Homework Assignment # 8, due Nov. 12

1. (10 points) Let M, N be smooth manifolds, and let $\pi_1\colon M\times N\to M$ and $\pi_2\colon M\times N\to N$ be the projection maps. Show that for any $(x,y)\in M\times N$ the map

$$\alpha\colon T_{(x,y)}(M\times N)\longrightarrow T_xM\oplus T_yN$$

defined by

$$\alpha(v)=((\pi_1)_*(v),(\pi_2)_*(v))$$

is an isomorphism. Hint: To prove this, it is unnecessary to "unpack" the definition of the tangent space of manifolds by using either the geometric or algebraic definition. Rather, only the functorial properties of the tangent space, i.e., the chain rule, is needed, applied to suitable projection/inclusion maps. Remark: Using this isomorphism, we will routinely identify T_xM and T_yN with subspaces of $T_{(x,y)}(M\times N)$.

2. (10 points) Let M be a smooth n manifold. For a point $p\in M$ let

$$DD^M\colon T_p^{\mathrm{geo}}M\ \longrightarrow\ T_p^{\mathrm{alg}}M=\mathrm{Der}(C_p^\infty(M),\mathbb{R})$$

be the map that sends $[\gamma]\in T_p^{\mathrm{geo}}M$ to the derivation DD_γ. More explicitly, if f is (the germ of) a function $f\colon M\to\mathbb{R}$ then $DD_\gamma f\in\mathbb{R}$ is the directional derivative of f in the direction of γ defined by

$$DD_\gamma f:=(f\circ\gamma)'(0)=\lim_{t\to 0}\frac{f(\gamma(t))-f(p)}{t}.$$

(a) Show that the geometric and the algebraic definition of the differential of a smooth map $F\colon M\to N$ are compatible in the sense that for $p\in M$ the following diagram is commutative:

$$\begin{array}{ccc}
T_p^{\mathrm{geo}}M & \xrightarrow{F_*^{\mathrm{geo}}} & T_{F(p)}^{\mathrm{geo}}N\\
{\scriptstyle DD^M}\big\downarrow & & \big\downarrow{\scriptstyle DD^N}\\
T_p^{\mathrm{alg}}M & \xrightarrow{F_*^{\mathrm{alg}}} & T_{F(p)}^{\mathrm{alg}}N
\end{array}$$

(b) Show that the map DD^M is a bijection for any manifold M. Hint: use a chart for M and part (a) to reduce to the case of open subsets $U\subset\mathbb{R}^n$; we have proved in class that the map $DD^U\colon T_p^{\mathrm{geo}}U\to T_p^{\mathrm{alg}}U$ is a bijection in that case.

3. (10 points) Let M, N be smooth manifolds, let $F\colon M\to N$ be a smooth map, and $p\in M$.

(a) Show that the map

$$F_p^*\colon C_{F(p)}^\infty(N)\longrightarrow C_p^\infty(M)\qquad\text{given by}\qquad [f\colon V\to\mathbb{R}]\mapsto[F^{-1}(V)\xrightarrow{F_|}V\xrightarrow{f}\mathbb{R}]$$

is well-defined (here $F_|$ denotes the restriction of F to $F^{-1}(V)\subset M$).

(b) Show that if $D\colon C_p^\infty(M) \to \mathbb{R}$ is a derivation, then the composition

$$C^\infty_{F(p)}(N) \xrightarrow{F_p^*} C_p^\infty(M) \xrightarrow{D} \mathbb{R}$$

is a derivation. In particular, we can define the (algebraic) differential

$$F_*^{\mathrm{alg}}\colon T_p^{\mathrm{alg}} M = \mathrm{Der}(C_p^\infty(M), \mathbb{R}) \longrightarrow T_{F(p)}^{\mathrm{alg}} N = \mathrm{Der}(C^\infty_{F(p)}(M), \mathbb{R})$$

by $F_*^{\mathrm{alg}}(D) := D \circ F_p^*$.

