Algebraic Topology

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These are incomplete notes of a second semester basic topology course taught in the Spring of 2016. A basic reference is Allen Hatcher's book [Ha].

Contents

1 Introduction

The following is an attempt at explaining what 'topology' is.

- Topology is the study of qualitative/global aspects of shapes, or – more generally – the study of qualitative/global aspects in mathematics.

A simple example of a 'shape' is a 2-dimensional surface in 3-space, like the surface of a ball, a football, or a donut. While a football is different from a ball (try kicking one...), it is *qualitatively* the same in the sense that you could squeeze a ball (say a balloon to make squeezing easier) into the shape of a football. While any surface is *locally homeomorphic* $\mathbb{R}^2$ (i.e., every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^2$) by definition of 'surface', the 'global shape' of two surfaces might be different meaning that they are not homeomorphic (e.g. the surface of a ball is not homeomorphic to the surface of a donut). The French mathematician Henry Poincaré (1854-1912) is regarded as one of the founders of topology, back then known as 'analysis situ'. He was interested in understanding qualitative aspects of the solutions of differential equation.

There are basically three flavors of topology:

1. Point set Topology: Study of general properties of topological spaces
2. Differential Topology: Study of manifolds (ideally: classification up to homeomorphism/diffeomorphism).
3. Algebraic topology: trying to distinguish topological spaces by assigning to them algebraic objects (e.g. a group, a ring, ...).

Let us go in more detail concerning algebraic topology, since that is the topic of this course. Before mentioning two examples of algebraic objects associated to topological spaces, let us make the purpose of assigning these algebraic objects clear: if X and Y are *homeomorphic* objects, we insist that the associated algebraic objects $A(X)$, $A(Y)$ are isomorphic. That means in particular, that if we find that $A(X)$ and $A(Y)$ are not isomorphic, then we can conclude that the spaces X and Y are not homeomorphic. In other words, the algebraic objects help us to *distinguish homeomorphism classes of topological spaces.*

Here are two examples of algebraic objects we can assign to topological spaces, which satisfy this requirement. We will discuss them in more detail below:

Homotopy groups To any topological space X equipped with a distinguished point $x_0 \in X$ (called the *base point*), we can associate groups $\pi_n(X, x_0)$ for $n = 1, 2, \dots$ called *homotopy groups* of X. These are *abelian* groups for $n \geq 2$.

Homology groups To any topological space X we can associate abelian groups $H_n(X)$ for $n = 0, 1, \dots,$ called *homology groups of* X.

The advantages/disadvantages of homotopy versus homology groups are

- homotopy groups are easy to define, but extremely hard to calculate;

- homology groups are harder to define, but comparatively easier to calculate (with the appropriate tools in place, which will take us about half the semester)

Let us illustrate these statements in a simple example. We will show (in about a month) that the homology group of spheres look as follows:

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$

The homotopy groups of spheres are much more involved; for example:

k	1	2	3	4	5	6	7	8	9
$\pi_k(S^2, x_0)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$

It is perhaps surprising that these homotopy groups are *not known* for large k (not only in the sense that we don't have a 'closed formula' for these groups, but also in the sense that we don't have an algorithm that would crank out these groups one after another on a computer if we just give it enough time...). This holds not only for S^2, but for any sphere S^n (except $n = 1$). In fact, the calculation of the homotopy groups of spheres is something akin to the 'holy grail' of algebraic topology.

1.1 Homotopy groups

Suppose f and g are continuous maps from a topological space X to a topological space Y. Then true to the motto that in topology we are interested in 'qualitative' properties we shouldn't try to distinguish between f and g if they can be 'deformed' into each other in the sense that for each $t \in [0,1]$ there is a map $f_t \colon X \to Y$ such that $f_0 = f$ and $f_1 = g$, and such that the family of maps f_t 'depends continuously on t'. The following definition makes precise what is meant by 'depending continuously on t' and introduces the technical terminology 'homotopic' for the informal 'can be deformed into each other'.

Definition 1.1. Two continuous maps $f, g \colon X \to Y$ between topological spaces X, Y are *homotopic* if there is a continuous map $H \colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The map H is called a *homotopy between f and g*. We will denote by $[X, Y]$ the set of homotopy classes of maps from X to Y.

We note that if H is a homotopy, then we have a family of maps $f_t \colon X \to Y$ parametrized by $t \in [0,1]$ interpolating between f and g, given by $f_t(x) = H(t,x)$. Conversely, if $f_t \colon X \to Y$ is a family of maps parametrized by $t \in [0,1]$, then we can define a map $H \colon [0,1] \times X \to Y$ by the above formula. We note that the continuity requirement for H implies not only that each map f_t is continuous, but also implies that for fixed $x \in X$ the map $t \mapsto f_t(x)$ is continuous. In other words, our idea of requiring that f_t should 'depend continuously on t' is made precise by requiring continuity of H.

Examples of homotopic maps.

