leq i \leq n} Map^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$

is a homotopy equivalence.

(iii) For every $n \geq 0$, the functors $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}$ induce an equivalence of ∞-categories $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^n$.

Remark A.13. Note that in Definition 2.1.1.10 of [10], Lurie gives a different version of condition (iii), but in Remark 2.1.1.14 he notes the equivalence to the above formulation.

Definition A.14 ([10], Defn 2.0.0.7)**.** A *symmetric monoidal ∞-cateogry* is a p-coCartesian fibration of simplicial sets $p : \mathcal{C}^{\otimes} \to N(\mathsf{Fin}_*)$, satisfying the following property:

- For $n \geq 0$, the maps $\{\rho^i : \langle m \rangle \to \langle 1 \rangle\}_{1 \leq i \leq m}$ induce an equivalence

$$\mathcal{C}^{\otimes}_{\langle m \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^m.$$

Remark A.15. Note that the definition of symmetric monoidal ∞-categories contains condition (iii) of the definition of ∞-operads; conditions (i) and (ii) are subsumed in the requirement that $p : \mathcal{C}^{\otimes} \to N(\mathsf{Fin}_*)$ is a coCartesian fibration (there are coCartesian lifts for all maps with a vertex over the target, rather than just for inert ones).

When the functor p is clear from the context, we will drop the notation of $p-$ and simply say 'coCartesian fibration/morphism'.

A.3 Maps of ∞-operads

We now look at maps between ∞-operads, which will give us another perspective on symmetric monoidal ∞-categories as well as the notion of symmetric monoidal ∞-functors.

Definition A.16 ([10], Defn 2.1.2.7)**.** Let $\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}$ be ∞-operads. An *∞-operad map* from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ is a map of simplicial sets $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ such that:

(1) The diagram

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{\quad f \quad} & \mathcal{O}'^{\otimes} \\ & \searrow \quad \swarrow & \\ & N(\mathsf{Fin}_*) & \end{array}$$

commutes.

(2) The functor f preserves inert morphisms.

Notation A.17. Lurie lets $Alg_{\mathcal{O}}(\mathcal{O}')$ denote the full subcategory of $\mathrm{Fun}_{N(\mathsf{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by the ∞-operad maps. Because we want to emphasize that these are simply functors of ∞-operads (highlighting that they preserve inert morphisms), we will denote this subcategory by $\mathrm{Fun}^{inert}(\mathcal{O}, \mathcal{O}')$.

Definition A.18 ([10], Defn 2.1.2.10)**.** A map of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a *fibration of ∞-operads* if q is a categorical fibration.

Proposition A.19 ([10], Propn 2.1.2.12)**.** *Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration (of ∞-categories). The following are equivalent:*

- *The composite $q : \mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes} \to N(\mathsf{Fin}_*)$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.*
- *For every $T \simeq T_1 \oplus \cdots \oplus T_n \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, the inert morphisms $T \to T_i$ induce an equivalence of ∞-categories*

$$\mathcal{C}^{\otimes}_{T} \to \prod_{1 \le i \le n} \mathcal{C}^{\otimes}_{T_i}.$$

Definition A.20 ([10], Defn 2.1.2.13)**.** Let $\mathcal{O}^{\otimes}$ be an ∞-operad. A map $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a *coCartesian fibration of ∞-operads* if it satisfies the hypotheses of A.19.

We say that p 'exhibits $\mathcal{C}^{\otimes}$ as an *$\mathcal{O}$-monoidal ∞-category*'.

Notation A.21. In the particular case of the above definition where $\mathcal{O}^\otimes = N(\mathsf{Fin}_*)$, we say that $\mathcal{C}^\otimes$ is a *symmetric monoidal* ∞*-category*; i.e. it is an ∞-category $\mathcal{C}^\otimes$ equipped with a coCartesian fibration of ∞-operads $p : \mathcal{C}^\otimes \to N(\mathsf{Fin}_*)$.

Example A.22. This generalizes the ordinary symmetric monoidal category case in the following sense. Let $\mathcal{C}$ be a symmetric monoidal category, with associated category $\mathcal{C}^\otimes$. Then $N(\mathcal{C}^\otimes)$ is a symmetric monoidal ∞-category, with underlying ∞-category $N(\mathcal{C}^\otimes)_{\langle 1 \rangle}$.

Remark A.23. One of the motivations for formulating our results in this framework comes from the flexibility built into this perspective of symmetric monoidal ∞-categories. In future work, we would like to explore replacing $\mathcal{O}^\otimes = N(\mathsf{Fin}_*)$ with a more general ∞-operad that would encode a parameterizing category for the family or smooth versions of factorization algebras.

