Homework Assignment # 2, due Sept. 5

1. Consider the following topological spaces

- The subspace $T_1 := \{v \in \mathbb{R}^3 \mid d(v, C) = r\} \subset \mathbb{R}^3$ equipped with the subspace topology, where $C = \{(x, y, 0) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ is the unit circle in the xy-plane and $0 < r < 1$.
- The product space $T_2 := S^1 \times S^1$ equipped with the product topology.
- The quotient space $T_3 := ([-1, 1] \times [-1, 1])/ \sim$ equipped with the quotient topology, where the equivalence relation is generated by $(s, -1) \sim (s, 1)$ and $(-1, t) \sim (1, t)$.

Show that these three topological spaces are homeomorphic.

Hints: Produce first two continuous bijections between pairs of these spaces (an important part of this problem is to figure out which of these three spaces should be the (co)domains of the two maps you are looking for). Then argue that these maps are in fact homeomorphisms by proving that their domains are compact and their codomains are Hausdorff. The Heine-Borel Theorem (that will be covered in class on Friday) is helpful to show compactness of some of these spaces.

2. Show that a closed subspace C of a compact topological space X is compact.

3. Let X be a topological space which is the union of two subspaces X_1 and X_2. Let $f\colon X \to Y$ be a (not necessarily continuous) map whose restriction to X_1 and X_2 is continuous.

(a) Show f is continuous if X_1 and X_2 are open subsets of X.

(b) Show f is continuous if X_1 and X_2 are closed subsets of X.

(c) Give an example showing that in general f is *not continuous.*

Remark. This result is needed to verify that various constructions (e.g., concatenations of paths) in fact lead to *continuous* maps. In a typical situation, we have continuous maps $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ which agree on $X_1 \cap X_2$ and hence there is a well-defined map

$$f\colon X \longrightarrow Y \qquad \text{given by} \qquad f(x) = \begin{cases} f_1(x) & x \in X_1 \\ f_2(x) & x \in X_2 \end{cases}$$

The above result then helps to show that this map is continuous.

4. Use the Heine-Borel Theorem to decide which of the topological groups $GL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are compact. Provide proofs for your statements. Hint: A strategy often useful for proving that a subset C of $\mathbb{R}^n$ is closed is to show that C is of the form $f^{-1}(C')$ for some closed subset $C' \subset \mathbb{R}^k$ (often C' consists of just one point) and some continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^k$.