1. Any two maps $f, g\colon X \to \mathbb{R}$ are homotopic; in other words, $[X, \mathbb{R}]$ is a one point set. A homotopy $H\colon X \times [0,1] \to \mathbb{R}$ is given by $H(x,t) = (1-t)f(x) + tg(x)$. We note that for fixed x the map $[0,1] \to \mathbb{R}$ given by $t \mapsto (1-t)f(x) + tg(x)$ is the affine linear path (aka straight line) from $f(x)$ to $g(x)$. For this reason, the homotopy H is called a *linear homotopy.* The construction of a linear homotopy can be done more generally for maps $f, g\colon X \to Y$ if Y is a vector space, or a convex subspace of a vector space.

2. A map $S^1 \to Y$ is a loop in the space Y. Physically, we can think of it as the trajectory of a particle that moves in the topological space Y, returning to its original position after some time. In general, there are maps $f, g\colon S^1 \to Y$ that are not homotopic. For example, given an integer $k \in \mathbb{Z}$, let

   $$f_k\colon S^1 \to S^1 \qquad \text{be the map given by} \qquad f_k(z) = z^k.$$

   Physically that describes a particle that moves $|k|$ times around the circle, going counterclockwise for $k > 0$ and clockwise for $k < 0$. We will prove that f_k and f_ℓ are homotopic if and only if $k = \ell$. Moreover, we will show that *any* map $f\colon S^1 \to S^1$ is homotopic to f_k for some $k \in \mathbb{Z}$. In other words, we will prove that there is a bijection

   $$\mathbb{Z} \leftrightarrow [S^1, S^1] \qquad \text{given by} \qquad k \mapsto f_k$$

   This fact will be used to prove the fundamental theorem of algebra.

Sometimes it is useful to consider pairs (X, A) of topological spaces, meaning that X is a topological space and A is a subspace of X. If (Y, B) is another pair, we write

$$f\colon (X, A) \longrightarrow (Y, B)$$

if f is a continuous map from X to Y which sends A to B. Two such maps $f, g\colon (X, A) \to (Y, B)$ are *homotopic* if there is a map

$$H\colon (X \times I, A \times I) \longrightarrow (Y, B)$$

with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We will use the notation $[(X, A), (Y, B)]$ for the set of homotopy classes of maps from (X, A) to (Y, B).

Definition 1.2. Let X be a topological space, and let x_0 be a point of X. Then the *n-th homotopy group of* (X, x_0) is by definition

$$\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)].$$

Here $I^n := \underbrace{I \times \cdots \times I}_{n} \subset \mathbb{R}^n$ is the n-dimensional cube, and ∂I^n is its boundary.

A map $f\colon (I,\partial) \to (X,x_0)$ is geometrically a path in X parametrized by the unit interval $I=[0,1]$ with starting point $f(0)=x_0$ and endpoint $f(1)=x_0$. Such maps are also called *based loops*. Similarly, a map $f\colon (I^2,\partial I^2) \to (X,x_0)$ is geometrically a membrane in X parametrized by the square I^2, such that the boundary of the square maps to the base point x_0.

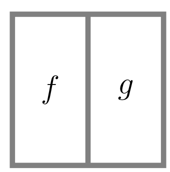
As suggested by the terminology of the above definition, the set $[(I^n,\partial I^n),(X,x_0)]$ has in fact the structure of a *group*. Given two maps $f,g\colon (I^n,\partial I^n)\to (X,x_0)$, their product $f*g\colon (I^n,\partial I^n)\to (X,x_0)$ is given by

$$(f*g)(t_1,\dots,t_n) := \begin{cases} f(2t_1,t_2,\dots,t_n) & \text{for } 0\le t_1\le \frac{1}{2} \\ g(2t_1-1,t_2,\dots,t_n) & \text{for } \frac{1}{2}\le t_1\le 1 \end{cases}$$

We note that this is a well-defined map, since for $t_1=\frac{1}{2}$ the points $(2t_1,t_2,\dots,t_n)$ and $(2t_1-1,t_2,\dots,t_n)$ both belong to the boundary ∂I^n, and hence both map to x_0 via f and g. Moreover, $f*g$ is continuous since its restriction to the closed subsets consisting of the points $t=(t_1,\dots,t_n)$ with $t_1\le\frac{1}{2}$ resp. $t_1\ge\frac{1}{2}$ is continuous. We will refer to $f*g$ as the *concatenation* of the maps f and g, since for $n=1$ the map $f*g\colon I\to X$ is usually referred to as the concatenation of the paths f and g.

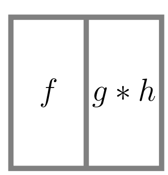
The following picture shows where $f*g$ maps points in the square I^2: if $t=(t_1,t_2)$ belongs to the left half of the square, it is mapped via f; points in the right half map via g (here we implicitly identify the left and right halves of the square again with I^2). In particular the boundaries of the two halves map to the base point x_0; this subset of I^2 is indicated by the gray lines in the picture.