A.3.1 Symmetric monoidal ∞-functors

What does it mean to have a symmetric monoidal functor in this ∞-category version? To start, suppose that $p : \mathcal{C}^\otimes \to N(\mathsf{Fin}_*), q : \mathcal{D}^\otimes \to N(\mathsf{Fin}_*)$ are two symmetric monoidal ∞-categories. Then an ∞-operad map $F \in \mathrm{Fun}^{inert}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ compatible with the symmetric monoidal stucture in a lax sense, i.e. we are given maps

$$F(C) \otimes F(C') \to F(C \otimes C') \qquad \mathbb{1} \to F(\mathbb{1})$$

which are compatible with commutativity and associativity properties for the monoidal structure on $\mathcal{C}$ and $\mathcal{D}$.

Definition A.24. Let $\mathcal{O}^\otimes$ be an ∞-operad; let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes, q : \mathcal{D}^\otimes \to \mathcal{O}^\otimes$ be coCartesian fibrations of ∞-operads (i.e. $\mathcal{O}$-monoidal ∞-categories). An ∞-operad map $f \in \mathrm{Fun}^{inert}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ is an *$\mathcal{O}$-monoidal functor* if it carries p-coCartesian morphisms to q-coCartesian morphisms. We denote the full subcategory spanned by $\mathcal{O}$-monoidal functors $\mathrm{Fun}^{\otimes}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \subset \mathrm{Fun}^{inert}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$.

In the case where $\mathcal{O}^{\otimes} = N(\mathsf{Fin}_*)$, we call these *symmetric monoidal functors* from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$; we denote these $\mathrm{Fun}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$.

APPENDIX B

A USEFUL CONSTRUCTION: THE MONOIDAL ENVELOPE

There is a forgetful functor from symmetric monoidal ∞-categories to ∞-operads, which forgets the extra structure a symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to N\mathsf{Fin}_*$ has (of being a coCartesian fibration of ∞-operads, as opposed to having weaker lifting properties). In Construction 2.2.4.1 of [10], Lurie gives an adjoint to this functor, creating from an ∞-operad, $\mathcal{O}^{\otimes}$, an associated symmetric monoidal ∞ category, $Env(\mathcal{O}^{\otimes})$, which he calls the *monoidal envelope*. This is an 'exterior completion' of the ∞-operad, where the objects consist of formal tensor products of the objects in $\mathcal{O}^{\otimes}$; it is analogous to the Stone-Čech compactification for topological spaces.

We first outline the construction of the monoidal envelope. It will use the following special type of morphisms in Fin_*:

Definition B.1. A morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in Fin_* is *active* if $\alpha^{-1}\{*\} = \{*\}$.

Let $p : \mathcal{O}^{\otimes} \to N(\mathsf{Fin}_*)$ be an ∞-operad. A morphism f in $\mathcal{O}^{\otimes}$ is *active* if $p(f)$ is active.

Notation B.2. Let $\mathrm{Act}(N\mathsf{Fin}_*) \subseteq \mathrm{Fun}(\Delta^1, N\mathsf{Fin}_*)$ denote the full subcategory spanned by the active morphisms in $N\mathsf{Fin}_*$.

Definition B.3. Let $p : \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$ be an ∞-operad. The *monoidal envelope* of $\mathcal{O}^{\otimes}$ is defined to be the fiber product:

$$Env(\mathcal{O}^{\otimes}) := \mathcal{O}^{\otimes} \times_{\mathrm{Fun}(\{0\}, N\mathsf{Fin}_*)} \mathrm{Act}(N\mathsf{Fin}_*)$$

where the map $\mathrm{Act}(N\mathsf{Fin}_*) \to \mathrm{Fun}(\{0\}, N\mathsf{Fin}_*)$ is given by taking the source of the active morphism.

Unpacking this definition, the 0-simplices of $Env(\mathcal{O}^{\otimes})$ consist of pairs (c, α) where $c \in \mathcal{O}^{\otimes}$ and $\alpha : p(c) \to \langle n \rangle$ is any active morphism in $N\mathsf{Fin}_*$.

Projection onto the target of the active morphisms (i.e. the target of α for an object (c, α)) induces a map $Env(\mathcal{O}^{\otimes}) \to N\mathsf{Fin}_*$; this is a coCartesian fibration of ∞-operads, making $Env(\mathcal{O}^{\otimes})$ a symmetric monoidal ∞-category. See [10], Propn 2.2.4.4 for the proof of this result.

