Homework Assignment # 11, due Dec. 3

1. (10 points) Let V be a vector space of dimension n, and let $\mathrm{Alt}^k(V,\mathbb{R})$ be the vector space of alternating multilinear maps $\omega\colon \underbrace{V\times\cdots\times V}_{k} \to \mathbb{R}$. Show that $\dim \mathrm{Alt}^k(V,\mathbb{R}) = \binom{n}{k}$.

Hint: Let $\{b_i\}_{i=1,\dots,n}$ be a basis for V, and for a multi-index $I=(i_1,\dots,i_k)$ let $b_I := (b_{i_1},\dots,b_{i_k}) \in V\times\cdots\times V$. Show that an element $\omega\in \mathrm{Alt}^k(V,\mathbb{R})$ is uniquely determined by the numbers $\omega(b_I)\in\mathbb{R}$ as I runs through all length k multi-indices that are strictly increasing, i.e., $i_1<i_2<\cdots<i_k$.

2. (10 points) Let V be a vector space and let $\mathrm{Alt}^k(V,\mathbb{R})$ be the vector space of alternating multilinear maps $\omega\colon \underbrace{V\times\cdots\times V}_{k} \to \mathbb{R}$. We recall that the *wedge product*

$$\mathrm{Alt}^k(V,\mathbb{R})\times \mathrm{Alt}^\ell(V,\mathbb{R}) \xrightarrow{\wedge} \mathrm{Alt}^{k+\ell}(V,\mathbb{R})$$

is a bilinear associative product defined by

$$(\omega\wedge\eta)(v_1,\dots,v_{k+\ell}) := \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)},\dots,v_{\sigma(k)})\eta(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)})$$

for $\omega\in \mathrm{Alt}^k l(V,\mathbb{R})$, $\eta\in \mathrm{Alt}^\ell(V,\mathbb{R})$, $v_1,\dots,v_{k+\ell}\in V$.

(a) Let $\{b_i\}_{i=1,\dots,n}$ be a basis of V, and let $\{b^i\}_{i=1,\dots,n}\}$ be the dual basis of the dual vector space $V^*=\mathrm{Alt}^1(V,\mathbb{R})$. Let $I=(i_1,\dots,i_k)$, $J=(j_1,\dots,j_k)$ be k-tupels with $1\le i_1<i_2<\cdots<i_k\le n$ and similarly for J. Show that

$$(b^{i_1}\wedge\cdots\wedge b^{i_k})(b_{j_1},\dots,b_{j_k}) = \begin{cases} 1 & I=J \\ 0 & I\neq J \end{cases}$$

In particular, $\{b^{i_1}\wedge\cdots\wedge b^{i_k}\}_{1\le i_1<i_2<\cdots<i_k\le n}$ is a basis of the vector space $\mathrm{Alt}^k(V,\mathbb{R})$. Hint: Use induction over k.

(b) Show that the wedge product is *graded commutative*, i.e.,

$$\eta\wedge\omega = (-1)^{k\ell}\omega\wedge\eta \qquad \text{for } \omega\in \mathrm{Alt}^k l(V,\mathbb{R}),\ \eta\in \mathrm{Alt}^\ell(V,\mathbb{R}).$$

Hint: First consider the case $k=\ell=1$, then argue that it suffices to prove the statement in the case $\omega=b^{i_1}\wedge\cdots\wedge b^{i_k}$, $\eta=b^{j_1}\wedge\cdots\wedge b^{j_\ell}$.

