Homework Assignment # 10, due Nov. 19

1. (10 points) Recall that the special linear group $SL_n(\mathbb{R})$ and the orthogonal group $O(n)$ are both submanifolds of the vector space $M_{n\times n}(\mathbb{R})$ of $n \times n$ matrices. In particular, the tangent spaces $T_A SL_n(\mathbb{R})$ for $A \in SL_n(\mathbb{R})$ and $T_A O(n)$ for $A \in O(n)$ are subspaces of the tangent space $T_A M_{n\times n}(\mathbb{R})$, which can be identified with $M_{n\times n}(\mathbb{R})$, since $M_{n\times n}(\mathbb{R})$ is a vector space.

(a) Show that $T_e SL_n(\mathbb{R}) = \{C \in M_{n\times n} \mid \operatorname{tr}(C) = 0\}$, where e is the identity matrix, and $\operatorname{tr}(C)$ denotes the trace of the matrix C.

(b) Show that $T_e O(n) = \{C \in M_{n\times n} \mid C^t = -C\}$.

Hint for parts (a) and (b): $SL_n(\mathbb{R})$ and $O(n)$ can be both be described as level sets $F^{-1}(c)$ of a regular value c for a suitable smooth map F (as we did in class for $SL_n(\mathbb{R})$ and you did for $O(n)$ in problem 5 of homework assignment # 9; note that $O(n)$ is equal to the Stiefel manifold $V_n(\mathbb{R}^n)$).

2. (10 points) Let M be a smooth manifold of dimension n. If $f\colon M \to \mathbb{R}$ is a smooth function, then for $p \in M$ its differential

$$f_*\colon T_pM \longrightarrow T_{f(p)}\mathbb{R} = \mathbb{R}$$

is an element of $\operatorname{Hom}(T_pM, \mathbb{R})$. This vector space dual to the tangent space T_pM is called the *cotangent space*, and is denoted T_p^*M. It is common to write $df_p \in T_p^*M$ for the differential $f_*\colon T_pM \to \mathbb{R}$.

(a) Let $x^i\colon \mathbb{R}^n \to \mathbb{R}$ be the i-th coordinate function, which maps $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ to $x_i \in \mathbb{R}$. Show that for any point $q \in \mathbb{R}^n$ a basis of the cotangent space $T_q^*\mathbb{R}^n$ is given by $\{dx_q^i\}_{i=1,\dots,n}$.

(b) If $M \supset U \xrightarrow{\phi} V \subset \mathbb{R}^n$ is a smooth chart of M, the component functions of ϕ, given by $y^i := x^i \circ \phi$ are called *local coordinates*. Show that for $p \in U$, a basis of the cotangent space T_p^*M is given by $\{dy_p^i\}_{i=1,\dots,n}$.

Hint for part (b): let $\phi^*\colon T_q^*\mathbb{R}^n \to T_p^*M$, $q = \phi(p)$ be the linear map dual to the differential $\phi_*\colon T_pM \to T_q\mathbb{R}^n$ defined by

$$(\phi^*\xi)(v) = \xi(\phi_*(v)) \qquad \text{for } \xi \in T_q^*\mathbb{R}^n \text{ and } v \in T_pM.$$

Show first that $\phi^*(dx_q^i) = dy_p^i$.

3. (10 points) Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Show that the differential df is a smooth section of the cotangent bundle T^*M. Hint: smoothness of a section s is a local property and hence to check smoothness it suffices to check that the composition $\Phi_\alpha \circ s$ is smooth for local trivializations Φ_α of the cotangent bundle T^*M.

