Homework Assignment # 2, due Sept. 10, 2021

1. (10 points) Show that a closed subspace C of a compact topological space X is compact.

2. (10 points) Show that the complex projective space $\mathbb{CP}^1$ is homeomorphic to the 2-sphere S^2. Hint: recall that $\mathbb{CP}^1$ is a quotient of $\mathbb{C}^2 \setminus \{0\}$ and hence a point of $\mathbb{CP}^1$ is an equivalence class $[z_0, z_1]$ of elements $(z_0, z_1) \in \mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{0\})$. How many points of $\mathbb{CP}^1$ are not of the form $[1, z]$?

3. (10 points) Let X be a topological space which is the union of two subspaces X_1 and X_2. Let $f\colon X \to Y$ be a (not necessarily continuous) map whose restriction to X_1 and X_2 is continuous.

(a) Show f is continuous if X_1 and X_2 are open subsets of X.

(b) Show f is continuous if X_1 and X_2 are closed subsets of X.

(c) Give an example showing that in general f is *not continuous*.

4. (10 points) Use the Heine-Borel Theorem to decide which of the topological groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are compact. Provide proofs for your statements. Hint: A strategy often useful for proving that a subset C of $\mathbb{R}^n$ is closed is to show that C is of the form $f^{-1}(C')$ for some closed subset $C' \subset \mathbb{R}^k$ (often C' consists of just one point) and some continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^k$.

5. (10 points) Which of the topological groups $GL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are connected? Hint: To show that one of these topological groups is connected, it might be easier to show that it is path-connected. Note that to prove this, it suffices to find a path connecting any element with the identity element (why?). Use without proof the fact that every element in $SO(n)$ (the group of linear maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ which are isometries with determinant one) for a suitable choice of basis of $\mathbb{R}^n$ is represented by a matrix of block diagonal form whose diagonal blocks are the 1×1 matrix with entry $+1$ and/or 2×2 rotational matrices

$$R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Here "block diagonal" means that all other entries are zero.