Homework Assignment # 3, due Sept. 20, 2019

1. (10 points) Which of the topological groups $GL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are connected? Hint: To show that one of these topological groups is connected, it might be easier to show that it is path-connected. Note that to prove this, it suffices to find a path connecting any element with the identity element (why?). Use without proof the fact that every element in $SO(n)$ (the group of linear maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ which are isometries with determinant one) for a suitable choice of basis of $\mathbb{R}^n$ is represented by a matrix of block diagonal form whose diagonal blocks are the 1×1 matrix with entry $+1$ and/or 2×2 rotational matrices

$$R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Here "block diagonal" means that all other entries are zero.

2. (10 points) The definition of a manifold involves the technical conditions of being Hausdorff and second countable. Show that these properties are "inherited" by subspaces in the following sense. Let X be a topological space and A a subspace.

(a) Show that if X is Hausdorff, then so is A.

(b) Show that if X is second countable, then so is A.

3. (10 points) Let M be a manifold of dimension m and let N be a manifold of dimension n. Show that the product $M \times N$ is a manifold of dimension $m + n$. Don't forget to check the technical conditions (Hausdorff and second countable) for $M \times N$.

4. (10 points) Show that the real projective space $\mathbb{RP}^n$ is a manifold of dimension n. Don't forget to check that $\mathbb{RP}^n$ is second countable (we have proved in class that the projective space is Hausdorff). Hint: to prove that $\mathbb{RP}^n$ is locally homeomorphic to $\mathbb{R}^n$ suitably modify the method we used for the sphere S^n.

5. (10 points) Show that the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ of two copies of the real projective plane is homeomorphic to the Klein bottle.