Homework Assignment # 10, due Nov. 22

1. (15 points) Let $U \subset \mathbb{R}^n$ be an open subset. Then the map

$$\frac{\partial}{\partial x_{i\,|p}} : C_p^\infty(U) \longrightarrow \mathbb{R}$$

given by mapping $f \in C_p^\infty(U)$ to the partial derivative $\frac{\partial f}{\partial x_{i\,|p}}$ at a point p is an element of $T_p^{\mathrm{alg}}U = \mathrm{Der}(C_p^\infty(U), \mathbb{R})$; in other words, $\frac{\partial}{\partial x_{i\,|p}}$ is a tangent vector of U at the point $p \in U$ (in the algebraic description of the tangent space).

(a) Show that the map $\mathbb{R}^n \to T_p^{\mathrm{alg}}U = \mathrm{Der}(C_p^\infty(U), \mathbb{R})$ given by $v \mapsto \sum_{i=1}^n v_i \frac{\partial}{\partial x_{i\,|p}}$ agrees with the isomorphism constructed in class (given by $v \mapsto d_v$).

(b) Let V be a smooth vector field on U, i.e., V is a smooth map $V : U \to \mathbb{R}^n$ with component functions $V_i \colon U \to \mathbb{R}$, i.e., $V(p) = (V_1(p), \dots, V_n(p))$. For $f \in C^\infty(U)$, let $D_V f \in C^\infty(U)$ be the smooth function whose value at $p \in U$ is given by

$$(D_V f)(p) = \sum_{i=1}^n V_i(p) \frac{\partial f}{\partial x_{i\,|p}}.$$

Show that the map $D_V \colon C^\infty(U) \longrightarrow C^\infty(U)$ defined by $f \mapsto D_V f$ is a derivation of the algebra $C^\infty(U)$, i.e., D_V is a linear map with the product rule property

$$D(f \cdot g) = D_V(f) \cdot g + f \cdot D_V(g) \qquad \text{for } f, g \in C^\infty(U).$$

Terminology and notation: the usual notation is

$$D_V = \sum_{i=1}^n V_i \frac{\partial}{\partial x_i}.$$

Geometrically, $D_V f$ is the derivative of the smooth function f in the direction of the vector field V.

(c) Let $C^\infty(U, \mathbb{R}^n)$ be the vector space of smooth maps $U \to \mathbb{R}^n$, and let $\mathrm{Der}(C^\infty(U))$ be the vector space of derivations of the algebra $C^\infty(U)$. Show that the map

$$C^\infty(U, \mathbb{R}^n) \to \mathrm{Der}(C^\infty(U)) \qquad \text{given by} \qquad V \mapsto D_V$$

is an isomorphism of vector spaces. Hint: use the isomorphisms $\mathbb{R}^n \cong \mathrm{Der}(C_p^\infty(U), \mathbb{R}) \cong \mathrm{Der}(C^\infty(U), \mathbb{R})$ we proved in class (the first is given by $v \mapsto d_v$, the second is induced by the restriction map $C^\infty(U)$ from to $C_p^\infty(U)$).

(d) For $X, Y \in \operatorname{Der}(C^\infty(U))$, define $[X,Y] := X \circ Y - Y \circ X \colon C^\infty(U) \to C^\infty(U)$. Show that the linear map $[X,Y]$ is again a derivation.

(e) Show that for $X, Y, Z \in \operatorname{Der}(C^\infty(U))$ the property

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{0.1}$$

holds.

Comments. Property (c) shows that a smooth vector field on U can alternatively be described as a derivation $X \colon C^\infty(U) \to C^\infty(U)$. This very abstract definition has two advantages:

- it immediately generalizes from open subsets of Euclidean space to smooth manifolds.
- It allows the definition of $[X,Y]$, called the *Lie bracket of the vector fields* X, Y.