4. (10 points) The goal of this problem is to prove the following result.

lem:VBCL **Lemma 0.1. (Vector Bundle Construction Lemma).** *Let M be a smooth manifold of dimension n, and let $\{E_p\}$ be a collection of vector spaces parametrized by $p \in M$. Let E be the set given by the disjoint union of all these vector spaces, which we write as*

$$E := \coprod_{p \in M} E_p = \{(p, v) \mid p \in M,\ v \in E_p\}$$

and let $\pi\colon E \to M$ be the projection map defined by $\pi(p, v) = p$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M, and let for each $\alpha \in A$, let $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ be maps with the following properties

(i) *The diagram*

$$\begin{array}{ccc} E_{|U_\alpha} := \pi^{-1}(U_\alpha) & \xrightarrow{\quad \Phi_\alpha \quad} & U_\alpha \times \mathbb{R}^k \\ & {\scriptstyle \pi} \searrow \qquad \swarrow {\scriptstyle \pi_1} & \\ & U_\alpha & \end{array} \tag{0.2}$$

eq:loc_tri

is commutative, where π_1 is the projection onto the first factor.

(ii) *For each $p \in U_\alpha$, the restriction of Φ_α to $E_p = \pi^{-1}(p)$ is a vector space isomorphism between E_p and $\{p\} \times \mathbb{R}^k = \mathbb{R}^k$ (which implies that Φ_α is a bijection).*

(iii) *For $\alpha, \beta \in A$, the composition*

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \xrightarrow{\Phi_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\Phi_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

is smooth.

Then the total space E can be equipped with the structure of a smooth manifold of dimension $n + k$ such that $\pi\colon E \to M$ is a smooth vector bundle of rank k with local trivializations Φ_α.

(a) Construct a topology on E by declaring $U \subset E$ to be *open* if $\Phi_\alpha(U \cap E_{|U_\alpha})$ is an open subset of $U_\alpha \times \mathbb{R}^k$ for all $\alpha \in A$. Show that this satisfies the conditions for a topology, and that with this topology on E the projection map $\pi\colon E \to M$ is continuous and the map Φ_α is a homeomorphism (for the subspace topology on $E_{|U_\alpha}$).

(b) Show that equipped with this topology E is a topological manifold of dimension $n + k$ (don't bother to check the technical conditions of being Hausdorff and second countable). Hint: Let $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ be an atlas for M. Show that the bundle chart Φ_α and the manifold chart ψ_β can be used to construct a chart

$$\chi_{\alpha,\beta}\colon E \underset{\text{open}}{\supset} E_{|U_\alpha \cap V_\beta} \longrightarrow \mathbb{R}^{n+k}.$$

(c) Show that the charts $\{(E_{|U_\alpha \cap V_\beta}), \chi_{\alpha,\beta})\}$ for $(\alpha, \beta) \in A \times B$ form a smooth atlas for E.

(d) Show that $\pi \colon E \to M$ is a smooth vector bundle of rank k with local trivializations provided by Φ_α.

5. (10 points) We recall that the projective space $\mathbb{RP}^n$ is a smooth manifold of dimension n whose underlying set is the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. In particular, each point $p \in \mathbb{RP}^n$ determines tautologically a 1-dimensional subspace $E_p \subset \mathbb{R}^{n+1}$. Let E be the disjoint union $E = \amalg_{p \in \mathbb{RP}^n} E_p$ of the vector spaces E_p. More explicitly,

$$E = \{([x], v) \mid [x] \in \mathbb{RP}^n, v \in \langle x \rangle\},$$

where $x \in \mathbb{R}^{n+1} \setminus \{0\}$, $\langle x \rangle \subset \mathbb{R}^{n+1}$ is the one-dimensional subspace spanned by x, and $[x] \in \mathbb{RP}^n$ is the corresponding point in the projective space.

(a) Use the Vector Bundle Construction Lemma to show that E is a smooth vector bundle of rank 1 over $\mathbb{RP}^n$ (which is called the *tautological line bundle over* $\mathbb{RP}^n$; *line bundle* is a synonym for *vector bundle of rank* 1). Hint: Construct local trivializations of E restricted to $U_i = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$.

(b) Show that the complement of the zero section in E is diffeomorphic to $\mathbb{R}^{n+1} \setminus \{0\}$.

(c) Show that the line bundle E is not isomorphic to the trivial line bundle. Hint: consider the complement of the zero-section of E and compare it with the complement of the zero-section of the trivial line bundle.