(c) If $G\colon N \to Q$ is a smooth map, show that

$$(G \circ F)_*^{\mathrm{alg}} = G_*^{\mathrm{alg}} \circ F_*^{\mathrm{alg}}.$$

We note that this statement is the *chain rule* (for the algebraic construction of the tangent space).

4. (10 points) Let $\mathbb{RP}^n = S^n/x \sim -x$ be the real projective space and let $h\colon \mathbb{RP}^n \to \mathbb{R}$ be the function defined by

$$h([x_0, \dots, x_n]) = \sum_{\ell=0}^n \ell x_\ell^2.$$

(a) Show that h is a well-defined smooth function.

(b) Determine the *critical points* of h, i.e., the points $p \in \mathbb{RP}^n$ where the differential $h_*\colon T_p\mathbb{RP}^n \to T_{h(p)}\mathbb{R} = \mathbb{R}$ vanishes.

Hint: Use the smooth atlas consisting of the charts $\mathbb{RP}^n \supset U_k \xrightarrow{\phi_k} B_1^n$ with

$$U_k = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_k \neq 0\} \quad \text{and} \quad \phi_k^{-1}(v_1, \dots, v_n) = [v_1, \dots, v_k, \sqrt{1 - ||v||^2}, v_{k+1}, \dots, v_n].$$

5. (10 points) Let $M_{n\times k}(\mathbb{R})$ be the vector space of $n \times k$-matrices. For $A \in M_{n\times k}(\mathbb{R})$ let $A^t \in M_{k\times n}(\mathbb{R})$ be the transpose of A, and let $\mathsf{Sym}(\mathbb{R}^k) = \{B \in M_{k\times k}(\mathbb{R}) \mid B^t = B\}$ be the vector space of *symmetric* $k \times k$*-matrices.*

(a) Show that the map $\Phi\colon M_{n\times k}(\mathbb{R}) \to \mathsf{Sym}(\mathbb{R}^k)$, $A \mapsto A^t A$ is smooth, and that its differential

$$\Phi_*\colon T_A M_{n\times k}(\mathbb{R}) = M_{n\times k}(\mathbb{R}) \longrightarrow T_{\Phi(A)}\mathsf{Sym}(\mathbb{R}^k) = \mathsf{Sym}(\mathbb{R}^k)$$

is given by $\Phi_*(C) = C^t A + A^t C$. Hint: Use the geometric description of tangent spaces. More explicitly, the tangent space $T_A M_{n\times k}(\mathbb{R})$ can be identified with $M_{n\times k}(\mathbb{R})$ by sending a matrix $C \in M_{n\times k}(\mathbb{R})$ to the path $\gamma(t) := A + tC$.

(b) Show that the identity matrix is a regular value of the map Φ. This implies in particular that the level set Φ^{-1}(identity matrix) is a smooth manifold. We recall that we showed in class that Φ^{-1}(identity matrix) is the Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$. Hint: to show that $\Phi_*\colon T_A M_{n\times n}(\mathbb{R}) \to T_e\mathsf{Sym}(\mathbb{R}^k)$ is surjective for $e =$ identity matrix, compute $\Phi_*(C)$ for $C = AB$ for $B \in \mathsf{Sym}(\mathbb{R}^k)$.

(c) What is the dimension of $V_k(\mathbb{R}^n)$?

We remark that identifying $M_{n\times k}(\mathbb{R})$ in the usual way with the vector space $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n)$ of linear maps $f\colon \mathbb{R}^k \to \mathbb{R}^n$, a matrix belongs to $V_k(\mathbb{R}^n)$ if and only if the corresponding linear map f is an *isometry*, that is, if f preserves the length of vectors in the sense that $||f(v)|| = ||v||$, or equivalently, if f preserves the scalar product in the sense that

$$\langle f(v), f(w)\rangle = \langle v, w\rangle \qquad \text{for all } v, w \in \mathbb{R}^k.$$

The manifold $V_k(\mathbb{R}^n)$ is called the *Stiefel manifold*. We observe that $V_n(\mathbb{R}^n)$ is the orthogonal group $O(n)$ of isometries $\mathbb{R}^n \to \mathbb{R}^n$.