Note that there is an map of ∞-operads:

$$\mathcal{O}^{\otimes} \hookrightarrow Env(\mathcal{O}^{\otimes}).$$

The monoidal envelope satisfies the following universal property which will be of particular use for our project:

Proposition B.4 ([10], Propn 2.2.4.9). *Let $\mathcal{O}^{\otimes} \to N\mathit{Fin}_*$ be an ∞-operad, $Env(\mathcal{O}^{\otimes}) \to N\mathit{Fin}_*$ the associated symmetric monoidal ∞-category and $\mathcal{C} \to N\mathit{Fin}_*$ another symmetric monoidal ∞-category. The inclusion $i : \mathcal{O}^{\otimes} \hookrightarrow Enc(\mathcal{O}^{\otimes})$ induces an equivalence of ∞-categories:*

$$Fun^{\otimes}(Env(\mathcal{O}^{\otimes}), \mathcal{C}) \xrightarrow{\sim} Fun^{inert}(\mathcal{O}^{\otimes}, \mathcal{C})$$

where $Fun^{\otimes}(Env(\mathcal{O}^{\otimes}), \mathcal{C})$ denotes the symmetric monoidal ∞-functors and $Fun^{inert}(\mathcal{M}, \mathcal{C})$ denotes the maps of ∞-operads.

APPENDIX C

ADJOINT EQUIVALENCES FOR ∞-CATEGORIES

C.1 Adjunctions for ∞-categories

Just as in the case of adjunctions for ordinary categories, there are several different equivalent ways of formulating adjunctions of ∞-categories. For ordinary categories, an adjunction between two categories $\mathcal{C}, \mathcal{D}$ can be defined to be a pair of functors $f : \mathcal{C} \to \mathcal{D}, g : \mathcal{D} \to \mathcal{C}$, plus a function $\phi : \mathcal{C} \times \mathcal{D} \to Set$ which assigns to any pair $(c, d) \in \mathcal{C} \times \mathcal{D}$ a bijection between the hom-sets: $\phi(c, d) : \mathcal{C}(c, gd) \cong \mathcal{D}(fc, d)$. This data (f, g, ϕ) determines two natural transformations, called the *unit*, $\eta : \mathbb{1}_{\mathcal{C}} \to gf$, and the *counit*, $\epsilon : fg \to \mathbb{1}_{\mathcal{D}}$, which satisfy certain compatibility conditions. Conversely, the adjunction is completely determined by the data (η, ϵ), plus the compatibility conditions. This is an example of the different perspectives possible for adjunctions: the hom-set definition, and the unit/counit definition.

An analogous situation holds in the quasi-category world. We will begin by giving what would be the analog of the hom-set definition, as this is how adjunctions between ∞-categories are defined in [9]. For our purposes, it will be more convenient to take the analog of the unit/counit definition, so we develop that perspective as well. We finally recall the definition of equivalences of ∞-categories. There are also multiple equivalent definitions of these; we take the one that fits well with the unit/counit adjunction set-up.

In what follows, let $\mathcal{C}, \mathcal{D}$ be quasicategories.

Definition C.1 ([9], 5.2.2.1)**.** An adjunction between $\mathcal{C}$ and $\mathcal{D}$ is a map

$$p : \mathcal{M} \to \Delta^1$$

which is both a Cartesian and coCartesian fibration, together with equivalences $\mathcal{C} \xrightarrow{\sim} \mathcal{M}_{\{0\}}$ and $\mathcal{D} \xrightarrow{\sim} \mathcal{M}_{\{1\}}$.

There are functors associated to this Cartesian-coCartesian fibration; this is an example of a correspondence and its associated functors (see [9], section 5.2.1).

We unpack this for the situation above. Take $p : \mathcal{M} \to \Delta^1$ to be an adjunction between $\mathcal{C}$ and $\mathcal{D}$ with associated functors $f : \mathcal{C} \to D, g : \mathcal{D} \to C$. Identify $\mathcal{M}_{\{0\}}$ with $\mathcal{C}$, and $\mathcal{M}_{\{1\}}$ with $\mathcal{D}$. The fact that f, g are associated with $\mathcal{M}$ means that we have commutative diagrams:

$$\begin{array}{ccc} \mathcal{C} \times \Delta^1 & \xrightarrow{F} & \mathcal{M} \\ & \searrow & \swarrow_{p} \\ & \Delta^1 & \end{array}$$