$f*g=$



Next we want to address the question whether given $f,g,h\colon (I^n,\partial I^n)\to (X,x_0)$ the maps $f*(g*h)$ and $(f*g)*h$ agree. Thinking in terms of pictures, we have

$f*(g*h) \;=$



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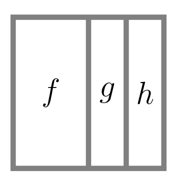
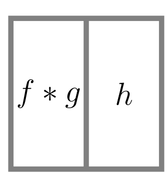


$(f*g)*h \;=$



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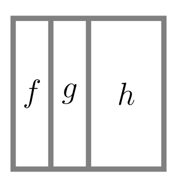


which shows that these two maps do not agree. However, they are homotopic to each other. We leave it to the reader to provide a proof of this. This implies the third of the following equalities in $\pi_n(X, x_0)$; the others hold by definition:

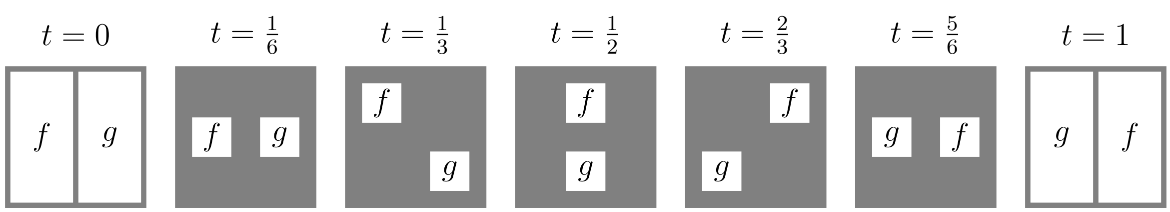
$$[f]([g][h]) = [f]([g * h]) = [f * (g * h)] = [(f * g) * h] = [f * g][h] = ([f][g])[h].$$

This shows that concatenation induces an associative product on $\pi_n(X, x_0)$. We leave it to the reader to show that this product gives $\pi_n(X, x_0)$ of a group where the unit element is represented by the constant map, and the inverse of an element $[f] \in \pi_n(X, x_0)$ is represented by $\bar{f}$, defined by $\bar{f}(t_1, \ldots, t_n) := f(1 - t_1, t_2, \ldots, t_n)$.

The group $\pi_1(X, x_0)$ is called the *fundamental group* of X, while the groups $\pi_n(X, x_0)$ for $n \geq 2$ are referred to as *higher homotopy groups*. Examples show that the fundamental group is in general not abelian. For example, the fundamental group of the "figure eight" is the free group generated by two elements. By contrast, for higher homotopy groups we have the following result.

Lemma 1.3. *For $n \geq 2$ the group $\pi_n(X, x_0)$ is abelian.*

Proof. We need to show that for maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ the concatenations $f * g$ and $g * f$ are homotopic to each other (as maps of pairs). Such a homotopy H is given by a continuous family of maps $H_t\colon (I^n, \partial I^n) \to (X, x_0)$ which agrees with $f * g$ for $t = 0$ and with $g * f$ for $t = 1$. Thinking of each such maps as a picture, like the one for $f * g$ above, such a homotopy H_t is a family of pictures parametrized by $t \in [0, 1]$. Interpreting t as "time", this means that the homotopy H_t is a *movie*! Here it is:



Here all points in the gray areas of the square map to the base point. So shrinking the rectangles inside of the square labeled f resp. g allows us to rotate them past each other, a move which is not possible for $n = 1$, but for all $n \geq 2$. $\square$

1.2 The Euler characteristic of closed surfaces

The goal of this section is to discuss the Euler characteristic of closed surfaces, that is, compact manifolds without boundary of dimension 2. We begin by recalling the definition of manifolds.

Definition 1.4. A *manifold of dimension* n or n*-manifold* is a topological space X which is locally homeomorphic to $\mathbb{R}^n$, that is, every point $x \in X$ has an open neighborhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$. Moreover, it is useful and customary to require that X is Hausdorff (see Definition 2.29) and second countable (see Definition 3.1). A *manifold with boundary of dimension* n is defined by replacing $\mathbb{R}^n$ in the definition above by the half-space $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$. If X is an n-manifold with boundary, its *boundary* ∂X consists of those points of X which via some homeomorphism $U \approx V \subset \mathbb{R}^n_+$ correspond to points in the hyperplane given by the equation $x_1 = 0$. The complement $X \setminus \partial X$ is called the *interior of* X. A *closed* n*-manifold* is a compact n-manifold without boundary.

Examples of manifolds of dimension 1. An open interval (a, b) is a 1-manifold. A closed interval $[a, b]$ is a 1-manifold with boundary $\{a, b\}$. A half-open interval $(a, b]$ is a 1-manifold with boundary $\{b\}$.

