Homework Assignment # 4, due Sept. 24

1. (10 points) Let $\alpha, \beta, \gamma\colon I \to X$ be paths in a topological space X. Assume that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$ which guarantees that the concatenated paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ can be formed. Show that these two paths are homotopic (relative endpoints). Verifying this shows that if α, β, γ are loops based at $x_0 \in X$ representing elements $a = [\alpha]$, $b = [\beta]$, $c = [\gamma]$ in $\pi_1(X; x_0)$, then $a(bc) = (ab)c$. In other words, this proves associativity of multiplication in $\pi_1(X; x_0)$, one of the last things to verify in order to prove that $\pi_1(X; x_0)$ is indeed a group.

Hint: Show that both paths can be written as the composition $\Psi \circ \phi$ of a suitable map $\phi\colon I \to [0, 3]$ and the map

$$[0,3] \xrightarrow{\Psi} X \qquad \text{defined by} \quad \Psi(s) := \begin{cases} \alpha(s) & 0 \le s \le 1 \\ \beta(s-1) & 1 \le s \le 2 \\ \gamma(s-2) & 2 \le s \le 3 \end{cases}$$

Avoid writing down explicit homotopies; instead use the handy fact that any two paths with the same endpoints in a convex subset of $\mathbb{R}^n$ are homotopic relative endpoints.

2. (10 points) Let X be a topological space and let β be a path from x_0 to x_1. Show that the map

$$\Phi_\beta\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1) \qquad [\gamma] \mapsto [\bar{\beta} * \gamma * \beta]$$

is an isomorphism of groups. In particular, the isomorphism class of the fundamental group $\pi(X, x_0)$ of a path connected space does not depend on the choice of the base point $x_0 \in X$. Hint: Recall from class that for any path γ in X, there are homotopies

$$\gamma * \bar{\gamma} \simeq c_{\gamma(0)} \qquad \bar{\gamma} * \gamma \simeq c_{\gamma(1)} \qquad c_{\gamma(0)} * \gamma \simeq \gamma, \qquad \gamma * c_{\gamma(1)} \simeq \gamma$$

where c_x for $x \in X$ denotes the constant path at x.

3. (10 points) Let (X, x_0), (Y, y_0) be pointed topological spaces. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the Cartesian product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ of the fundamental groups of (X, x_0) and (Y, y_0).

4. (10 points) Let $f\colon S^1 \to S^1$ be defined by $f(z) = z^n$ for some $n \in \mathbb{Z}$. Calculate the induced homomorphism

$$f_*\colon \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1).$$

Hint: We have proved in class that the fundamental group $\pi_1(S^1, 1)$ is isomorphic to $\mathbb{Z}$ by mapping the element $[\gamma] \in \pi_1(S^1, 1)$ represented by a loop $\gamma\colon I \to S^1$ based at $1 \in S^1$ to its winding $W(\gamma) \in \mathbb{Z}$, defined by $W(\gamma) = \widetilde{\gamma}(1)$, where $\widetilde{\gamma}\colon I \to \mathbb{R}$ is the unique lift of γ with $\widetilde{\gamma}(0) = 0$. Via the isomorphism $W\colon \pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ the endomorphism $f_* \in \text{End}(\pi_1(S^1, 1))$ corresponds to an endomorphisms $g \in \text{End}(\mathbb{Z})$. What is the most general endomorphism g?

5. (10 points) Let (X, x_0) be a pointed topological space. Then each based loop $\gamma\colon (I, \partial I) \to (X, x_0)$ yields a well-defined map $\widehat{\gamma}\colon I/\partial I \to X$.

(a) Show that a path homotopy between based loops $\gamma, \gamma'\colon (I, \partial I) \to (X, x_0)$ yields a homotopy between the associated maps $\widehat{\gamma}, \widehat{\gamma}'\colon I/\partial I \to X$.

(b) Let $\Psi\colon \pi_1(X, x_0) \to [I/\partial I, X]$ be the map given by $[\gamma]_{\mathrm{ph}} \mapsto [\widehat{\gamma}]_{\mathrm{h}}$, where we write $[\gamma]_{\mathrm{ph}}$ for the path homotopy class of γ and $[\widehat{\gamma}]_{\mathrm{h}}$ for the homotopy class of $\widehat{\gamma}$ (this is a well-defined map by part (a)). Show that Ψ is surjective if X is path-connected.

(c) Show that Ψ is injective provided the fundamental group $\pi_1(X, x_0)$ is abelian.

Hint: The proof will involve constructing homotopies $H\colon I \times I \to X$. The restriction of H to the boundary of the square is typically given, and the issue is to extend it to the whole square. To do this for the homotopies in parts (b) and (c), it is useful to decompose the square into three regions, construct a suitable map to X on each of them, and show that these maps fit together to a well-defined, continuous map $I \times I \to X$.