A *Lie algebra* is a vector space L equipped with a map $[\ ,\ ] \colon L \times L \to L$ which is *linear in each slot*, which is *alternating* in the sense that $[X,Y] = -[Y,X]$, and which satisfies equation (0.1), called the *Jacobi identity*. The Lie bracket of smooth vector fields is evidently linear in each slot and alternating; so the result of part (e) can be summarized by saying that the vector space of smooth vector fields on a smooth manifold is a Lie algebra with respect to the Lie bracket of vector fields.

2. (10 points) Let U be an open subset of $\mathbb{R}^n$ and let $\gamma \colon \mathbb{R} \times U \to U$ be a smooth action of the group $\mathbb{R}$ on U. Let $V \colon U \to \mathbb{R}^n$ be the vector field given by $V(p) := \gamma_p'(0)$, where $\gamma_p \colon \mathbb{R} \to U$ is the smooth path given by $\gamma_p(t) = \gamma(t,p)$.

(a) Show that V is a smooth vector field and write the associated derivation D_V explicitly in the form $D_V = \sum_{i=1}^n V_i \frac{\partial}{\partial x_i}$ for smooth functions $V_i \in C^\infty(U)$.

(b) Find the explicit formula for the action $\gamma \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ where $\gamma(t,x,y)$ is the point $(x,y) \in \mathbb{R}^2$ rotated counterclockwise by the angle t.

(c) Calculate the corresponding vector field $R = R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y}$, where $R_1, R_2 \in C^\infty(\mathbb{R}^2)$.

(d) Give the explicit formula of the action $\gamma^z \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, where $\gamma^z(t,(x,y,z))$ describes the rotation of $(x,y,z) \in \mathbb{R}^3$ around the z-axis by the angle t. Give an explicit formula for the corresponding vector field R^z as a linear combination of the vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ whose coefficients are smooth functions on $\mathbb{R}^3$. Use symmetry considerations (cyclic permutation of the x, y, z coordinates) to write down the vector fields R^x and R^y corresponding to rotation around the x resp. y axis.

(e) Show that $[R^x, R^y] = -R^z$, $[R^y, R^z] = -R^x$ and $[R^z, R^x] = -R^y$. Hint: calculate one of these bracket relations and deduce the other two by symmetry arguments.

Comments: The vector field V corresponding to an $\mathbb{R}$-action is called the *infinitesimal generator* of the action. The group $SO(3)$ is the group of rotations of $\mathbb{R}^3$; its Lie algebra is the 3-dimensional vector space with basis R^x, R^y and R^z and the Lie algebra structure determined by the Lie brackets determined in part (e).

3. (10 points) Let V and W be real vector spaces with bases $\{v_i\}_{i=1,\ldots,m}$ and $\{w_j\}_{j=1,\ldots,n}$, respectively. Using these bases, construct bases for the following vector spaces that can be constructed from V, W and determine the dimension of these vector spaces. These are standard constructions; there is no need to verify that the collections of vectors you write down in fact form a basis. The point of this exercise is to make sure you know these bases. Feel free to consult the literature to see how this is done.

(a) The *dual vector space* $V^* := \mathrm{Hom}(V, \mathbb{R})$ of linear maps $f\colon V \to \mathbb{R}$.

(b) The vector space $\mathrm{Hom}(V, W)$ of linear maps $f\colon V \to W$.

(c) The vector space $\mathrm{Mult}(V, W; \mathbb{R})$ of *multilinear maps* (or *bilinear*, since here f has two slots) $f\colon V \times W \to \mathbb{R}$, i.e., $f(v, w) \in \mathbb{R}$ is linear in each slot, that is, $f(v, w)$ is a linear function of $v \in V$ (for fixed $w \in W$) and $f(v, w)$ is a linear function of $w \in W$ (for fixed $v \in V$).