A non-example. The subspace $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0\}$ of $\mathbb{R}^2$ consisting of the x-axis and y-axis is not a 1-dimensional manifold, since X is not locally homeomorphic to $\mathbb{R}^1$ at the origin $x = (0, 0)$. To prove this intuitively obvious fact, suppose that U is an open neighborhood of $(0, 0)$ which is homeomorphic to an open subset $V \subset \mathbb{R}$. Replacing U by the connected component of U containing $(0, 0)$, and V by the image of that component, we can assume that U and V are connected. This implies that V is an open interval. Restricting the homeomorphism $f \colon U \to V$, we obtain a homeomorphism $U \setminus \{(0, 0)\} \approx V \setminus f(0, 0)$. This is the desired contradiction, since $U \setminus \{(0, 0)\}$ has four connected components, while $V \setminus f(0, 0)$ has two.

Examples of higher dimensional manifolds.

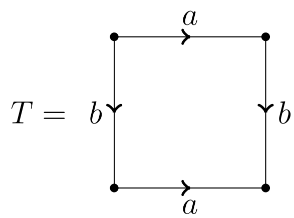
1. Any open subset $U \subset \mathbb{R}^n_+$ is an n-manifold whose boundary ∂U is the intersection of U with the hyperplane $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$.

2. The n-sphere $S^n := \{x \in \mathbb{R}^n \mid ||x|| = 1\}$ is an n-manifold.

3. The n-disk $D^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is an n-manifold with boundary $\partial D^n = S^{n-1} = \{x \in \mathbb{R}^n \mid ||x|| = 1\}$.

4. The *torus* $T := S^1 \times S^1$ is a manifold of dimension 2. There at least two other ways to describe the torus. The usual picture we draw describes the torus as a subspace of $\mathbb{R}^3$. It can also be constructed as a quotient space of the square I^2: we identify the two horizontal edges of the square to obtain a cylinder, and then the two boundary circles to obtain the torus T. From a formal point of view, the last sentence describes an equivalence relation $\sim$ on I^2, and the claim is that the quotient space $I^2/\sim$ is homeomorphic to $S^1 \times S^1$. It will be convenient to use pictures for this and similar

quotient spaces. Here is the picture for the quotient space $I^2/\sim$ described above:



(1.5)

Question. Is the sphere S^2 homeomorphic to the torus T?

It seems intuitively clear that the answer is "no", but how do we prove that rigorously? The usual strategy for showing that two topological spaces aren't homeomorphic involves thinking of adjectives for topological spaces (e.g., compact, connected, Hausdorff, second countable, etc), and to show that one has some such property while the other doesn't. It turns out that the usual adjectives from point-set topology will not manage to distinguish these space. Instead, we will associate an integer to closed surfaces, called the Euler characteristic, that will distinguish S^2 and T. This is the most basic "algebraic topological invariant" for spaces.

The definition of the Euler characteristic of a closed 2-manifold Σ will involve choosing a "pattern of polygons" on Σ. By this we mean a graph Γ (consisting of vertices and edges) on Σ, such that all connected components of the complement $\Sigma \setminus \Gamma$ are homeomorphic to open discs. For example, the boundary of the 3-dimensional cube is a 2-dimensional manifold homeomorphic to S^2. The 8 vertices and 12 edges of the cube form a graph Γ on S^2; the complement $S^2 \setminus \Gamma$ consists of the 6 faces of the cube.

Given a pattern of polygons Γ on a surface Σ, we define the integer

$$\chi(\Sigma, \Gamma) := \#V - \#E + \#F,$$

where V is the set of vertices, E is the set of edges, and F is the set of faces.

Lemma 1.6. $\chi(\Sigma, \Gamma) = \chi(\Sigma, \Gamma')$ *for any two choices of graphs* Γ, Γ'.

Before proving this lemma, let us illustrate the statement in the example of two patterns on the 2-sphere S^2:

1. Let Γ be the graph described above obtained by identifying S^2 with the boundary of the cube. Then $\chi(S^2, \Gamma) = 8 - 12 + 6 = 2$.

2. Let Γ' be the graph obtained by identifying S^2 with the boundary of the tetrahedron. Then $\chi(S^2, \Gamma') = 4 - 6 + 4 = 2$.

Proof. We begin by proving the statement in the special case were the graph Γ' is obtained from Γ by adding an new edge. Then the number of vertices is the same for Γ and Γ', and the number of edges Γ' is the number of edges for Γ plus one. Similarly, the number of faces of Γ' is one larger than that for Γ, since the new edge subdivides one face for Γ into two faces for Γ'. Hence $\chi(\Gamma') = \chi(\Gamma)$.