such that

$$\begin{aligned} F|_{\mathcal{C}\times\{0\}} &= id_{\mathcal{C}} \\ F|_{\mathcal{C}\times\{1\}} &= f \\ F|_{\{c\}\times\Delta^1} & \text{ is coCartesian for every } c \in \mathcal{C} \end{aligned}$$

and

$$\begin{array}{ccc} \mathcal{D} \times \Delta^1 & \xrightarrow{G} & \mathcal{M} \\ & \searrow & \swarrow_{p} \\ & \Delta^1 & \end{array}$$

such that

$$G|_{\mathcal{D}\times\{0\}} = g$$

$$G|_{\mathcal{D}\times\{1\}} = id_{\mathcal{D}}$$

$$G|_{\{d\}\times\Delta^1} \text{ is Cartesian for every } d \in \mathcal{D}$$

It will be convenient for our purposes to look at an alternative description of adjunctions between ∞-categories, using the notion of unit and counit transformations.

Definition C.2 ([9], 5.2.2.7)**.** Suppose $f : \mathcal{C} \to \mathcal{D}, g : \mathcal{D} \to \mathcal{C}$ are functors between quasicategories. A *unit transformation* for (f, g) is a morphism $u : id_{\mathcal{C}} \to g \circ f$ in $Fun(\mathcal{C}, \mathcal{C})$ such that for every $c \in \mathcal{C}, d \in \mathcal{D}$, the composition

$$Map_{\mathcal{D}}(f(c), d) \to Map_{\mathcal{C}}(gf(c), g(d)) \xrightarrow{u(c)} Map_{\mathcal{C}}(c, g(d))$$

is an isomorphism in the homotopy category of spaces, $\mathcal{H}$.

Dually, one has:

Definition C.3. Suppose $f : \mathcal{C} \to \mathcal{D}, g : \mathcal{D} \to \mathcal{C}$ are functors between quasicategories. A *counit transformation* for (f, g) is a morphism $\epsilon : f \circ g \to id_{\mathcal{D}}$ in $Fun(\mathcal{D}, \mathcal{D})$ such that for every $c \in \mathcal{C}, d \in \mathcal{D}$, the composition

$$Map_{\mathcal{C}}(c, g(d)) \to Map_{\mathcal{D}}(f(c), fg(d)) \xrightarrow{\epsilon(d)} Map_{\mathcal{D}}(f(c), d)$$

is an isomorphism in the homotopy category of spaces, $\mathcal{H}$.

The definition of an adjunction between ∞-categories in terms of the correspondence $p : \mathcal{M} \to \Delta^1$ is equivalent to a definitions in terms of the unit transformation:

Proposition C.4 ([9], 5.2.2.8)**.** *Let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ be a pair of functors between quasicategories. The following conditions are equivalent:*

(1) The functor f is left adjoint to g.

(2) There exists a unit transformation $u : id_{\mathcal{C}} \to g \circ f$.

Dually, one gets an equivalent definition in terms of the counit transformation:

Proposition C.5. *Let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ be a pair of functors between quasicategories. The following conditions are equivalent:*

(1) The functor f is left adjoint to g.

(2) There exists a counit transformation $\epsilon : f \circ g \to id_{\mathcal{D}}$.

This proof is completely analogous to the proof that Lurie gives, see [9], Propn 5.2.2.8.

C.2 Equivalences of ∞-categories

We will use the notion of equivalence of ∞-categories as formulated in [13].

Definition C.6 ([13], Defn 19.1)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. A functor $g : \mathcal{D} \to \mathcal{C}$ is a *categorical inverse* to f if there are natural isomorphisms:

$$gf \cong \mathbb{1}_{\mathcal{C}} \qquad fg \cong \mathbb{1}_{\mathcal{D}}.$$

A functor f is a *categorical equivalence* if it admits a categorical inverse.

Note that this is equivalent to the definition of equivalences of ∞-categories given in [9], see [13], Rmk 19.6, Propn 22.11). We will use the definition in [13] because it more naturally fits our situation: we are dealing with an adjunction (between the restriction map and the left Kan extension) with unit and counit maps which give the desired natural isomorphisms.

