Homework Assignment # 3, due Sept. 17, 2021

1. (10 points) The definition of a manifold involves the technical conditions of being Hausdorff and second countable. Show that these properties are "inherited" by subspaces in the following sense. Let X be a topological space and A a subspace.

(a) Show that if X is Hausdorff, then so is A.

(b) Show that if X is second countable, then so is A.

2. (10 points) Let M be a manifold of dimension m and let N be a manifold of dimension n. Show that the product $M \times N$ is a manifold of dimension $m + n$. Don't forget to check the technical conditions (Hausdorff and second countable) for $M \times N$.

3. (10 points) Show that the real projective space $\mathbb{RP}^n$ is a manifold of dimension n. Don't forget to check that $\mathbb{RP}^n$ is second countable (we have proved in class that the projective space is Hausdorff). Hint: to prove that $\mathbb{RP}^n$ is locally homeomorphic to $\mathbb{R}^n$ suitably modify the method we used for the sphere S^n. For showing that $\mathbb{RP}^n$ is second countable, recall that we proved in class that if X is second countable, and $p\colon X \twoheadrightarrow Y$ is an open surjection, then Y is second countable.

4. (10 points) Let Σ, Σ' be compact 2-manifolds. Show that the Euler characteristic of the connected sum $\Sigma \# \Sigma'$ is given by the following formula:

$$\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2.$$

5. (10 points) Show that the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ of two copies of the real projective plane is homeomorphic to the Klein bottle K.