Homework Assignment # 8, due Nov. 8

1. (10 points) Let M be a topological manifold.

(a) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be a smooth atlas for M. Show that if (U, ϕ) and (V, ψ) are two charts for M, both of which are smoothly compatible with the smooth atlas, then they are smoothly compatible with each other.

(b) Show that every smooth atlas for M is contained in a unique maximal smooth atlas.

(c) Show that two smooth atlases for M determine the same maximal smooth atlas if and only if their union is a smooth atlas.

2. (10 points) We recall that the stereographic projection provides a homeomorphism between the open subsets $U_\pm := S^n \setminus \{(\mp 1, 0, \dots, 0)\}$ of S^n and $\mathbb{R}^n$. More explicitly, the stereographic projection is the map

$$\psi_\pm \colon U_\pm \longrightarrow \mathbb{R}^n \qquad \text{is defined by} \qquad \psi_\pm(x_0, \dots, x_n) := \frac{1}{1 \pm x_0}(x_1, \dots, x_n),$$

and its inverse $\psi_\pm^{-1} \colon \mathbb{R}^n \to U_\pm$ is given by the formula

$$\psi_\pm^{-1}(y_1, \dots, y_n) = \frac{1}{||y||^2 + 1}(\pm(1 - ||y||^2), 2y_1, \dots, 2y_n).$$

In particular, the two charts (U_+, ψ_+), (U_-, ψ_+) form an atlas for S^n.

(a) Show that $\{(U_+, \psi_+), (U_-, \psi_-)\}$ is a *smooth* atlas for S^n.

(b) Show that the atlas above is smoothly compatible with the smooth atlas we've discussed in class, consisting of the charts $(U_{i,\epsilon}, \phi_{i,\epsilon})$, $\epsilon \in \{\pm 1\}$, where $U_{i,\epsilon} \subset S^n$ consists of the points $(x_0, \dots, x_n) \in S^n$ such that $\epsilon x_i > 0$, and

$$\phi_{i,\epsilon} \colon U_{i,\epsilon} \xrightarrow{\approx} B_1^n \qquad \text{is given by} \qquad \phi_{i,\epsilon}(x_0, \dots, x_n) = (x_0, \dots, \widehat{x_i}, \dots, x_n)$$

with inverse given by $\phi_{i,\epsilon}^{-1}(y_1, \dots, y_n) = (y_1, \dots, y_i, \epsilon\sqrt{1 - ||y||^2}, y_{i+1}, \dots, y_n)$.

(c) Let $V \subset \mathbb{R}^{n+1}$ be an open subset containing S^n and let $f \colon V \to \mathbb{R}$ be a smooth map. Show that the restriction of f to S^n, i.e., the composition $S^n \hookrightarrow V \xrightarrow{f} \mathbb{R}$, is a smooth function on the smooth manifold S^n (equipped by the "standard smooth structure" on S^n given by either of the two smooth atlases described above).

3. (10 points) There are two common ways to describe the complex projective space $\mathbb{CP}^n$ as topological space, namely as

$$S^{2n+1}/z \sim \lambda z \quad \text{for } z \in S^{2n+1},\ \lambda \in S^1$$

or as

$$\left(\mathbb{C}^{n+1} \setminus \{0\}\right)/z \sim \lambda z \quad \text{for } z \in \mathbb{C}^{n+1} \setminus \{0\},\ \lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

(a) Show that these two quotient spaces are homeomorphic.

(b) Using the description of $\mathbb{CP}^n$ as quotient of $\mathbb{C} \setminus \{0\}$, let $U_k := \{[z_0, z_1, \dots, z_n] \in \mathbb{CP}^n \mid z_k \neq 0\}$ for $k = 0, \dots, n$ and let

$$\phi_k \colon U_k \longrightarrow \mathbb{C}^n \qquad \text{be defined by} \qquad \phi_k([z_0, z_1, \dots, z_n]) = \left(\frac{z_0}{z_k}, \dots, \widehat{\frac{z_k}{z_k}}, \dots, \frac{z_n}{z_k}\right).$$

Show that (U_k, ϕ_k) is a chart for $\mathbb{CP}^n$.

(c) Show that $\{(U_i, \phi_i) \mid i = 0, \dots, n\}$ is a smooth atlas for $\mathbb{CP}^n$.

4. (10 points) Show that the Cartesian product of $M \times N$ of smooth manifolds of dimension m resp. n is again a smooth manifold of dimension $m + n$.

5. (10 points) Let $\mathbb{RP}^n = S^n/x \sim -x$ be the real projective space and let $h \colon \mathbb{RP}^n \to \mathbb{R}$ be the function defined by

$$h([x_0, \dots, x_n]) = \sum_{\ell=0}^{n} \ell x_\ell^2.$$

(a) Show that h is a well-defined smooth function.

(b) Determine the *critical points* of h.

Explanation: a *critical point* of a smooth function $\mathbb{R}^n \underset{\text{open}}{\supset} U \xrightarrow{f} \mathbb{R}$ is a point $x \in U$ such that the gradient of f vanishes at the point x. More generally, a critical point of a smooth function $f \colon M \to \mathbb{R}$ on a smooth manifold M is a point $x \in M$ such that for some smooth chart (U, ϕ) with $x \in U$ the point $\phi(x) \in \phi(U) \subset \mathbb{R}^n$ is a critical point of the composition $f \circ \phi^{-1} \colon \phi(U) \to \mathbb{R}$ (use without proof the fact that this is independent of the choice of the smooth chart (U, ϕ)). Here a "smooth chart" means any chart belonging to the maximal smooth atlas defining the smooth structure of M.

Hint: Use the smooth atlas consisting of the charts $\mathbb{RP}^n \supset U_k \xrightarrow{\phi_k} B_1^n$ (the open ball of radius 1 in $\mathbb{R}^n$) with $U_k = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_k \neq 0\}$ and

$$\phi_k^{-1}(v_1, \dots, v_n) = [v_1, \dots, v_k, \sqrt{1 - ||v||^2}, v_{k+1}, \dots, v_n].$$