As in the case of ordinary categories, natural isomorphisms for quasi-categories can be checked objectwise:

Proposition C.7 ([13], 18.3-18.5)**.** *A natural transformation* $\alpha : \mathcal{C} \times \Delta^1 \to \mathcal{D}$ *of functors between quasicategories,* $f_0, f_1 : \mathcal{C} \to \mathcal{D}$*, is a natural isomorphism if and only if for every object* $c \in \mathcal{C}$ *the map* $\alpha(c) : f_0(c) \to f_1(c)$ *is an isomorphism in* $\mathcal{D}$*.*

Equivalently, α *is a natural isomorphism if and only if* $h\alpha$ *is an isomorphism of the homotopy categories.*

APPENDIX D

ANOTHER USEFUL CONSTRUCTION: OPERADIC LEFT KAN EXTENSION

The theory of left Kan extensions has an analog in the world of ∞-categories. We sketch some of the background here, starting with some general definitions and then focusing on the case that is applicable to our situation, where we are dealing with symmetric monoidal ∞-categories.

D.1 Background about operadic left Kan extension

For more details about limits and colimits in the context of ∞-categories, see [9], Chapter 4.

Let $\mathcal{C}$ be an ∞-category and $\mathcal{C}^0$ a full subcategory of $\mathcal{C}$.

Definition D.1 ([9], Defn 4.3.2.2)**.** Given a commutative diagram of ∞-categories of the following form

$$\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \big\downarrow & \overset{F}{\nearrow} & \big\downarrow p \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

where p is an inner fibration and the left vertical map is the inclusion of the full subcategory, then F is a *p-left Kan extension of F_0 at $c \in \mathcal{C}$* if the induced diagram

$$\begin{array}{ccc} \mathcal{C}^0/c & \xrightarrow{F_c} & \mathcal{D} \\ \big\downarrow & \nearrow & \big\downarrow p \\ (\mathcal{C}^0/c)^{\triangleright} & \longrightarrow & \mathcal{D}' \end{array}$$

exhibits $F(c)$ as a p-colimit of F_c.

We say that F is a *p-left Kan extension of* F_0 if it is a p-left Kan extension of F_0 at c for every object $c \in \mathcal{C}$.

Left Kan extensions for ∞-categories satisfy various stability conditions which we would want. Of particular note for our purposes is the following existence result:

Lemma D.2 ([9], Lemma 4.3.2.13)**.** *Suppose we are given a diagram of the form in D.1*

$$\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \big\downarrow & \overset{F}{\nearrow} & \big\downarrow p \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

The following are equivalent:

(1) There exists a functor $F : \mathcal{C} \to \mathcal{D}$ filling the diagram, such that F is a p-left Kan extension of F_0.

(2) For every object $c \in \mathcal{C}$,

$$\mathcal{C}^0/c \to \mathcal{C}^0 \xrightarrow{F_0} \mathcal{D}$$

admits a p-colimit.

The following results allow us to talk about a left Kan extension functor:

Proposition D.3 ([9], Propn 4.3.2.15)**.** *Suppose we are given a diagram of ∞-categories:*

$$\begin{array}{ccc} \mathcal{C}^0 & & \mathcal{D} \\ i\big\downarrow & & \big\downarrow p \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

where p is a categorical fibration and $\mathcal{C}^0$ is a full subcategory of $\mathcal{C}$.

Let $\mathcal{K} \subseteq Map_{\mathcal{D}'}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by functors $F : \mathcal{C} \to \mathcal{D}$ which are p-left Kan extensions of $F|_{\mathcal{C}^0}$.

Let $\mathcal{K}' \subseteq Map_{\mathcal{D}'}(\mathcal{C}^0, \mathcal{D})$ be the full subcategory spanned by functors $F_0 : \mathcal{C}^0 \to \mathcal{D}$ such that, for each $c \in \mathcal{C}$, $\mathcal{C}^0/c \to \mathcal{D}$ has a p-colimit.

Then the restriction functor $\mathcal{K} \to \mathcal{K}'$ is a trivial fibration of simplicial sets.

Corollary D.4 ([9], Cor 4.3.2.16)**.** *Suppose we are given a diagram of ∞-categories as before:*

$$\begin{array}{ccc} \mathcal{C}^0 & & \mathcal{D} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

where p is a categorical fibration and $\mathcal{C}^0$ is a full subcategory of $\mathcal{C}$.

Suppose for every $F_0 \in Map_{\mathcal{D}'}(\mathcal{C}^0, \mathcal{D})$ there exists an $F \in Map_{\mathcal{D}'}(\mathcal{C}, \mathcal{D})$ which is a p-left Kan extension of F_0.

