Homework Assignment # 7, due Nov. 1

1. (10 points) Let M, N be path-connected manifolds of dimension $n \geq 3$. The goal of this problem is to compute the fundamental group of their connected sum $M\#N$ in terms of the fundamental groups of M and N. We recall that for the construction of the connected sum we picked points $x_0 \in M$, $y_0 \in N$ and maps $\phi\colon B^n_2 \to M$, $\psi\colon B^n_2 \to N$ which are are homeomorphisms onto their image with $\phi(0) = x_0$, $\psi(0) = y_0$; here $B^n_2 = \{v \in \mathbb{R}^n \mid ||v|| < 2\} \subset \mathbb{R}^n$ is the open ball of radius 2. Then we defined

$$M\#N := (M \setminus \phi(B^n_1) \ \amalg \ N \setminus \psi(B^n_1))/\sim$$

where the equivalence relation is given by identifying for $v \in S^{n-1}$ the point $\phi(v) \in M \setminus \phi(B^n_1)$ with the point $\psi(v) \in N \setminus \psi(B^n_1))$.

For the problem at hand, as well as for defining the connected sum for smooth manifolds, a modification of the definition of the connected sum is convenient. Let

$$\alpha\colon S^{n-1} \times (-1,1) \xrightarrow{\approx} B^n_2 \setminus \{0\} \qquad \text{be given by} \qquad (v,t) \mapsto (1-t)v.$$

This is a homeomorphism with inverse given by $B^n_2 \setminus \{0\} \ni w \mapsto (w/||w||, 1 - ||w||)$. Let

$$M\widetilde{\#}N := (M \setminus \{x_0\} \amalg N \setminus \{y_0\})/\sim,$$

where the equivalence relation identifies $\phi(\alpha(v,-t))$ and $\psi(\alpha(v,t))$ for $(v,t) \in S^{n-1} \times (-1,1)$ (warning: $M\widetilde{\#}N$ is just an ad hoc notation).

(a) Show that $M\widetilde{\#}N$ is homeomorphic to $M\#N$. Hint: it might be helpful to draw a picture of $M\widetilde{\#}N$, indicating the image of $\phi(B^n_2 \setminus \{0\}) = \psi(B^n_2 \setminus \{0\})$.

(b) How are the fundamental groups of M and $M \setminus \{x_0\}$ related? Hint: Use the Seifert van Kampen Theorem.

(c) Express the fundamental group of $M\#N$ in terms of the fundamental groups of $M \setminus \{x_0\}$ and $N \setminus \{y_0\}$.

2. (10 points)

(a) Show that the projection map $p\colon S^n \to \mathbb{RP}^n$ is a double covering.

(b) Calculate the fundamental group of $\mathbb{RP}^n$ for $n \geq 2$. Hint: use part (a).

3. (10 points) Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Let Y be a path-connected and let $f\colon (Y, y_0) \to (X, x_0$ be a map such that the image $f_*\pi_1(Y, y_0)$ is contained in the image

$p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$. We proved in class that then there exists a unique (not necessarily continuous) map $\widetilde{f}$ making the diagram

$$\begin{array}{ccc} & & (\widetilde{X}, \widetilde{x}_0) \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}$$

commutative. We constructed $\widetilde{f}(y)$ by picking a path $\gamma\colon I \to Y$ from y_0 to y, composed with the map $f\colon Y \to X$ to obtain the path $f\gamma\colon I \to X$, and defined $\widetilde{f}(y) := \widetilde{f\gamma}(1)$, where $\widetilde{f\gamma}\colon I \to \widetilde{X}$ is the unique lift of $f\gamma$ with starting point $\widetilde{x}_0$.
Show that $\widetilde{f}$ is continuous under the additional assumption that Y is locally path-connected. Hint: It suffices to show that $\widetilde{f}$ is continuous in some open neighborhood V of every point $y \in Y$. Show that the assumption that Y is locally path-connected can be used to choose for every point $y \in Y$ a path-connected neighborhood V such that $f(V)$ is contained in a evenly covered open subset $U \subset X$. To analyze $\widetilde{f}(y')$ for $y' \in V$, use the concatenation $\gamma * \delta$ of a path γ from y_0 to y and $\delta\colon I \to V$ from y to y'.

4. (10 points) Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the universal covering of a path-connected and locally path-connected space X.

(a) It follows from the General Lifting Criterion that for $g \in G := \pi_1(X, x_0)$ there is a unique map $\phi_g\colon (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, g\widetilde{x}_0)$ making the diagram

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{x}_0) & \xrightarrow{\phi_g} & (\widetilde{X}, g\widetilde{x}_0) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & (X, x_0) & \end{array}$$

commutative. Here $g\widetilde{x}_0 := \widetilde{\gamma}(1)$ is the endpoint of a lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$ of any based loop γ in (X, x_0) which represents $g \in \pi_1(X, x_0)$ (we have shown that $\widetilde{\gamma}(1)$ depends only on $[\gamma] \in \pi_1(X, x_0)$, not on the particular loop γ). Show that the map

$$G \times \widetilde{X} \longrightarrow \widetilde{X} \qquad (g, \widetilde{x}) \mapsto \phi_g(\widetilde{x})$$

is an action map.

(b) Show that the action is free, i.e., for every $\widetilde{x} \in \widetilde{X}$, the only element of $g \in G$ with $g\widetilde{x} = \widetilde{x}$ is the identity element.

(c) Show that the action is transitive on the fiber $p^{-1}(x)$ for all $x \in X$, i.e., for $\widetilde{x}, \widetilde{x}' \in p^{-1}(x)$ there is some $g \in G$ such that $g\widetilde{x} = \widetilde{x}'$.

5. (10 points) Let (X, x_0) be a pointed space which is path-connected, locally path-connected, and semilocally simply connected. The goal of this assignment is to show that there is a bijection Ψ between

$$\{\text{based coverings } p\colon (E, e_0) \to (X, x_0 \text{ with } E \text{ path-connected}\}/\text{isomorphism}$$

and

$$\{\text{subgroups of } \pi_1(X, x_0)\}.$$

It is given by sending a covering p to the subgroup $p_*\pi_1(E, e_0) \subset \pi_1(X, x_0)$.

(a) Show that Ψ is injective. Hint: use the general lifting criterion to show that any two path-connected based coverings $p\colon (E, e_0) \to (X, x_0)$ and $p'\colon (E', e_0') \to (X, x_0)$ are isomorphic.

(b) Let $p\colon \widetilde{X} \to X$ be the universal covering of X, on which the fundamental group $G = \pi_1(X, x_0)$ acts freely by covering maps; this action is transitive on all fibers $p^{-1}(x)$ for $x \in X$. Let H be a subgroup of G and let $H\backslash\widetilde{X}$ be the orbit space of action of the subgroup H and let $p^H\colon (H\backslash\widetilde{X}, [\widetilde{x}_0]) \to (X, x_0)$, $[\widetilde{x}] \mapsto p(\widetilde{x})$ be the projection map. Here $[\widetilde{x}] = H\widetilde{x}$ denotes the orbit through the point $\widetilde{x}$. Show that p^H is a covering and that $p^H_*\pi_1(H\backslash\widetilde{X}, [\widetilde{x}_0]) \subset G$ is the subgroup H.