Similarly, if Γ' is obtained from Γ by introducing a new vertex on one of the edges of Γ, then the number of vertices and edges goes up by one, while the number of faces doesn't change. Again, this implies that $\chi(\Gamma') = \chi(\Gamma)$ in this case as well. More generally, if Γ' is a *refinement* of Γ in the sense that Γ' is obtained from Γ by adding new edges and vertices, we see that $\chi(\Gamma') = \chi(\Gamma)$.

Finally, for general graphs Γ, Γ' we may assume without changing the number of vertices, edges or faces that Γ, Γ' are in general position to each other in the sense that the vertex sets $V(\Gamma)$, $V(\Gamma')$ are disjoint and that edges of Γ intersect those of Γ' in finitely many points. Then the union of the graphs Γ and Γ' can again be viewed as a graph Γ'' on Σ. For example, the vertices of Γ'' consist of the vertices of Γ, the vertices of Γ' and the intersection points of edges in Γ with edges in Γ'. The graph Γ'' is a refinement of both Γ and Γ', and hence $\chi(\Gamma') = \chi(\Gamma'') = \chi(\Gamma)$. □

Definition 1.7. The *Euler characteristic* of a closed surface Σ, denoted $\chi(\Sigma) \in \mathbb{Z}$ is defined to be $\chi(\Sigma, \Gamma)$ for any pattern of polygons on Σ.

The following is a simple consequence of Lemma 1.6 and the definition of the Euler characteristic.

Corollary 1.8. *If Σ, Σ' are closed surfaces with $\chi(\Sigma) \neq \chi(\Sigma')$, then these surfaces are not homeomorphic.*

Proof. Suppose that there is a homeomorphism $f\colon \Sigma \xrightarrow{\approx} \Sigma'$. If Γ is a pattern of polygons on Σ, let Γ' be the pattern of polygons on Σ' whose vertices (resp. edges resp. faces) are the images of vertices (resp. edges resp. faces) of Γ under the map f. Then

$$\chi(\Sigma) = \chi(\Sigma, \Gamma) = \chi(\Sigma', \Gamma') = \chi(\Sigma')$$

is the desired contradiction. □

Corollary 1.9. *The sphere S^2 is not homeomorphic to the torus T.*

Proof. By the previous result it suffices to show $\chi(S^2) \neq \chi(T)$. By our calculations above we know $\chi(S^2) = 2$. We claim that $\chi(T) = 0$. To prove this, we use as "pattern of polygons" on T the picture (1.5), which has

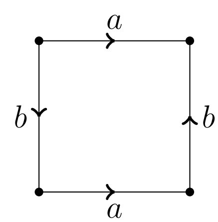
- one face (the square);

- two edges (labeled a resp. b). There are only two rather than four edges since the two edges of the square labeled a lead to the *same edge* on the torus;
- one vertex; all four vertices of the square lead to the same vertex on the torus.

It follows that $\chi(T) = 1 - 2 + 1 = 0$. □

Other examples of closed surfaces and their Euler characteristic.

Klein bottle Like the torus, the Klein bottle K can be constructed as the quotient space of the square I^2 by identifying opposite edges of the square. Here is the picture:

$K =$ 

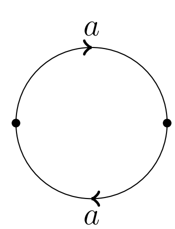
(1.10)

Like for the torus, the Euler characteristic of the Klein bottle can be calculated by using the "pattern of polygons" on K given by the above picture to obtain

$$\chi(K) = 1 - 2 + 1 = 0.$$

Real projective plane From an algebraic perspective, the real projective plane $\mathbb{RP}^2$ is the set of lines (= 1-dimensional subspaces of $\mathbb{R}^3$). To describe the usual topology on $\mathbb{RP}^2$, it is useful to identify this set of lines with the quotient $S^2/x \sim -x$ of the 2-sphere obtained by identifying antipodal points (the bijection is given by sending a point $x \in S^2$ to the line spanned by the unit vector x; since x and $-x$ span the same line, this gives a well-defined map $S^2/\sim \to \{\text{lines in } \mathbb{R}^3\}$ which is easily seen to be bijective). The usual topology of S^2 then induces the quotient topology on $\mathbb{RP}^2 = S^2/\sim$.

Another way to think of $\mathbb{RP}^2$ comes from noting that any equivalence class $[x] \in S^2/\sim$ is represented by a point in the upper hemisphere $S^2_+ = \{(x_1, x_2, x_3) \in S^2 \mid x_3 \geq 0\}$ which can be identified with the disk D^2 by sending $(x_1, x_2, x_3) \in S^2_+$ to $(x_1, x_2) \in D^2$. This shows that $\mathbb{RP}^2$ is homeomorphic to the quotient $D^2/\sim$, where the equivalence relation identifies antipodal points on the boundary of D^2. In other words, it identifies points of the upper semicircle with the corresponding points in the lower semicircle as indicated by the following picture.