Then

$$i^* : Map_{\mathcal{D}'}(\mathcal{C}, \mathcal{D}) \to Map_{\mathcal{D}'}(\mathcal{C}^0, \mathcal{D})$$

admits a section $i_!$. The essential image of $i_!$ is precisely those F which are p-left Kan extensions of $F|_{\mathcal{C}^0}$.

In this case, we call $i_!$ a left Kan extension functor*.*

The previous proposition proves the existence of $i_!$ and its uniqueness up to homotopy.

D.2 Operadic Lan in the case of inclusion functors

In the case where $i : \mathcal{A} \to \mathcal{B}$ is the inclusion of a full subcategory, the adjunction between i^* and $i_!$ follows naturally:

Proposition D.5 ([9], Propn 4.3.2.17)**.** *Suppose one is given a diagram of ∞-categories:*

$$\mathcal{B} \to \mathcal{C}' \stackrel{p}{\leftarrow} \mathcal{C},$$

where p is a categorical fibration. Suppose $i : \mathcal{A} \to B$ is the inclusion of a full subcategory and that every functor $F \in \mathrm{Fun}_{\mathcal{C}'}(\mathcal{A}, \mathcal{C})$ admits a p-left Kan extension. Then the left Kan extension functor $i_! : \mathrm{Fun}_{\mathcal{C}'}(\mathcal{A}, \mathcal{C}) \to \mathrm{Fun}_{\mathcal{C}'}(\mathcal{B}, \mathcal{C})$ is a left adjoint to the the restriction functor $i^* : \mathrm{Fun}_{\mathcal{C}'}(\mathcal{B}, \mathcal{C}) \to \mathrm{Fun}_{\mathcal{C}'}(\mathcal{A}, \mathcal{C})$.

The proof of this proposition consists of noting that $i^*i_!$ is equivalent to identity functor on $Fun(\mathcal{A}, \mathcal{C})$ and that there is a particularly nice candidate for the unit transformation

$$\eta : id \Rightarrow i^*i_!,$$

namely the identity transformation.

We unpack the claim that $i^*i_!$ is equivalent to the identity functor here.

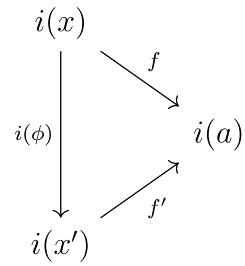
Proposition D.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category admitting sifted colimits; let $i : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor of symmetric monoidal ∞-categories. Consider the restriction functor $i^* : \mathrm{Fun}^{\otimes}(\mathcal{B}, \mathcal{C}) \to \mathrm{Fun}^{\otimes}(\mathcal{A}, \mathcal{C})$ and left Kan extension functor $i_! : \mathrm{Fun}^{\otimes}(\mathcal{A}, \mathcal{C}) \to \mathrm{Fun}^{\otimes}(\mathcal{B}, \mathcal{C})$. Then*

$$i^*i_! \cong \mathbb{1}_{Fun(\mathcal{A},\mathcal{C})}.$$

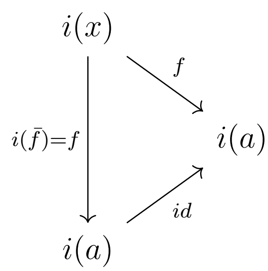
Proof. Recall that showing there is a natural isomorphism $i^*i_! \cong \mathbb{1}_{Fun^{\otimes}(\mathcal{A},\mathcal{C})}$ amounts to showing that for every $F \in Fun(\mathcal{A}, \mathcal{C})$ and for every $a \in \mathcal{A}$, $\mathcal{F}(a) \simeq i^*i_!F(a)$. Note that on the left hand side, the left Kan extension formula gives us

$$\begin{aligned} i^*i_!F &= i_!F(i(a)) \\ &= colim(\mathcal{A}_{/i(a)} \to \mathcal{A} \xrightarrow{F} \mathcal{C}) \end{aligned}$$

Note that the category $\mathcal{A}_{/i(a)}$ consists of objects $x \in \mathcal{A}$ together with a map in $\mathcal{B}$, $f : i(x) \to i(a)$. A morphism between (x, f) and (x', f') consists of a morphism in $\mathcal{A}$, $\phi : x \to x'$, such that



The object $(a, id_{i(a)})$ is a terminal object in $\mathcal{A}_{/i(a)}$. To see this, consider any (x, f) in $\mathcal{A}_{/i(a)}$. Because i is fully faithful, $f \in Hom_{\mathcal{B}}(i(x), i(a))$ corresponds to an $\bar{f} \in Hom_{\mathcal{A}}(x, a)$. Take $\phi := \bar{f}$; by construction we have



,

thus giving that $i^* i_! \cong \mathbb{1}_{Fun(\mathcal{A},\mathcal{C})}$ are naturally isomorphic.

