Homework Assignment # 8, due Nov. 5

1. (10 points) Let $\mathbb{CP}^n$ be the complex projective space, described as

$$\mathbb{CP}^n := \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/z \sim \lambda z \quad \text{for } z \in \mathbb{C}^{n+1} \setminus \{0\},\ \lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

Let $U_i := \{[z_0, z_1, \dots, z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$ for $i = 0, \dots, n$ and let

$$\phi_i \colon U_i \longrightarrow \mathbb{C}^n \qquad \text{be defined by} \qquad \phi_i([z_0, z_1, \dots, z_n]) = \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\right).$$

(a) Show that (U_i, ϕ_i) is a chart for $\mathbb{CP}^n$.

(b) Show that $\{(U_i, \phi_i) \mid i = 0, \dots, n\}$ is a smooth atlas for $\mathbb{CP}^n$.

2. (10 points) Show that the Cartesian product of $M \times N$ of smooth manifolds of dimension m resp. n is a smooth manifold of dimension $m + n$.

3. (10 points) We recall that a smooth structure on a topological manifold M is a maximal smooth atlas $\mathcal{A}_M$ on M. A smooth structure on M can be restricted to a smooth structure on an open subset $W \subset M$ given by the maximal smooth atlas $\mathcal{A}_W$ consisting of those charts $(U, \phi) \in \mathcal{A}_M$ with $U \subset W$.

(a) Let M be a topological manifold, and let U, V be open subsets of M with $U \cup V = M$. Show that smooth structures $\mathcal{A}_U$ on U and $\mathcal{A}_V$ on V which agree on $U \cap V$ determine a unique smooth structure $\mathcal{A}_M$ on M which restricts to the given smooth structure on U resp. V.

(b) Let $f_1 \colon A \to M_1$ and $f_2 \colon A \to M_2$ be continuous maps. We recall that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f_1} & M_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow \\ M_2 & \longrightarrow & M_1 \cup_A M_2 \end{array}$$

is a pushout diagram, where the space $M_1 \cup_A M_2$ is the quotient of the disjoint union $M_1 \amalg M_2$ modulo the equivalence relation that identifies for each point $a \in A$ the elements $f_1(a)$ and $f_2(a)$. Show that $M_1 \cup_A M_2$ is a smooth n-manifold provided A, M_1 and M_2 are smooth n-manifolds and f_1, f_2 are open maps which are diffeomorphisms onto their images (don't bother with proving that $M_1 \cup_A M_2$ is Hausdorff and second countable).

(c) Show that the connected sum $M \# N$ of two smooth connected n-manifolds is a smooth n-manifold. Hint: Use the description of $M \# N$ from problem (1) of the homework assignment # 6 as the pushout $(M \setminus \{x_0\}) \cup_{S^{n-1} \times (-1,1)} (N \setminus \{y_0\})$.

4. (10 points) For any topological space M, let $C(M)$ denote the vector space of continuous functions $f\colon M \to \mathbb{R}$. If M is equipped with the structure of a smooth n-manifold, let $C^\infty(M) \subset C(M)$ be the subspace of smooth functions. If $F\colon M \to N$ is a continuous map, define $F^*\colon C(N) \to C(M)$ by $F^*(f) := f \circ F$.

(a) Show that F^* is linear.

(b) If M and N are smooth manifolds, show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) If $F\colon M \to N$ is a homeomorphism between smooth manifolds, show that it is a diffeomorphism if and only if $F^*\colon C^\infty(N) \to C^\infty(M)$ is an isomorphism.

Thus in a certain sense the entire smooth structure of M is encoded in $C^\infty(M)$.

5. (10 points) We recall that for an open subset $U \subset \mathbb{R}^n$ and $p \in U$ the map

$$T_p^{\mathrm{geo}}U \longrightarrow \mathbb{R}^n \qquad \text{given by} \qquad [\gamma] \mapsto \gamma'(0)$$

is a bijection. Show that for a smooth map $\mathbb{R}^m \underset{\text{open}}{\supset} U \xrightarrow{F} V \underset{\text{open}}{\subset} \mathbb{R}^n$ and $p \in U$, the diagram

$$\begin{array}{ccc} T_p^{\mathrm{geo}}U & \xrightarrow{F_*^{\mathrm{geo}}} & T_{F(p)}^{\mathrm{geo}}V \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{R}^m & \xrightarrow{dF(p)} & \mathbb{R}^n \end{array}$$

is commutative. We note that this expresses the compatibility of $dF(p)$ (the traditional calculus definition of the derivative of the map F) and F_*^{geo} (the new definition of the derivative, which generalizes to manifolds).