Homework Assignment # 1, due Sept. 3, 2021

1. (10 points) Let $GL_n(\mathbb{R})$ be the set of invertible $n \times n$ matrices.

(a) Show that $GL_n(\mathbb{R})$ is an open subset of the topological space $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ of all $n \times n$ matrices. Hint: use the map $A \mapsto \det(A)$.

(b) Show that the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is a continuous map.

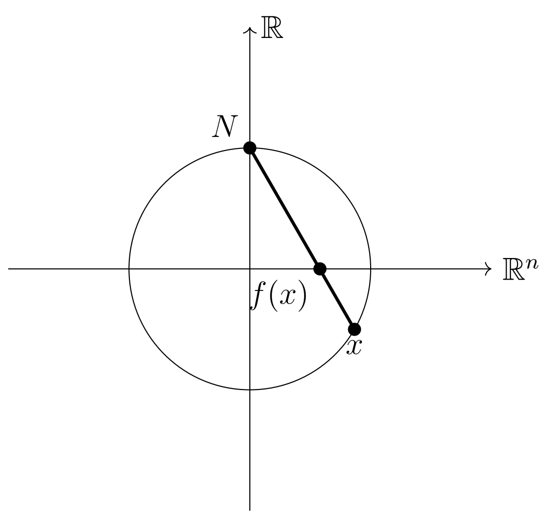
2. (10 points) The point of this problem is to show that the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ agrees with the product topology (where each factor is equipped with the metric topology). Since both, the metric topology and the product topology, are defined via a basis, it is good to know how to compare two topologies given in terms of bases. This is provided by the statement of part (a).

(a) Let X be a set, and let $\mathcal{T}$, $\mathcal{T}'$ be topologies generated by a basis $\mathcal{B}$ resp. $\mathcal{B}'$. Show that $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for each $B \in \mathcal{B}$ and $x \in B$ there is some $B' \in \mathcal{B}'$ with $x \in B'$ and $B' \subset B$.

(b) Show that the products of balls $B_r(x) \times B_s(y) \subset \mathbb{R}^m \times \mathbb{R}^n$ for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, $s, r > 0$ form a basis for the product topology on $\mathbb{R}^m \times \mathbb{R}^n$.

(c) Show that the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ agrees with the product topology. Hint: it might be helpful to draw pictures of a ball around $(x, y) \in \mathbb{R}^{m+n}$ and a product of balls $B_r(x) \times B_s(y) \subset \mathbb{R}^{m+n}$ for $m = n = 1$.

3. (10 points) Let $N \in S^n$ be the "north pole" of S^n, i.e., $N = (0, \dots, 0, 1) \in S^n$. The *stereographic projection* is the map $f\colon S^n \setminus \{N\}$ which sends a point $x \in S^n \setminus \{N\}$ to the intersection point of the straight line L_x in $\mathbb{R}^{n+1}$ with endpoint N and x with $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. Here is a picture of the situation for $n = 1$.



The map $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$ is a bijection. Explicitly, the map f and its inverse are given by the explicit formulas

$$f(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n) \qquad f^{-1}(y_1, \dots, y_n) = \frac{1}{||y||^2 + 1}(2y_1, \dots, 2y_n, ||y||^2 - 1)$$

for $(x_1, \dots, x_{n+1}) \in S^n$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$. Provide a careful argument for the continuity of f and f^{-1} (you can use freely that recognize that certain maps $\mathbb{R} \supset U \to \mathbb{R}$ are continuous, but each time you use one of our "continuity criteria" for maps involving sub-spaces, products and quotients, you should be explicit about it).

4. (10 points) Show that the quotient space D^n/S^{n-1} is homeomorphic to the sphere S^n. Hint: produce a bijective map f relating these spaces by writing down an explicit formula, paying attention to have this map go the "natural direction" to make proving its continuity simple. Then show that f is a homeomorphism by verifying that domain and range of f satisfy the conditions that make continuity of f^{-1} automatic.

5. (10 points) Consider the following topological spaces

- The subspace $T_1 := \{v \in \mathbb{R}^3 \mid \mathrm{dist}(v, K) = r\} \subset \mathbb{R}^3$ equipped with the subspace topology, where $K = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $0 < r < 1$.
- The product space $T_2 := S^1 \times S^1$ equipped with the product topology.
- The quotient space $T_3 := ([-1, 1] \times [-1, 1])/\sim$ equipped with the quotient topology, where the equivalence relation is generated by $(s, -1) \sim (s, 1)$ and $(-1, t) \sim (1, t)$.

Show that these three spaces are homeomorphic. Hint: It suffices to produce two homeomorphisms between pairs of these spaces. First construct suitable continuous bijections, making sure to pick maps going in a direction that makes it easy to verify continuity using the Continuity Criterions for maps to/from subspaces, product spaces resp. quotient spaces. Then verify that continuity of the inverse is automatic since domain resp. codomain is compact resp. Hausdorff.