**Homework Assignment # 4, due Sept. 19**

1. Show that the connected sum $\mathbb{RP}^2\#\mathbb{RP}^2$ of two copies of the real projective plane is homeomorphic to the Klein bottle $K$.

2. Let $\Sigma$, $\Sigma'$ be compact 2-manifolds. Show that the Euler characteristic of the connected sum $\Sigma\#\Sigma'$ is given by the following formula:

$$\chi(\Sigma\#\Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2.$$

3. Let $\gamma\colon I \to X$ be a path in a topological space $X$, and let $\bar{\gamma}\colon I \to X$ be the path $\gamma$ run backwards, that is, $\bar{\gamma}(s) = \gamma(1-s)$. Then there are the following homotopies

$$\gamma * \bar{\gamma} \simeq c_{\gamma(0)} \qquad \bar{\gamma} * \gamma \simeq c_{\gamma(1)} \qquad c_{\gamma(0)} * \gamma \simeq \gamma, \qquad \gamma * c_{\gamma(1)} \simeq \gamma \tag{1}$$

where $c_x$ for $x \in X$ denotes the constant path at $x$.

(a) Prove the first and the third homotopy.

(b) Use the homotopies (1) to finish our proof that $\pi_1(X, x_0)$ is a group. In other words, show that the constant map $c_{x_0}$ gives an identity element for $\pi_1(X, x_0)$, and that $[\bar{\gamma}]$ is the inverse for $[\gamma] \in \pi_1(X, x_0)$.

(c) Let $\beta$ be a path from $x_0$ to $x_1$. Show that the map

$$\Phi_\beta\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1) \qquad [\gamma] \mapsto [\bar{\beta} * \gamma * \beta]$$

is an isomorphism of groups. Note that this implies in particular that the isomorphism class of the fundamental group $\pi(X, x_0)$ of a path connected space does not depend on the choice of the base point $x_0 \in X$.

4. Let $\omega_n\colon (I, \partial I) \to (S^1, 1)$ be the based loop defined by $\omega_n(s) = e^{2\pi i n s}$. Show that the map $\Phi\colon \mathbb{Z} \to \pi_1(S^1, 1)$ given by $n \mapsto [\omega_n]$ is a group homomorphism.