Homework Assignment # 1, due Sept. 6, 2019

1. (10 points) Let $GL_n(\mathbb{R})$ be the set of invertible $n \times n$ matrices.

(a) Show that $GL_n(\mathbb{R})$ is an open subset of the topological space $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ of all $n\times n$ matrices.

(b) Show that the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is a continuous map.

2. (10 points) Prove the *Continuity criterion for maps to a subspace*: Let Y be a topological space, and A a subspace of Y, i.e., a subset $A \subset Y$ equipped with the subspace topology.

(a) Show that the inclusion map $i\colon A \to Y$ is continuous.

(b) Show that a map $f\colon X \to A$ from a topological space X is continuous if and only if the composition $X \xrightarrow{f} A \xrightarrow{i} Y$ is continuous.

3. (10 points) The point of this problem is to show that the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ agrees with the product topology (where each factor is equipped with the metric topology). Since both, the metric topology and the product topology, are defined via a basis, it is good to know how to compare two topologies given in terms of bases. This is provided by the statement of part (a).

(a) Let X be a set, and let $\mathcal{T}$, $\mathcal{T}'$ be topologies generated by a basis $\mathcal{B}$ resp. $\mathcal{B}'$. Show that $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for each $B \in \mathcal{B}$ and $x \in B$ there is some $B' \in \mathcal{B}'$ with $x \in B'$ and $B' \subset B$.

(b) Show that the products of balls $B_r(x) \times B_s(y) \subset \mathbb{R}^m \times \mathbb{R}^n$ for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, $s, r > 0$ form a basis for the product topology on $\mathbb{R}^m \times \mathbb{R}^n$.

(c) Show that the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ agrees with the product topology. Hint: it might be helpful to draw pictures of a ball around $(x, y) \in \mathbb{R}^{m+n}$ and a product of balls $B_r(x) \times B_s(y) \subset \mathbb{R}^{m+n}$ for $m = n = 1$.

4. (10 points) Prove the *Continuity criterion for maps out of a quotient space*: let X be a topological space, let $Y = X/\sim$ be the quotient of X by a equivalence relation equipped with the quotient topology, and let $p\colon X \to Y$ be the projection map.

(a) Show that the projection map p is continuous.

(b) Show that a map $f\colon Y \to Z$ to a topological space Z is continuous if and only if the composition $X \xrightarrow{p} Y \xrightarrow{f} Z$ is continuous.