Homework Assignment # 3, due Sept. 12

1. Which of the topological groups $GL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are connected? Hint: Use without proof the fact that every element in $SO(n)$ (the group of linear maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ which are isometries with determinant one) for a suitable choice of basis of $\mathbb{R}^n$ is represented by a matrix of block diagonal form whose diagonal blocks are the 1×1 matrix with entry $+1$ and/or 2×2 rotational matrices

$$R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Here "block diagonal" means that all other entries are zero.

2. Let M be a manifold of dimension m and let N be a manifold of dimension n. Show that the product $M \times N$ is a manifold of dimension $m+n$. Don't forget to check the technical conditions (Hausdorff and second countable) for $M \times N$.

3. Let $\sim$ be the equivalence relation on the disk D^n generated by $v \sim -v$ for $v \in S^{n-1} \subset D^n$. Show that the quotient space $D^n/\sim$ is homeomorphic to the real projective space $\mathbb{RP}^n$.

4. Let $N := (0, \dots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$ be the "north pole" of the n-sphere. Let

$$f\colon S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$$

be the *stereographic projection*, which maps a point $x \in S^n \setminus N$ to the unique intersection point of the straight line through the points N and x with the subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ (draw a picture for $n = 2$). Show that f is a homeomorphism.

Remark: This homework problem provides an alternative proof that S^n is a manifold of dimension n: the open subset $U^+ := S^n \setminus (0, \dots, 0, 1)$ is homeomorphic to $\mathbb{R}^n$ via the stereographic projection; since $U^- := S^n \setminus (0, \dots, 0, -1)$ is homeomorphic to U^+ by mapping $(x_0, \dots, x_{n-1}, x_n)$ to $(x_0, \dots, x_{n-1}, -x_n)$, then also U^- is homeomorphic to $\mathbb{R}^n$. Since the union of U^+ and U^- is all of S^n, this shows that S^n is locally homeomorphic to S^n.