(d) The vector space $\mathsf{Sym}^2(V; \mathbb{R}) \subset \mathrm{Mult}(V, V; \mathbb{R})$ of symmetric bilinear maps $f\colon V \times V \to \mathbb{R}$, i.e., f is a bilinear map which is *symmetric* in the sense that $f(v_1, v_2) = f(v_2, v_1)$ for all $v_1, v_2 \in V$.

(e) The vector space $\mathrm{Alt}^2(V; \mathbb{R}) \subset \mathrm{Mult}(V, V; \mathbb{R})$ of alternating bilinear maps $f\colon V \times V \to \mathbb{R}$, i.e., f is a bilinear map which is *alternating* in the sense that $f(v_1, v_2) = -f(v_2, v_1)$ for all $v_1, v_2 \in V$.

(f) The vector space $\mathrm{Alt}^3(V; \mathbb{R})$ of *alternating multilinear maps* $f\colon V \times V \times V \to \mathbb{R}$, i.e., f is linear in each slot and is alternating in the sense that

$$f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) = \mathrm{sign}(\sigma) f(v_1, v_2, v_3)$$

for all $v_1, v_2, v_3 \in V$ and any permutation $\sigma \in S_3$. Here $\mathrm{sign}(\sigma) \in \{\pm 1\}$ is the sign of the permutation σ.

4. (15 points) The goal of this problem is to prove the following result.

Lemma 0.2. (Vector Bundle Construction Lemma). *Let M be a smooth manifold of dimension n, and let $\{E_p\}$ be a collection of vector spaces parametrized by $p \in M$. Let E be the set given by the disjoint union of all these vector spaces, which we write as*

$$E := \coprod_{p \in M} E_p = \{(p, v) \mid p \in M,\ v \in E_p\}$$

and let $\pi \colon E \to M$ be the projection map defined by $\pi(p, v) = p$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M, and let for each $\alpha \in A$, let $\Phi_\alpha \colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ be maps with the following properties

(i) The diagram

$$\begin{array}{ccc} E_{|U_\alpha} := \pi^{-1}(U_\alpha) & \xrightarrow{\quad \Phi_\alpha \quad} & U_\alpha \times \mathbb{R}^k \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U_\alpha & \end{array}$$

is commutative, where π_1 is the projection onto the first factor.

(ii) For each $p \in U_\alpha$, the restriction of Φ_α to $E_p = \pi^{-1}(p)$ is a vector space isomorphism between E_p and $\{p\} \times \mathbb{R}^k = \mathbb{R}^k$ (which implies that Φ_α is a bijection).

(iii) For $\alpha, \beta \in A$, the composition

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \xrightarrow{\Phi_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\Phi_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

is smooth.

Then the total space E can be equipped with the structure of a smooth manifold of dimension $n + k$ such that $\pi \colon E \to M$ is a smooth vector bundle of rank k with local trivializations Φ_α.

(a) Construct a topology on E by declaring $U \subset E$ to be *open* if $\Phi_\alpha(U \cap E_{|U_\alpha})$ is an open subset of $U_\alpha \times \mathbb{R}^k$ for all $\alpha \in A$. Show that this satisfies the conditions for a topology and that with this topology on E the map Φ_α is a homeomorphism (for the subspace topology on $E_{|U_\alpha}$).

(b) Show that equipped with this topology E is a topological manifold of dimension $n + k$ (don't bother to check the technical conditions of being Hausdorff and second countable). Hint: Let $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ be an atlas for M. Show that the bundle chart Φ_α and the manifold chart ψ_β can be used to construct a chart

$$\chi_{\alpha,\beta} \colon E \underset{\text{open}}{\supset} E_{|U_\alpha \cap V_\beta} \longrightarrow \mathbb{R}^{n+k}.$$

(c) Show that the charts $\{(E_{|U_\alpha \cap V_\beta}), \chi_{\alpha,\beta})\}$ for $(\alpha, \beta) \in A \times B$ form a smooth atlas for E.

(d) Show that $\pi\colon E \to M$ is a smooth vector bundle of rank k with local trivializations provided by Φ_α.