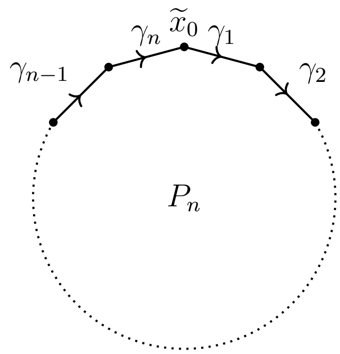
Homework Assignment # 5, due Oct. 1

1. (10 points) A subspace $A \subset X$ of a topological space X is called a *retract of* X if there is a continuous map $r\colon X \to A$ whose restriction to A is the identity.

   1. Show that S^1 is not a retract of D^2. Hint: Show that the assumption that there is a continuous map $r\colon D^2 \to S^1$ which restricts to the identity on S^1 leads to a contradiction by contemplating the induced map r_* of fundamental groups.

   2. Brouwer's Fixed Point Theorem states that every continuous map $f\colon D^n \to D^n$ has a fixed point, i.e., a point x with $f(x) = x$. Prove this for $n = 2$. Hint: show that if f has no fixed point, then a retraction map $r\colon D^2 \to S^1$ can be constructed out of f.

2. (10 points) Let W be an n-letter word with letters from the set $\{a_1, a_1^{-1}, \dots, a_k, a_k^{-1}\}$. Let $\Sigma(W) = P_n/\sim$ be the corresponding quotient space of the regular n-gon P_n (the regular polygon with n edges) obtained by identifying all edges labeled by $a_i^{\pm 1}$ with each other. Let $\gamma_i\colon I \to P_n$ be the straight line path along the edge from the i-th vertex to the $(i+1)$-th vertex, where the vertices are enumerated clockwise, starting with some fixed vertex $\widetilde{x}_0$ as the first vertex as shown in the picture.



Assume that the equivalence relation $\sim_W$ determined by the word W is such that all vertices are identified. In other words, the projection map $p\colon P_n \to \Sigma(W) = P_n/\sim_W$ maps every vertex to the same point $x_0 \in \Sigma(W)$. In particular, the paths γ_i in P_n project to based loops $p \circ \gamma_i$ in $(\Sigma(W), x_0)$. Moreover, for any label a_k that occurs in the word W, edge paths γ_i with that label project to the *same* based loop $\alpha_k := p \circ \gamma_i$ and any edge path γ_j with label a_k^{-1} projects to $\bar{\alpha}_k$. In this way, any label a_k that occurs in W determines a based loop α_k in $(\Sigma(W), x_0)$ and hence an element of the fundamental group $a_k := [\alpha_k] \in \pi_1(\Sigma(W), x_0)$ (called a_k by abuse of language).

Show that these elements of $\pi_1(\Sigma(W), x_0)$ satisfy the relation $W = 1$. Hint: Don't worry, the proof is shorter than the statement of this problem.

3. (10 points) Two topological spaces X, Y are *homotopy equivalent* if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f\colon X \to X$ is homotopic to id_X and $f \circ g\colon Y \to Y$ is homotopic to id_Y. Show that the following five topological spaces are all homotopy equivalent:

1. the circle S^1,
2. the open cylinder $S^1 \times \mathbb{R}$,
3. the annulus $A = \{(x, y) \mid 1 \le x^2 + y^2 \le 2\}$,
4. the solid torus $S^1 \times D^2$,
5. the Möbius strip

Hint: A subspace $A \subset X$ is a *retract of* X if there is map $r\colon X \to A$ which restricts to the identity on A. It is a *deformation retract* of X if in addition the composition $X \xrightarrow{r} A \xhookrightarrow{i} X$ with the inclusion map i is homotopic to the identity on X. Note that if A is a deformation retract of X, then $r \circ i = \mathrm{id}_A$ and $i \circ r \sim \mathrm{id}_X$. In particular, A is homotopy equivalent to X. Show that each of the spaces (2)-(5) contains a subspace S homeomorphic to the circle S^1 which is a deformation retract of the bigger space it is contained in.

4. (10 points) Let G_1 and G_2 be groups. Show that the free product $G_1 * G_2$ is the coproduct of G_1 and G_2 in the category of groups.

5. (10 points) We recall that if $G \times X \to X$ is the action of a group G on a set X, then the subgroup $G_x := \{g \in G \mid gx = x\} \subseteq G$ is the *isotropy subgroup* of the point $x \in X$. The action is called *free* if the isotropy subgroup G_x is the trivial group for all $x \in X$. If X is a topological space, the action is called *continuous* if for every $g \in G$ the map $X \to X$ given by $x \mapsto gx$ is continuous.

(a) Show that if G is a finite group which acts freely and continuously on a Hausdorff space X, then the projection map $p\colon X \to X/G$ to the orbit space X/G is a covering map. Hint: Use the assumptions that the action is free and X is Hausdorff to show that for every $x \in X$ there is an open neighborhood U such that the subsets $gU \subset X$ for $g \in G$ are mutually disjoint.

(b) Show that if X is a manifold of dimension n, then also the orbit space X/G is a manifold of dimension n (in order to make this problem a little shorter, don't worry about proving that X/G is Hausdorff and second countable).

(c) Show that the map $\mathbb{Z}/2 \times S^n \to S^n$ given by $(m, v) \mapsto (-1)^m v$ is a continuous free action. We note that the orbit space $S^n/\mathbb{Z}/2$ is the real projective space $\mathbb{RP}^n$, and hence part (b) of this problem provides a different way to show that $\mathbb{RP}^n$ is a manifold of dimension n.

(d) Show that the map

$$\mathbb{Z}/k \times S^{2n-1} \longrightarrow S^{2n-1} \qquad \text{given by} \qquad (m, v) \mapsto e^{2\pi i m/k} v$$

is a continuous free action of the cyclic group $\mathbb{Z}/k$ on the sphere $S^{2n-1} \subset \mathbb{C}^n$. By part (b) the orbit space $S^{2n-1}/\mathbb{Z}/k$ is then a manifold of dimension $2n-1$, which is known as a *lens space*. Note that for $k = 2$, this is the real projective space $\mathbb{RP}^{2n-1}$.