Homework Assignment # 6, due Oct. 8

1. (10 points) Let M, N be path-connected manifolds of dimension $n \geq 3$. The goal of this problem is to compute the fundamental group of their connected sum $M\#N$ in terms of the fundamental groups of M and N. We provide an alternative description of the connected sum $M\#N$, which is easier for the problem at hand, works for smooth manifolds, and uses pushout diagrams (it is not hard to show that this version of $M\#N$ is homeomorphic to the version presented in class).

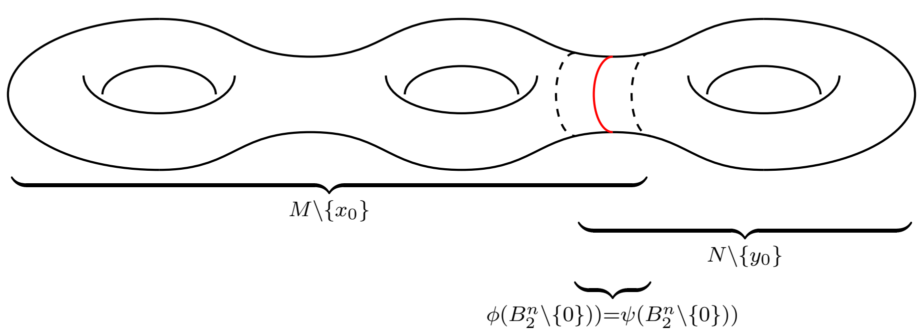
For the construction of the connected sum we pick points $x_0 \in M$, $y_0 \in N$ and maps $\phi\colon B^n_2 \to M$, $\psi\colon B^n_2 \to N$ which are are homeomorphisms onto their image with $\phi(0) = x_0$, $\psi(0) = y_0$; here $B^n_2 = \{v \in \mathbb{R}^n \mid ||v|| < 2\} \subset \mathbb{R}^n$ is the open ball of radius 2. Let α be the homeomorphism

$$\alpha\colon S^{n-1} \times (-1,1) \xrightarrow{\approx} B^n_2 \setminus \{0\} \qquad \text{given by} \qquad (v,t) \mapsto (1-t)v,$$

and let $g\colon S^{n-1} \times (-1,1) \xrightarrow{\approx} S^{n-1} \times (-1,1)$ be the homeomorphism given by $g(v,t) = g(v,-t)$. Let $M\#N$ be the space determined by the pushout diagram

$$\begin{array}{ccc} S^{n-1} \times (-1,1) & \xrightarrow{\psi\circ\alpha} & N \setminus \{y_0\} \\ {\scriptstyle \phi\circ\alpha\circ g}\downarrow & & \downarrow \\ M \setminus \{x_0\} & \longrightarrow & M\#N. \end{array}$$

In other words, $M\#N = (M \setminus \{x_0\}) \cup_{S^{n-1}\times(-1,1)} (N \setminus \{y_0\})$ is obtained from the disjoint union $(M \setminus \{x_0\}) \amalg (N \setminus \{y_0\})$ by identifying the point $\phi \circ \alpha \circ g(v,t) \in M \setminus \{x_0\}$ with the point $\psi \circ \alpha(v,t) \in N \setminus \{y_0\}$ for $(v,t) \in S^{n-1} \times (-1,1)$. Here is a picture of $M\#N$, where the red circle is the image of $S^{n-1} \times \{0\} \subset S^{n-1} \times (-1,1)$ under either map in the commutative diagram above.



(a) Determine the fundamental group of $M \setminus \{x_0\}$ in terms of the fundamental group of M. Hint: use the Seifert van Kampen Theorem.

(b) Determine the fundamental group of $M\#N$ in terms of the fundamental groups of M and N.

2. (10 points) Let X be the subspace of $\mathbb{R}^3$ given by the union of the 2-sphere S^2 and the segment S of the x-axis given by $S = \{(t,0,0) \in \mathbb{R}^3 \mid -1 \leq t \leq 1\}$. Calculate the fundamental group of X. Hint: use the Seifert van Kampen Theorem.

3. (10 points)

(a) Show that

$$\pi_1(\underbrace{T\#\ldots\#T}_{g}) \cong \langle a_1,\ldots,a_g,b_1,\ldots,b_g \mid \prod_{i=1}^{g}[a_i,b_i]\rangle,$$

where $[a_i,b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator.

(b) Show that

$$\pi_1(\underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{k}) \cong \langle a_1,\ldots,a_k \mid a_1a_1a_2a_2\ldots a_ka_k\rangle.$$

4. (10 points) Let G be a group. The *abelianization of* G, is the abelian group G^{ab} obtained as the quotient of G modulo the commutator subgroup $[G,G]$, the normal subgroup generated by all commutators $[g,h] := ghg^{-1}h^{-1}$ for $g,h \in G$.