□

D.3 Operadic Lan in the case of symmetric monoidal ∞-categories

When $\mathcal{A}, \mathcal{B}$ are symmetric monoidal ∞-categories, we can ask whether $i : \mathcal{A} \to \mathcal{B}$ gives an adjunction on the categories of symmetric monoidal functors. [1] give the conditions for this to hold, as well as an explicit description of the operadic left Kan extension, in the following result:

Lemma D.7 ([1], Lemma 2.16)**.** *Suppose $\mathcal{C}$ is a symmetric monoidal ∞-category. Let $i : \mathcal{A} \to \mathcal{B}$ be a symmetric monoidal functor of symmetric monoidal ∞-categories where $\mathcal{A}$ is small and $\mathcal{B}$ is locally small. Consider the commutative diagram of solid arrows:*

$$\begin{array}{ccc} & \overset{i_!^{\otimes}}{\dashrightarrow} & \\ Fun^{\otimes}(\mathcal{A},\mathcal{C}) & \underset{i^*}{\longleftarrow} & Fun^{\otimes}(\mathcal{B},\mathcal{C}) \\ \downarrow & & \downarrow \\ & \overset{i_!}{\dashrightarrow} & \\ Fun(\mathcal{A},\mathcal{C}) & \overset{i^*}{\longleftarrow} & Fun(\mathcal{B},\mathcal{C}) \end{array}$$

where the vertical arrows forget that a given functor was symmetric monoidal (i.e. forget that it preserves coCartesian morphisms).

If

(1) the underlying ∞-category of $\mathcal{C}$ admits sifted colimits;

(2) for each $b \in \mathcal{B}$, the slice ∞-category $\mathcal{A}/b$ is sifted

Then i^* has a left adjoint $i_!$, which can be calculated as

$$i_!F : b \mapsto colim\Big(\mathcal{A}/b \to \mathcal{A} \xrightarrow{F} \mathcal{C}\Big) \simeq i^*b \otimes_{\mathcal{A}} F.$$

If additionally

(3) the symmetric monoidal structure for $\mathcal{C}$ distributes over sifted colimits;

(4) the functor $\mathcal{A}/(\mathbb{1}_{\mathcal{A}}) \to \mathcal{A}/(\mathbb{1}_{\mathcal{B}})$ is final; and

(5) for each pair $b, b' \in \mathcal{B}$, the tensor product functor

$$\bigotimes : \mathcal{A}/b \times \mathcal{A}/b' \to \mathcal{A}/(b \otimes b')$$

is final,

then there is a left adjoint $i_!^{\otimes}$ and the 'downward right square commutes'.

If i is fully faithful then so are $i_!$ and $i_!^{\otimes}$.

We apply this lemma to $\mathcal{A} = Env(Open^G(M)), \mathcal{B} = \mathcal{G}Man$ and the inclusion functor $i : \mathcal{A} \to \mathcal{B}$. It gives us an adjunction between multiplicative pre-factorization algebras on $Env(Open_G(M))$ and $\mathcal{G}Man$, which we then show lifts to an equivalence of $(\infty, 1)$-categories when we impose the descent condition (in chapter 5).

Corollary D.8. *Applying the above to $i : Env((Open^G(M)^{\otimes}) \to \mathcal{G}Man$, one gets an adjunction:*

$$\mathrm{Fun}^{\otimes}(Env(Open^G(M)^{\otimes}), Ch) \quad \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \quad \mathrm{Fun}^{\otimes}(\mathcal{G}Man, Ch)$$

Proof. Let $\mathcal{A} = Env(Open^G(M)), \mathcal{B} = \mathcal{G}Man$ and i be the inclusion functor $i : \mathcal{A} \to \mathcal{B}$. Conditions (1) and (3) follow from the fact that $(Ch_k, \otimes)$ is $\otimes$-sifted cocomplete (see [1], Defn 1.15, Ex 1.18).

