Homework Assignment # 10, due Nov. 14

1. Let $M_{n\times k}(\mathbb{R})$ be the vector space of $n \times k$-matrices. For $A \in M_{n\times k}(\mathbb{R})$ let $A^t \in M_{k\times n}(\mathbb{R})$ be the transpose of A, and let $\mathsf{Sym}(\mathbb{R}^k) = \{B \in M_{k\times k}(\mathbb{R}) \mid B^t = B\}$ be the vector space of *symmetric $k \times k$-matrices.*

(a) Show that the map $\Phi\colon M_{n\times k}(\mathbb{R}) \to \mathsf{Sym}(\mathbb{R}^k)$, $A \mapsto A^tA$ is smooth, and that its differential

$$\Phi_*\colon T_AM_{n\times k}(\mathbb{R}) = M_{n\times k}(\mathbb{R}) \longrightarrow T_{\Phi(A)}\mathsf{Sym}(\mathbb{R}^k) = \mathsf{Sym}(\mathbb{R}^k)$$

is given by $\Phi_*(C) = C^tA + A^tC$. Hint: Use the geometric description of tangent spaces. More explicitly, the tangent space $T_AM_{n\times k}(\mathbb{R})$ can be identified with $M_{n\times n}(\mathbb{R})$ by sending a matrix $C \in M_{n\times n}(\mathbb{R})$ to the path $\gamma(t) := A + tC$.

(b) Show that the identity matrix is a regular value of the map Φ.

(c) What is the dimension of the smooth manifold $V_k(\mathbb{R}^n)$, which we showed in class is equal to Φ^{-1}(identity matrix)?

We remark that identifying $M_{n\times k}(\mathbb{R})$ in the usual way with the vector space $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n)$ of linear maps $f\colon \mathbb{R}^k \to \mathbb{R}^n$, a matrix belongs to $V_k(\mathbb{R}^n)$ if and only if the corresponding linear map f is an *isometry*, that is, if f preserves the length of vectors in the sense that $||f(v)|| = ||v||$, or equivalently, if f preserves the scalar product in the sense that

$$\langle f(v), f(w)\rangle = \langle v, w\rangle \qquad \text{for all } v, w \in \mathbb{R}^k.$$

The manifold $V_k(\mathbb{R}^n)$ is called the *Stiefel manifold*. We observe that $V_n(\mathbb{R}^n)$ is the orthogonal group $O(n)$ of isometries $\mathbb{R}^n \to R^n$.

2. Recall that the special linear group $SL_n(\mathbb{R})$ and the orthogonal group $O(n)$ are both submanifolds of the vector space $M_{n\times n}(\mathbb{R})$ of $n \times n$ matrices. In particular, the tangent spaces $T_ASL_n(\mathbb{R})$ for $A \in SL_n(\mathbb{R})$ and $T_AO(n)$ for $A \in O(n)$ are subspaces of the tangent space $T_AM_{n\times n}(\mathbb{R})$, which can be identified with $M_{n\times n}(\mathbb{R})$, since $M_{n\times n}(\mathbb{R})$ is a vector space.

(a) Show that $T_eSL_n(\mathbb{R}) = \{C \in M_{n\times n} \mid \mathrm{tr}(C) = 0\}$, where e is the identity matrix, and $\mathrm{tr}(C)$ denotes the trace of the matrix C. Hint for parts (a) and (b): $SL_n(\mathbb{R})$ and $O(n)$ can be both be described as level sets $F^{-1}(c)$ of a regular value c for a suitable smooth map F.

(b) Show that $T_eO(n) = \{C \in M_{n\times n} \mid C^t = -C\}$.

(c) Let $G \subset M_{n\times n}(\mathbb{R})$ be either the group $SL_n(\mathbb{R})$ or the group $O(n)$. For $A \in G$ let $L_A\colon G \to G$ be the map given by left multiplication by A, i.e., $B \mapsto AB$. Show that the differential

$$(L_A)_*\colon T_BG \longrightarrow T_{AB}G \qquad \text{is given by} \qquad C \mapsto AC,$$

where we identify all of these tangent spaces as subspaces of $M_{n\times n}(\mathbb{R})$. Hint: Compute first the differential of the map $\mathbb{M}_{n\times n}(\mathbb{R}) \to \mathbb{M}_{n\times n}(\mathbb{R})$, $B \mapsto AB$, and then compare with $(L_A)_*$.

(d) Use parts (a)–(c) to determine the tangent space $T_AG \subset M_{n\times n}(\mathbb{R})$ for $A \in G$ and $G = SL_n(\mathbb{R})$, as well as $G = O(n)$.

3. Show that the projection map $p\colon S^{2n+1} \to \mathbb{CP}^n$ is a *submersion*, i.e., the differential $p_*\colon T_z S^{2n+1} \to T_{p(z)}\mathbb{CP}^n$ is surjective for every point $z \in S^{2n+1}$. (In particular, each fiber $p^{-1}(L)$, $L \in \mathbb{CP}^n$ is a submanifold of dimension 1). Hint: The projection map p extends to a projection map $P\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Argue first that P is a submersion, using the obvious identification of tangent spaces of the domain with $\mathbb{C}^{n+1}$, and using the differential of our standard charts for $\mathbb{CP}^n$ to identify the tangent spaces of the range with $\mathbb{C}^n$.

4. Let M be a smooth manifold of dimension n. If $f\colon M \to \mathbb{R}$ is a smooth function, then for $p \in M$ its differential

$$f_*\colon T_pM \longrightarrow T_{f(p)}\mathbb{R} = \mathbb{R}$$

is an element of $\operatorname{Hom}(T_pM, \mathbb{R})$. This vector space dual to the tangent space T_pM is called the *cotangent space*, and is denoted T_p^*M. It is common to write $df_p \in T_p^*M$ for the differential $f_*\colon T_pM \to \mathbb{R}$.

(a) Let $x^i\colon \mathbb{R}^n \to \mathbb{R}$ be the i-th coordinate function, which maps $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ to $x_i \in \mathbb{R}$. Show that for any point $q \in \mathbb{R}^n$ a basis of the cotangent space $T_q^*\mathbb{R}^n$ is given by $\{dx_q^i\}_{i=1,\dots,n}$.

(b) If $M \supset U \xrightarrow{\phi} V \subset \mathbb{R}^n$ is a smooth chart of M, the component functions of ϕ, given by $y^i := x^i \circ \phi$ are called *local coordinates*. Show that for $p \in U$, a basis of the cotangent space T_p^*M is given by $\{dy_p^i\}_{i=1,\dots,n}$.

Hint for part (b): let $\phi^*\colon T_q^*\mathbb{R}^n \to T_p^*M$, $q = \phi(p)$ be the linear map dual to the differential $\phi_*\colon T_pM \to T_q\mathbb{R}^n$ defined by

$$(\phi^*\xi)(v) = \xi(\phi_*(v)) \qquad \text{for } \xi \in T_q^*\mathbb{R}^n \text{ and } v \in T_pM.$$

Show first that $\phi^*(dx_q^i) = dy_p^i$.