$\mathbb{RP}^2 \approx$ 

(1.11)

Interpreting this picture as a pattern of polygons on $\mathbb{RP}^2$, we see that

$$\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$$

Another way to calculate the Euler characteristic of the real projective plane is to note that the projection map

$$S^2 \longrightarrow S^2/\sim \quad = \quad \mathbb{RP}^2$$

is a double covering. The following implies that $\chi(S^2) = 2\chi(\mathbb{RP}^2)$ and hence $\chi(\mathbb{RP}^2) = 1$ by our previous calculation of the Euler characteristic of the sphere.

Lemma 1.12. *If Σ is a closed surface and $p\colon \widetilde{\Sigma} \to \Sigma$ is a d-fold covering map, then $\chi(\widetilde{\Sigma}) = d\chi(\Sigma)$.*

The proof of this lemma is a homework problem.

1.3 Homology groups of surfaces

The goal in this section is to define homology groups for closed surfaces. These are abelian groups $H_n(\Sigma)$ associated to a closed surface Σ for $n \in \mathbb{Z}$. We begin by a quick review of abelian groups.

Digression on abelian groups. We will write abelian groups A additively, that is, we write $a + b \in A$ for the group operation applied to two elements $a, b \in A$, and $-a$ for the inverse of a. As usual, we write na as shorthand for the n-fold sum $a + \cdots + a$ of an element $a \in A$, and $-na$ for the n-fold sum $(-a) + \cdots + (-a)$. In particular, we can multiply an element $a \in A$ with any integer. This multiplication gives A the structure of a $\mathbb{Z}$-module. In fact, *abelian group* and *$\mathbb{Z}$-module* are just two different names for the same mathematical structure.

Here are some examples of abelian groups (aka $\mathbb{Z}$-modules).

- The infinite cyclic group $\mathbb{Z}$;
- The cyclic group $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$ of order n. Here $n\mathbb{Z} \subset \mathbb{Z}$ is the subgroup consisting of integers divisible by n. We write $[i] \in \mathbb{Z}/n$ for the coset represented by $i \in \mathbb{Z}$.
- If A, B are abelian groups, we can form their sum $A \oplus B$. The elements of this abelian group are all pairs (a, b) with $a \in A$, $b \in B$. The sum of two such pairs is given by $(a, b) + (a', b') = (a + a', b + b')$.
- If S is a set, the *free abelian group generated by S* or *free $\mathbb{Z}$-module generated by S*, denoted $\mathbb{Z}[S]$ is defined by

$$\mathbb{Z}[S] := \left\{ \sum_{s \in S} n_s s \mid n_s \in \mathbb{Z},\ n_s \neq 0 \text{ for only finitely many } s \in S \right\}$$

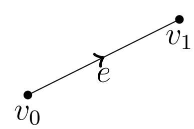
In other words, the elements of $\mathbb{Z}[S]$ consist of the finite linear combinations of elements of S with integer coefficients.

We recall the following important facts about abelian groups:

- The sum $\mathbb{Z}/m \oplus \mathbb{Z}/n$ is isomorphic to $\mathbb{Z}/mn$ if and only if m is prime to n.
- Any finitely generated group is isomorphic to a sum of $\mathbb{Z}$'s and $\mathbb{Z}/n$'s. Without loss of generality we can assume that the n's are powers of primes. We recall that an abelian group A is *finitely generated* if there are finitely many elements $a_i \in A$ such that every element $a \in A$ can be expressed as a linear combination of the a_i's.

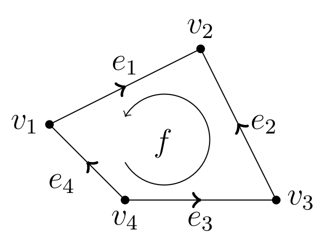
Like for the definition of the Euler characteristic, the definition of the homology group $H_n(\Sigma)$ for a closed surface Σ requires us to first choose some additional structure on the surface. To construct the homology groups, we need to make the following choices:

1. The choice of a pattern of polygons Γ on Σ;
2. We need to choose an "orientation" for each edge and each face.
   (a) For an edge, this means giving it a direction, which we indicate in pictorially by an arrow:



   Thinking of the edge e as an arrow, we will refer to the vertices v_1 resp. v_0 as the *tip* resp. *tail* of the edge e, and write $\mathrm{tip}(e) = v_1$, $\mathrm{tail}(e) = v_0$.

   (b) An orientation for a face means a direction for the boundary circle (clockwise or anti-clockwise), which we indicate in pictures as follows:



These choices allow us to construct the following abelian groups and homomorphisms:

$$\mathbb{Z}[V] \xleftarrow{\partial_1} \mathbb{Z}[E] \xleftarrow{\partial_2} \mathbb{Z}[F] \tag{1.13}$$

Here V (resp. E resp. F) is the set of vertices (resp. edges resp. faces) of the pattern of polygons that we picked, and $\mathbb{Z}[V]$ (resp. $\mathbb{Z}[E]$ resp. $\mathbb{Z}[F]$) is the free abelian group generated by these sets. Given an edge $e \in E$, then

$$\partial_1(e) = \mathrm{tip}(e) - \mathrm{tail}(e) \in \mathbb{Z}[V],$$

and this determines the map ∂_1 by linearity. If $f \in F$ is a face,

$$\partial_2(f) = \sum_{e \in \partial f} \pm e \in \mathbb{Z}[E].$$

Here the sum is over all edges e of the polygon f; the sign of an edge e is positive if the direction of e and the direction of f agree, and negative otherwise.