(a) Show that the abelianization of the free group $\langle S\rangle$ generated by a set $S = \{s_1,\ldots,s_k\}$ is the free abelian group

$$\mathbb{Z}[S] := \mathbb{Z}s_1 \oplus \cdots \oplus \mathbb{Z}s_k$$

whose elements are the linear combinations $\sum_{i=1}^k n_i s_i$ of the elements of S with coefficients $n_i \in \mathbb{Z}$ (the group structure is given by the evident sum of such linear combinations). More precisely, show that an isomorphism

$$\Psi\colon \langle S\rangle^{\mathrm{ab}} \longrightarrow \mathbb{Z}[S]$$

is given by sending a word W in the letters s_i, s_i^{-1} to the linear combination $\sum_{i=1}^k n_i s_i$, where

$$n_i = \#\{\text{occurrences of } s_i \text{ in } W\} \quad - \quad \#\{\text{occurrences of } s_i^{-1} \text{ in } W\}.$$

(b) Let $R_1,\ldots,R_\ell$ be elements of the free group $\langle S\rangle$, and let $\langle S \mid R_1,\ldots,R_\ell\rangle$ be the quotient group of $\langle S\rangle$ modulo the normal subgroup generated by the elements $R_1,\ldots,R_\ell$. Show that the there is an isomorphism

$$\langle S \mid R_1,\ldots,R_\ell\rangle^{\mathrm{ab}} \cong \mathbb{Z}[S]/(\Psi\circ p(R_1),\ldots,\Psi\circ p(R_\ell)).$$

Here $p\colon \langle S\rangle \to \langle S\rangle^{\mathrm{ab}}$ is the projection map and $(\Psi\circ p(R_1),\ldots,\Psi\circ p(R_\ell)) \subset \mathbb{Z}[S]$ is the subgroup generated by $\Psi\circ p(R_1),\ldots,\Psi\circ p(R_\ell)$.

(c) Show that $\pi_1(\Sigma_g)^{\text{ab}} \cong \mathbb{Z}^{2g}$.

(d) Show that $\pi_1(\underbrace{\mathbb{RP}^2\#\dots\#\mathbb{RP}^2}_{k})^{\text{ab}} \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2$. Hint: By problem 3(b), this fundamental group is generated by $a_1, \dots, a_k$. For the free abelian group $\mathbb{Z}a_1 \oplus \dots \oplus \mathbb{Z}a_k$, also known as free $\mathbb{Z}$-module, it will be convenient to use the basis $a_1, \dots, a_{k-1}, c$, where $c = a_1 + \dots + a_k$.

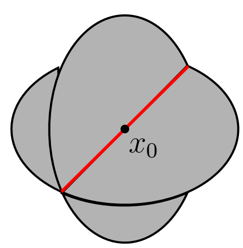
Remark. In general it is very difficult to determine whether two groups G, G' are isomorphic. By contrast, this is easy to determine for finitely generated *abelian* groups, since by the *Fundamental Theorem of finitely generated abelian groups* such a group G is isomorphic to the direct product of the infinite cyclic group $\mathbb{Z}$ and finite cyclic groups $\mathbb{Z}/q = \mathbb{Z}/q\mathbb{Z}$ whose order q is a prime power. Moreover, two finitely generated abelian groups are isomorphic if and only if their direct sum decomposition contains the same number of summands of order q for any prime power q and $q = \infty$. Hence the simplest way to show that two groups (e.g., the fundamental groups of topological spaces X, X') are *not* isomorphic, is to show that their abelianizations are not isomorphic.

5. (10 points)

(a) Let $B^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid ||x|| < 1,\ x_1 \geq 0\}$ be the open half ball, which includes the boundary edge $E = \{(x_1, 0) \in \mathbb{R}^2 \mid -1 < x_1 < 1\}$. For $k \geq 1$, let

$$X_k := (\underbrace{B^2_+ \amalg \dots \amalg B^2_+}_{k})/\sim$$

the quotient of k disjoint copies of the half ball B^2_+ obtained by identifying all the boundary edges with each other. Below is a picture of X_k for $k = 4$, with the red line given by the equivalence classes of the edges of each half ball, and x_0 the equivalence class of the midpoints of these lines.

$X_4 =$ 

Show that X_k is not homeomorphic to the open ball B^2 unless $k = 2$. Hint: If $f\colon X_k \to B^2$ were a homeomorphism, it would restrict to a homeomorphism

$$X_k \setminus \{x_0\} \xrightarrow{\approx} B^2 \setminus \{f(x_0)\},$$

where $x_0 \in X_k$ is represented by the midpoint of the edge. Show that this is impossible for $k \neq 2$.

(b) Show that X_k is not locally Euclidean at the point x_0 for $k \neq 2$. Hint: If X_k were locally Euclidean at the point x_0, there would be a homeomorphism from B^2 to an open neighborhood U of $x_0 \in X_k$, and hence a homeomorphism $\phi\colon B^2 \setminus \{0\} \to U \setminus \{x_0\}$. Verify that this is impossible by showing that $\phi(U \setminus \{x_0\}) \subset X_k$ contains a subspace X_k' which is a deformation retract of X_k and contemplating the inclusion maps

$$X_k' \hookrightarrow \phi(U \setminus \{x_0\}) \hookrightarrow X_k$$

and their induced homomorphisms on π_1.

(c) Let $\Sigma(W) = P_n/\sim_W$ be the quotient space of a polygon P_n determined by the edge identification determined by an n-letter word $w = a_{i_1}^{\epsilon_1} \dots a_{i_n}^{\epsilon_n}$. Show that $\Sigma(w)$ is not a 2-manifold unless each label a_k occurs exactly twice in the word w (e.g., in the word $w = aba^{-1}b$, the letter a occurs only once, but the label a occurs twice: in the first and third letter of the word w). Hint: Use part (b).