3. (10 points) Let M, N be smooth manifolds and $F\colon M\to N$ a smooth map. Then a differential form $\omega\in\Omega^k(N)$ leads to a form $F^*\omega\in\Omega^k(M)$, called the *pullback of ω along F* which is defined by

$$(F^*\omega)_p(v_1,\dots,v_k) := \omega_p(F_*v_1,\dots,F_*v_k) \qquad \text{for } p\in M,\ v_1,\dots,v_k\in T_pM.$$

In more detail: the k-form $F^*\omega$ is a section of the vector bundle $\mathrm{Alt}^k(TM;\mathbb{R})$, and hence it can be evaluated at $p \in M$ to obtain an element $(F^*\omega)_p$ in the fiber of that vector bundle over p, which is $\mathrm{Alt}^k(T_pM;\mathbb{R})$. In other words, $(F^*\omega)_p$ is an alternating multilinear map

$$(F^*\omega)_p \colon \underbrace{T_pM \times \cdots \times T_pM}_{k} \longrightarrow \mathbb{R},$$

and hence it can be evaluated on the k tangent vectors $v_1, \dots, v_k \in T_pM$ to obtain a real number $(F^*\omega)_p(v_1, \dots, v_k)$. On the right hand side to the equation defining $F^*\omega$, the map $F_* \colon T_pM \to T_{F(p)}N$ is the differential of F. Hence the alternating multilinear map $\omega_{F(p)} \in \mathrm{Alt}^k(T_{F(p)}N;\mathbb{R})$ can be evaluated on $F_*v_1, \dots, F_*v_k$ to obtain the real number $\omega_p(F_*v_1, \dots, F_*v_k)$.

Let $F \colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Show that

$$F^*(dx^1 \wedge \cdots \wedge dx^n) = \det(dF_x)\; dx^1 \wedge \cdots \wedge dx^n \tag{0.1}$$

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Here $dF_x \colon \mathbb{R}^n \to \mathbb{R}^n$ is the differential of F at the point $x \in \mathbb{R}^n$.

4. (10 points) For any smooth manifold M the *de Rham differential* (also called *exterior differential*) is the unique map $d \colon \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) d is linear.

(ii) For a function $f \in C^\infty(M) = \Omega^0(M)$ the 1-form $df \in \Omega^1(M) = C^\infty(M, T^*M)$ is the usual differential of f.

(iii) d is a graded derivation with respect to the wedge product; i.e.,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta \qquad \text{for } \omega \in \Omega^k(M),\ \eta \in \Omega^l(M).$$

(iv) $d^2 = 0$.

Show that for $M = \mathbb{R}^n$ the de Rham differential of the k-form

$$\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \text{for } f \in C^\infty(\mathbb{R}^n),\ i_1 < i_2 < \cdots < i_k$$

is given explicitly by the formula

$$d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We note that *every* k-form $\eta \in \Omega^k(\mathbb{R}^n)$ can be written uniquely in the form

$$\eta = \sum_{i_1 < \cdots < i_k} f_{i_1, \dots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for smooth functions $f_{i_1,\ldots,i_k} \in C^\infty(\mathbb{R}^n)$.

5. (10 points) Show that the exterior derivative for differential forms on $\mathbb{R}^3$ corresponds to the classical operations of *gradient* resp. *curl* resp. *divergence*. More precisely, show that there is a commutative diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, equal] & \mathsf{Vect}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\cong"] & \mathsf{Vect}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\cong"] & C^\infty(\mathbb{R}^3) \arrow[d, "\cong"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

Here $\mathsf{Vect}(\mathbb{R}^3)$ is the space of vector fields on $\mathbb{R}^3$, and we recall that grad, curl and divergence are given by the formulas

$$\begin{aligned}
\mathrm{grad}(f) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\mathrm{curl}(f_1, f_2, f_3) &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
\mathrm{div}(f_1, f_2, f_3) &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}
\end{aligned}$$

Here we identify a vector field on $\mathbb{R}^3$ with a triple (f_1, f_2, f_3) of smooth functions on $\mathbb{R}^3$. The vertical isomorphisms are given by

$$\begin{aligned}
(f_1, f_2, f_3) &\mapsto f_1 dx + f_2 dy + f_3 dz \\
(f_1, f_2, f_3) &\mapsto f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \\
f &\mapsto f dx \wedge dy \wedge dz
\end{aligned}$$