Lemma 1.14. $\partial_1 \circ \partial_2 = 0$.

Definition 1.15. A *chain complex* is a sequence

$$\xleftarrow{\partial_{k-1}} C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \xleftarrow{\partial_{k+2}}$$

of $\mathbb{Z}$-modules C_k and module maps $\partial_k \colon C_k \to C_{k-1}$ for $k \in \mathbb{Z}$, such that $\partial_k \circ \partial_{k+1} = 0$ for all $k \in \mathbb{Z}$. We typically abbreviate by writing (C_*, ∂_*) or just C_* for a chain complex, where $*$ is a placeholder for an index $n \in \mathbb{Z}$.

We note that (1.13) can be interpreted as a chain complex by setting

$$C_k := \begin{cases} \mathbb{Z}[V] & k = 0 \\ \mathbb{Z}[E] & k = 1 \\ \mathbb{Z}[F] & k = 2 \\ 0 & k \neq 0, 1, 2 \end{cases}.$$

Terminology. Motivated by this example, the maps ∂_i are called *boundary maps.* An element $c \in C_k$ is a k-chain. If c is in the kernel of $\partial_k \colon C_k \to C_{k-1}$, it is a k-cycle, and if it is in the image of $\partial_{k+1} \colon C_{k+1} \to C_k$, it is a *$k$-boundary.* We note that the condition $\partial_k \circ \partial_{k+1} = 0$ implies that any k-boundary is a k-cycle. Furthermore, we write

$$Z_k := \{k\text{-cycles}\} = \ker(\partial_k \colon C_k \to C_{k-1})$$

for the $\mathbb{Z}$-module of k-cycles and

$$B_k := \{k\text{-boundaries}\} = \mathrm{im}(\partial_{k+1} \colon C_{k+1} \to C_k)$$

for the submodule of k-boundaries. The the *k-homology* is defined to be the quotient module

$$H_k := \frac{Z_k}{B_k} = \frac{\{k\text{-cycles}\}}{\{k\text{-boundaries}\}}.$$

If there is more than one chain complex around we use the notation

$$Z_k(C_*, \partial_*),\ B_k(C_*, \partial_*),\ H_k(C_*, \partial_*) \qquad \text{or} \qquad Z_k(C_*),\ B_k(C_*),\ H_k(C_*)$$

to indicate that we are talking about cycles, boundaries, or homology classes of the chain complex $C_* = (C_*, \partial_*)$.

2 Appendix: Pointset Topology

2.1 Metric spaces

We recall that a map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ between Euclidean spaces is *continuous* if and only if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y \in X \quad d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon, \tag{2.1}$$

where

$$d(x,y) = ||x - y|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \in \mathbb{R}_{\geq 0}$$

is the *Euclidean distance* between two points x, y in $\mathbb{R}^n$.

Example 2.2. (Examples of continuous maps.)

1. The addition map $a\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 + x_2$;
2. The multiplication map $m\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 x_2$;

The proofs that these maps are continuous are simple estimates that you probably remember from calculus. Since the continuity of *all* the maps we'll look at in these notes is proved by expressing them in terms of the maps a and m, we include the proofs of continuity of a and m for completeness.

Proof. To prove that the addition map a is continuous, suppose $x = (x_1, x_2) \in \mathbb{R}^2$ and $\epsilon > 0$ are given. We claim that for $\delta := \epsilon/2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x,y) < \delta$ we have $d(a(x), a(y)) < \epsilon$ and hence a is a continuous function. To prove the claim, we note that

$$d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$

and hence $|x_1 - y_1| \leq d(x,y)$, $|x_1 - y_1| \leq d(x,y)$. It follows that

$$d(a(x), a(y)) = |a(x) - a(y)| = |x_1 + x_2 - y_1 - y_2| \leq |x_1 - y_1| + |x_2 - y_2| \leq 2d(x,y) < 2\delta = \epsilon.$$

To prove that the multiplication map m is continuous, we claim that for

$$\delta := \min\{1, \epsilon/(|x_1| + |x_2| + 1)\}$$

and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x, y) < \delta$ we have $d(m(x), m(y)) < \epsilon$ and hence m is a continuous function. The claim follows from the following estimates:

$$\begin{aligned}
d(m(y), m(x)) &= |y_1y_2 - x_1x_2| = |y_1y_2 - x_1y_2 + x_1y_2 - x_1x_2| \\
&\leq |y_1y_2 - x_1y_2| + |x_1y_2 - x_1x_2| = |y_1 - x_1||y_2| + |x_1||y_2 - x_2| \\
&\leq d(x, y)(|y_2| + |x_1|) \leq d(x, y)(|x_2| + |y_2 - x_2| + |x_1|) \\
&\leq d(x, y)(|x_1| + |x_2| + 1) < \delta(|x_1| + |x_2| + 1) \leq \epsilon
\end{aligned}$$

