Homework Assignment # 7, due Oct. 10

1. Let X be the subspace of $\mathbb{R}^3$ given by the union of the 2-sphere S^2 and the segment S of the x-axis given by $S = \{(t,0,0) \in \mathbb{R}^3 \mid -1 \leq t \leq 1\}$. Calculate the fundamental group of X. Hint: use the Seifert van Kampen Theorem.

2. Let $p\colon \widetilde{X} \longrightarrow X$ be a covering space such that for each $x \in X$ the fiber $p^{-1}(x)$ is countable.

(a) Show that if X is a manifold of dimension n, then so is $\widetilde{X}$. Hint: to show that $\widetilde{X}$ has a countable basis, first show that X has a countable basis consisting of evenly covered open subsets.

(b) If X is compact manifold of dimension 2 and p is a d-fold covering map, what is the Euler characteristic of $\widetilde{X}$ in terms of the Euler characteristic of X? Here a d-fold covering means that each fiber $p^{-1}(x)$ consists of $d \in \mathbb{N}$ points.

(c) Suppose that $p\colon \widetilde{X} \longrightarrow X$ is a d-fold covering, $d \in \mathbb{N}$ where X is a surface of genus g and $\widetilde{X}$ is a surface of genus $\widetilde{g}$. Give a formula expressing $\widetilde{g}$ in terms of g and d.

3. Let $p\colon (E, e_0) \to (B, b_0)$ be a covering space, and let $f\colon (X, x_0) \to (B, b_0)$ be a map with X path-connected and locally path-connected. Show:

(a) A lift $\widetilde{f}\colon (X, x_0) \to (E, e_0)$ of f exists if and only if $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0))$. Hint: use the path lifting property for the covering space $E \to B$.

(b) There is at most one such lift.

4. Let $p\colon \widetilde{X} \to X$ be a covering map. Assume that $\widetilde{X}$ is path connected, let $\widetilde{x}_0 \in \widetilde{X}$ and let $x_0 = p(\widetilde{x}_0) \in X$.

(a) Show that the kernel of the induced homomorphism $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is trivial. Hint: use the lifting property for families of paths.

(b) Show that the map

$$\Phi\colon \pi_1(X, x_0) \longrightarrow p^{-1}(x_0) \qquad \text{defined by} \qquad [\gamma] \mapsto \widetilde{\gamma}(1)$$

is a well-defined surjection. Here $\widetilde{\gamma}\colon I \to \widetilde{X}$ is the lift of γ with $\widetilde{\gamma}(0) = \widetilde{x}_0$.

(c) Show that $\Phi(g_1) = \Phi(g_2)$ for $g_1, g_2 \in G := \pi_1(X, x_0)$ if and only if $Hg_1 = Hg_2$, where $Hg_i \subset G$ are right cosets for the subgroup $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq G$.

We note that putting parts (b) and (c) together, we obtain a bijection between the set of right cosets $H\backslash G$ and the fiber $p^{-1}(x_0)$.