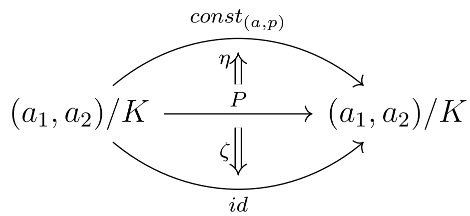
For condition (2), $\mathcal{A}/b$ is sifted if it is non-empty and the diagonal functor $K : \mathcal{A}/b \to \mathcal{A}/b \times \mathcal{A}/b$ is final. For every $b \in \mathcal{B}$, $\mathcal{A}/b$ is non-empty because each $\mathcal{G}$-manifold has a covering by elements of $\mathcal{A}$. To see that the diagonal functor is final, take an arbitrary $((a_1 \xrightarrow{p_1} b), (a_2 \xrightarrow{p_2} b)) \in \mathcal{A}/b \times \mathcal{A}/b$ and consider the slice category $(a_1, a_2)/K$. Objects of this category consist of $(a \xrightarrow{p} b) \in \mathcal{A}/b$ equipped with the following maps in $\mathcal{A}/b$:

$$\begin{array}{ccccc} a_1 & \xrightarrow{\phi_1} & a & \xleftarrow{\phi_2} & a_2 \\ & {\scriptstyle p_1}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle p_2} & \\ & & b & & \end{array}$$

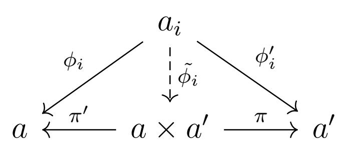
Morphisms of this category consist of $f : (a, p) \to (a', p')$, making the following diagrams commute:

$$\begin{array}{ccc} a & & \\ & \searrow^{p} & \\ {\scriptstyle f}\downarrow & & b \\ & \nearrow_{p'} & \\ a' & & \end{array} \qquad\qquad \begin{array}{ccccc} & & a & & \\ & {\scriptstyle \phi_1}\nearrow & & \nwarrow{\scriptstyle \phi_2} & \\ a_1 & & \downarrow{\scriptstyle f} & & a_2 \\ & {\scriptstyle \phi_1'}\searrow & & \swarrow{\scriptstyle \phi_2'} & \\ & & a' & & \end{array}$$

Take an arbitrary object $(a, p) \in (a_1, a_2)/K$. Consider the following endofunctors:



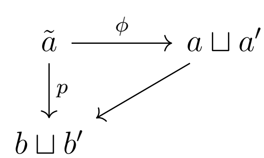
where id is the identity functor; $const_{(a,p)}$ is the constant functor sending all objects to (a, p); and P is the functor that sends (a', p') to the categorical product in $\mathcal{A}/b$, $(a, p) \times (a', p') =: (a \times a', \tilde{p})$, where the maps $\tilde{\phi}_i$ $(i = 1, 2)$ are given by the universal property of the product, as illustrated in the diagram below (where all objects and morphisms should be considered in $\mathcal{A}/b$):



There are natural transformations $\eta : P \Rightarrow const_{(a,\phi)}$ and $\zeta : P \Rightarrow id$, given by $\eta_{(a',\phi')} := \pi$ and $\zeta_{(a',\phi')} := \pi'$. Upon taking the nerve, these natural transformations give a zig-zag of homotopies between the identity map and a constant map, giving the contractibility of $N((a_1, a_2)/K)$ and the desired finality of the diagonal functor.

For condition (4), note that $\mathbb{1}_{\mathcal{A}} = \emptyset = \mathbb{1}_{\mathcal{B}}$, and $\mathcal{A}/\emptyset$ only contains the trivial element, $\emptyset$; thus the functor is final trivially.

For condition (5), take an arbitrary object $(\tilde{a} \xrightarrow{p} b \sqcup b') \in \mathcal{A}/(b \otimes b')$. Then consider the undercategory $\tilde{a}/\bigotimes$. An object of this category consists of $a \to b, a' \to b'$, plus a map making the following diagram commute:



Because the maps in $\mathcal{B} = \mathcal{G}Man$ are embeddings, we can decompose $\tilde{a}$ into the components over b, b', respectively: $\tilde{a} = \tilde{a}_b \sqcup \tilde{a}_{b'}$. This gives an initial object for $\tilde{a}/\bigotimes$; for any other object (a, a', ϕ), the map ϕ gives the desired morphism, as illustrated in the diagram below:

$$
\begin{array}{ccccc}
 & & \overset{\phi}{\curvearrowright} & & \\
\tilde{a}_b \sqcup \tilde{a}_{b'} & \xleftarrow{\;id\;} & \tilde{a} & \xrightarrow{\;\phi\;} & a \sqcup a' \\
 & {\scriptstyle p}\searrow & \downarrow{\scriptstyle p} & \swarrow & \\
 & & b \sqcup b' & &
\end{array}
$$

□

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