Homework Assignment # 12, Dec. 5

1. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Show that

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \det(T_x f) dx_1 \wedge \cdots \wedge dx_n,$$

where $T_x f\colon T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$ is the differential of f at the point $x \in \mathbb{R}^n$.

2. For any smooth manifold M the *de Rham differential* (also called *exterior differential*) is the unique map $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) d is linear.

(ii) For a function $f \in C^\infty(M) = \Omega^0(M)$ the 1-form $df \in \Omega^1(M) = \Gamma^\infty(M, T^*M)$ is the usual differential of f.

(iii) d is a graded derivation with respect to the wedge product; i.e.,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta \qquad \text{for } \omega \in \Omega^k(M),\ \eta \in \Omega^l(M).$$

(iv) $d^2 = 0$.

Show that for $M = \mathbb{R}^n$ the de Rham differential of the k-form

$$\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \text{for } f \in C^\infty(\mathbb{R}^n),\ i_1 < i_2 < \cdots < i_k$$

is given explicitly by the formula

$$d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We note that *every* k-form $\eta \in \Omega^k(\mathbb{R}^n)$ can be written uniquely in the form

$$\eta = \sum_{i_1 < \cdots < i_k} f_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for smooth functions $f_{i_1,\dots,i_k} \in C^\infty(\mathbb{R}^n)$.

3. Show that the exterior derivative for differential forms on $\mathbb{R}^3$ corresponds to the classical operations of *gradient* resp. *curl* resp. *divergence*. More precisely, show that there is a commutative diagram

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\text{curl}} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\
\| & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}$$

Here $\mathfrak{X}(\mathbb{R}^3)$ is the space of vector fields on $\mathbb{R}^3$, and we recall that grad, curl and divergence are given by the formulas

$$\begin{aligned}
\text{grad}(f) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\text{curl}(f_1, f_2, f_3) &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
\text{div}(f_1, f_2, f_3) &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}
\end{aligned}$$

Here we identify a vector field on $\mathbb{R}^3$ with a triple (f_1, f_2, f_3) of smooth functions on $\mathbb{R}^3$. The vertical isomorphisms are given by

$$(f_1, f_2, f_3) \mapsto f_1 dx + f_2 dy + f_3 dz \qquad (f_1, f_2, f_3) \mapsto f_1 dy\, dz + f_2 dz\, dx + f_3 dx\, dy \qquad f \mapsto f dx\, dy\, dz$$