Homework Assignment # 9, due Nov. 7

1. Let M, N be smooth manifolds, let $F\colon M \to N$ be a smooth map, and $p \in M$.

(a) Show that the map

$$F^*\colon C^\infty_{F(p)}(N) \longrightarrow C^\infty_p(M) \qquad \text{given by} \qquad [f\colon V \to \mathbb{R}] \mapsto [F^{-1}(V) \xrightarrow{F_|} V \xrightarrow{f} \mathbb{R}]$$

is well-defined (here $F_|$ denotes the restriction of F to $F^{-1}(V) \subset M$).

(b) Show that if $\Phi\colon C^\infty_p(M) \to \mathbb{R}$ is a derivation, then the composition $C^\infty_{F(p)}(N) \xrightarrow{F^*} C^\infty_p(M) \xrightarrow{\Phi} \mathbb{R}$ is a derivation. In particular, we can define the (algebraic) differential

$$F^{\mathrm{alg}}_*\colon T^{\mathrm{alg}}_p M = \mathrm{Der}(C^\infty_p(M), \mathbb{R}) \longrightarrow T^{\mathrm{alg}}_{F(p)} N = \mathrm{Der}(C^\infty_{F(p)}(M), \mathbb{R})$$

by $F^{\mathrm{alg}}_*(\Phi) := \Phi \circ F^*$.

(c) If $G\colon N \to Q$ is a smooth map, show that

$$(G \circ F)^{\mathrm{alg}}_* = G^{\mathrm{alg}}_* \circ F^{\mathrm{alg}}_*.$$

We note that this statement is the *chain rule* (for the algebraic construction of the tangent space).

(d) Show that if $F\colon M \to N$ is a diffeomorphism, then the differential $F_*\colon T^{\mathrm{alg}}_p M \to T^{\mathrm{alg}}_{F(p)} N$ is an isomorphism of vector spaces. We note that this in particular implies that diffeomorphic smooth manifolds have the same dimension. We remark that the analogous statement holds for topological manifolds (i.e., homeomorphic topological manifolds have the same dimension), but the proof of this perhaps intuitively seemingly obvious statement is quite subtle and involves tools from algebraic topology (we could prove it for manifolds of dimension 2 using the fundamental group).

2. Let $F\colon M \to N$ be a smooth map, and let $F^{\mathrm{geo}}_*\colon T^{\mathrm{geo}}_p M \to T^{\mathrm{geo}}_{F(p)} N$ be the induced map on the *geometric tangent spaces*. We recall that an element of $T^{\mathrm{geo}}_p M$ is given by the equivalence class $[\gamma]$ of a smooth path $\gamma\colon (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and that F^{geo}_* sends $[\gamma]$ to $[F \circ \gamma] \in T^{\mathrm{geo}}_{F(p)} N$.

(a) Show that the chain rule holds for the differential (defined geometrically), i.e., if $F\colon M \to N$ and $G\colon N \to Q$ are smooth maps, then $(G \circ F)^{\mathrm{geo}}_* = G^{\mathrm{geo}}_* \circ F^{\mathrm{geo}}_*$.

(b) Let $D_M\colon T^{\mathrm{geo}}_p M \to T^{\mathrm{alg}}_p M$ be defined by $D_M([\gamma]) = d_\gamma$, where $d_\gamma\colon C^\infty_p(M) \to \mathbb{R}$ is the derivation given by $d_\gamma(f) = (f \circ \gamma)'(0)$ (in class we showed that D_M is a bijection).

Show that the geometrically defined differential F_*^{geo} is compatible with the algebraically defined differential F_*^{alg} in the sense that that following diagram is commutative:

$$\begin{tikzcd}
T_p^{\mathrm{geo}} M \arrow[r, "F_*^{\mathrm{geo}}"] \arrow[d, "D_M"'] & T_{F(p)}^{\mathrm{geo}} N \arrow[d, "D_N"] \\
T_p^{\mathrm{alg}} M \arrow[r, "F_*^{\mathrm{alg}}"] & T_{F(p)}^{\mathrm{alg}} N
\end{tikzcd}$$

3. Let M, N be smooth manifolds, and let $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ be the projection maps. Show that for any $(x, y) \in M \times N$ the map

$$\alpha \colon T_{(x,y)}(M \times N) \longrightarrow T_x M \oplus T_y N$$

defined by

$$\alpha(v) = ((\pi_1)_*(v), (\pi_2)_*(v))$$

is an isomorphism. Remark: Using this isomorphism, we will routinely identify $T_x M$ and $T_y N$ with subspaces of $T_{(x,y)}(M \times N)$.

4. Let $h \colon \mathbb{RP}^n \to \mathbb{R}$ be the function defined by

$$h([x_0, \dots, x_n]) = (\sum_{i=0}^{n} i x_i^2)(x_0^2 + x_1^2 + \cdots + x_n^2)^{-1}.$$

(a) Show that h is a well-defined smooth function.

(b) Determine the *critical points* of h, i.e., the points $p \in \mathbb{RP}^n$ where the differential $h_* \colon T_p\mathbb{RP}^n \to T_{h(p)}\mathbb{R} = \mathbb{R}$ vanishes.

Hint: Use the smooth atlas consisting of the charts $\mathbb{RP}^n \supset U_i \xrightarrow{\phi_i} \mathring{D}^n$ with

$$U_i = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_i \neq 0\} \quad \text{and} \quad \phi_i^{-1}(v_1, \dots, v_n) = [v_1, \dots, v_i, \sqrt{1 - ||v||^2}, v_{i+1}, \dots, v_n].$$