Homework Assignment # 1, due Aug. 29, 2018

1. Show that the ϵ-δ definition of continuity of a map $f\colon X \to Y$ between metric spaces is equivalent to the definition of continuity in terms of open subsets.

2. a) Suppose $\mathcal{T}$, $\mathcal{T}'$ are topologies on a set X which are generated by bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Show that $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for each $B \in \mathcal{B}$ and $x \in B$ there is some $B' \in \mathcal{B}'$ with $x \in B'$ and $B' \subset B$.

b) Show that equivalent metrics d, d' on a set X lead to the same metric topology on X.

c) Show that the metric topology on $\mathbb{R}^n$ determined by the Euclidean metric is equal to the metric topology determined by the maximum-metric

$$d_{\max}(x, y) := \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

d) Show that the product topology on $\mathbb{R}^m \times \mathbb{R}^n$ (with each factor equipped with the metric topology) agrees with the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. Hint: Use the freedom provided by part (c) to choose whether to work with the Euclidean metric or the maximum-metric to make this easier.

3. Let X, Y_1, Y_2 be topological spaces. Show that

(a) the projection maps $p_i\colon Y_1 \times Y_2 \to Y_i$ are continuous, and

(b) a map $f\colon X \to Y_1 \times Y_2$ is continuous if and only if the *component maps* $f_i := p_i \circ f$ are continuous for $i = 1, 2$.

4. Show that the map $f\colon GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is continuous (here $GL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ is equipped with the metric topology).