Homework Assignment # 7, due Oct. 29

The following two problems give some insight into the universal covering space of a nice space X. The first problem identifies some evenly covered subsets U of X, and describes the path-components of the preimage of U in terms of paths in X. The second problem constructs the universal covering space, basically by "reverse engineering".

1. (10 points) Let X be a nice space, i.e., X is path-connected, locally path-connected, and semi-locally simply connected. Let $q\colon (E, e_0) \to (X, x_0)$ be a universal covering, i.e., the total space E is simply connected.

(a) Let $\widetilde{X}^u := \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$, where $[\gamma]$ denotes the homotopy class of γ relative endpoints (this is a *set* in this problem; in the next problem we will construct a topology on this set). Let $p\colon \widetilde{X}^u \to X$ be the map defined by $p([\gamma]) = \gamma(1) \in X$. Let

$$\Phi\colon \widetilde{X}^u \longrightarrow E \qquad \text{be the map defined by} \qquad [\gamma] \mapsto \widetilde{\gamma}(1),$$

where $\widetilde{\gamma}\colon I \to E$ is the unique lift of γ starting at the basepoint e_0. Show that Φ is a well-defined bijection that makes the diagram

$$\begin{array}{ccc} \widetilde{X}^u & \xrightarrow{\Phi} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

commutative.

(b) Let $U \subset X$ be an open path-connected subset such that the induced map $\pi_1(U) \to \pi_1(X)$ is trivial. Show that U is evenly covered for the covering map $q\colon E \to X$, and that the path-connected component of $q^{-1}(U)$ containing the point $\Phi([\gamma])$ is of the form $\Phi(U_{[\gamma]})$, where

$$U_{[\gamma]} = \{[\gamma * \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}.$$

(c) Let $\mathcal{U}$ be the collection of open path-connected subsets $U \subset X$ such that the induced map $\pi_1(U) \to \pi_1(X)$ is trivial. Show that the subsets $\Phi(U_{[\gamma]}) \subset E$ generate the given topology on E.

The following homework problem constructs a topology on the set $\widetilde{X}^u$, basically by reverse engineering from the previous problem.

2. (10 points) Let X be a nice space with basepoint x_0 and let $p\colon \widetilde{X}^u \to X$ be as in the previous problem.

(a) Show that the collection of subsets $U_{[\gamma]} \subset \widetilde{X}^u$ for $U \in \mathcal{U}$ and γ a path in X from x_0 to a point in U form the basis of a topology for the set $\widetilde{X}^u$.

(b) Show that $p\colon \widetilde{X}^u \to X$ is a covering map.

(c) Show that $\widetilde{X}^u$ is simply connected. Hint: to show that $\pi_1(\widetilde{X}, [x_0])$ is trivial, it suffices to that its image under p_* is trivial.

3. (10 points) Let X be a nice space, and let $p\colon \widetilde{X}^u \to X$ be the universal covering associated to a basepoint $x_0 \in X$. We recall that the fundamental group $G := \pi_1(X, x_0)$ acts on $\widetilde{X}^u$ by covering transformations

$$G \times \widetilde{X}^u \longrightarrow \widetilde{X}^u \qquad ([\alpha], [\gamma]) \mapsto [\alpha * \gamma].$$

(a) Let H be a subgroup of G, and let $H\backslash\widetilde{X}^u$ be the orbit space of the H-action on $\widetilde{X}^u$ obtained by restricting the G-action to the subgroup H. Show that the natural projection map $q\colon H\backslash\widetilde{X}^u \to X$ is a covering map.

(b) Show that the image of the fundamental group of $H\backslash\widetilde{X}^u$ under q_* is the subgroup $H < G$.

4. (10 points) For $i = 0, 1, 2, 3$, let $\phi_i\colon (-1, +1) \to \mathbb{R}$ be the following maps

$$\phi_0(x) = x \qquad \phi_1(x) = \begin{cases} x & x \le 0 \\ 2x & x \ge 0 \end{cases} \qquad \phi_2(x) = x^3 \qquad \phi_3(x) = \tan\frac{\pi x}{2}$$

All of these maps are homeomorphisms between $M := (-1, +1)$ and an open subset of $\mathbb{R}$, allowing us to interpret (M, ϕ_i) as a *chart* for M. For $S \subset \{0, 1, 2, 3\}$ let $\mathcal{A}_S = \{(M, \phi_i) \mid i \in S\}$ be the atlas consisting of the charts (M, ϕ_i) with $i \in S$.

(a) For which subsets $S \subset \{0, 1, 2, 3\}$ is $\mathcal{A}_S$ a smooth atlas of M?

(b) Assuming that $\mathcal{A}_S$ is a smooth atlas, let M_S be the smooth manifold given by the topological space M equipped with the smooth structure determined by $\mathcal{A}_S$. For those subsets S determined in part (a), which of the manifolds M_S are diffeomorphic? Hint: If M is a smooth n manifold, and (U, ϕ) is a chart belonging to the maximal smooth atlas of M, then ϕ is a diffeomorphism from U to $\phi(U) \subset \mathbb{R}^n$.

5. We recall that the stereographic projection provides a homeomorphism between the open subsets $U_\pm := S^n \setminus \{(\mp 1, 0, \dots, 0)\}$ of S^n and $\mathbb{R}^n$. More explicitly, the stereographic projection is the map

$$\psi_\pm\colon U_\pm \longrightarrow \mathbb{R}^n \qquad \text{is defined by} \qquad \psi_\pm(x_0, \dots, x_n) := \frac{1}{1 \pm x_0}(x_1, \dots, x_n),$$

and its inverse $\psi_\pm^{-1}\colon \mathbb{R}^n \to U_\pm$ is given by the formula

$$\psi_\pm^{-1}(y_1, \dots, y_n) = \frac{1}{||y||^2 + 1}(\pm(1 - ||y||^2), 2y_1, \dots, 2y_n).$$

In particular, the two charts (U_+, ψ_+), (U_-, ψ_+) form an atlas for S^n.

(a) Show that $\{(U_+, \psi_+), (U_-, \psi_-)\}$ is a *smooth* atlas for S^n.

(b) Show that the atlas above is smoothly compatible with the smooth atlas we've discussed in class, consisting of the charts $(U_{i,\epsilon}, \phi_{i,\epsilon})$, $\epsilon \in \{\pm 1\}$, where $U_{i,\epsilon} \subset S^n$ consists of the points $(x_0, \ldots, x_n) \in S^n$ such that $\epsilon x_i > 0$, and

$$\phi_{i,\epsilon} \colon U_{i,\epsilon} \xrightarrow{\approx} B_1^n \qquad \text{is given by} \qquad \phi_{i,\epsilon}(x_0, \ldots, x_n) = (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$$

with inverse given by $\phi_{i,\epsilon}^{-1}(y_1, \ldots, y_n) = (y_1, \ldots, y_i, \epsilon\sqrt{1 - ||y||^2}, y_{i+1}, \ldots, y_n)$.

(c) Let $V \subset \mathbb{R}^{n+1}$ be an open subset containing S^n and let $f \colon V \to \mathbb{R}$ be a smooth map. Show that the restriction of f to S^n, i.e., the composition $S^n \hookrightarrow V \xrightarrow{f} \mathbb{R}$, is a smooth function on the smooth manifold S^n (equipped by the "standard smooth structure" on S^n given by either of the two smooth atlases described above).