□

Lemma 2.3. *The function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ has the following properties:*

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$ (symmetry);*
3. *$d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)*

Definition 2.4. A *metric space* is a set X equipped with a map

$$d\colon X \times X \to \mathbb{R}_{\geq 0}$$

with properties (1)-(3) above. A map $f\colon X \to Y$ between metric spaces X, Y is

continuous if condition (2.1) is satisfied.

an isometry if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$;

Two metric spaces X, Y are *homeomorphic* (resp. *isometric*) if there are continuous maps (resp. isometries) $f\colon X \to Y$ and $g\colon Y \to X$ which are inverses of each other.

Example 2.5. An important class of examples of metric spaces are subsets of $\mathbb{R}^n$. Here are particular examples we will be talking about during the semester:

1. The *n-disk* $D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\} \subset \mathbb{R}^n$, and $D^n_r := \{x \in \mathbb{R}^n \mid |x| \leq r\}$, the *n-disk of radius* $r > 0$.

   The dilation map

   $$D^n \longrightarrow D^n_r \qquad x \mapsto rx$$

is a homeomorphism between D^n and D^n_r with inverse given by multiplication by $1/r$. However, these two metric spaces are *not* isometric for $r \neq 1$. To see this, define the *diameter* $\operatorname{diam}(X)$ of a metric space X by

$$\operatorname{diam}(X) := \sup\{d(x,y) \mid x, y \in X\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

For example, $\operatorname{diam}(D^n_r) = 2r$. It is easy to see that if two metric spaces X, Y are isometric, then their diameters agree. In particular, the disks D^n_r and $D^n_{r'}$ are not isometric unless $r = r'$.

2. The *n-sphere* $S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} \subset \mathbb{R}^{n+1}$.

3. The *torus* $T = \{v \in \mathbb{R}^3 \mid d(v, C) = r\}$ for $0 < r < 1$. Here

$$C = \{(x, y, 0) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^3$$

is the unit circle in the xy-plane, and $d(v, C) = \inf_{w \in C} d(v, w)$ is the distance between v and C.

4. The *general linear group*

$$\begin{aligned} GL_n(\mathbb{R}) &= \{\text{vector space isomorphisms } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &\longleftrightarrow \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) \neq 0\} \\ &= \{\text{invertible } n \times n\text{-matrices}\} \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} = \mathbb{R}^{n^2} \end{aligned}$$

Here we think of $(v_1, \dots, v_n)$ as an $n \times n$-matrix with column vectors v_i, and the bijection is the usual one in linear algebra that sends a linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ to the matrix $(f(e_1), \dots, f(e_n))$ whose column vectors are the images of the standard basis elements $e_i \in \mathbb{R}^n$.

5. The *special linear group*

$$SL_n(\mathbb{R}) = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

6. The *orthogonal group*

$$\begin{aligned} O(n) &= \{\text{linear isometries } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{n^2} \end{aligned}$$

We recall that a collection of vectors $v_i \in \mathbb{R}^n$ is *orthonormal* if $|v_i| = 1$ for all i, and v_i is perpendicular to v_j for $i \neq j$.

7. The *special orthogonal group*

$$SO(n) = \{(v_1, \dots, v_n) \in O(n) \mid \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

8. The *Stiefel manifold*

$$\begin{aligned} V_k(\mathbb{R}^n) &= \{\text{linear isometries } f\colon \mathbb{R}^k \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{kn} \end{aligned}$$

Example 2.6. The following maps between metric spaces are continuous. While it is possible to prove their continuity using the definition of continuity, it will be much simpler to prove their continuity by 'building' these maps using compositions and products from the continuous maps a and m of Example 2.2. We will do this below in Lemma 2.22.

1. Every polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous. We recall that a polynomial function is of the form $f(x_1, \dots, x_n) = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} x_1^{i_1} \cdot \dots \cdot x_n^{i_n}$ for $a_{i_1,\dots,i_n} \in \mathbb{R}$.

2. Let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices. Then the map

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

given by matrix multiplication is continuous. Here we use the fact that a map to the product $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2} = \mathbb{R} \times \dots \times \mathbb{R}$ is continuous if and only if each component map is continuous (see Lemma 2.21), and each matrix entry of AB is a polynomial and hence a continuous function of the matrix entries of A and B. Restricting to the invertible matrices $GL_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$, we see that the multiplication map

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

is continuous. The same holds for the subgroups $SO(n) \subset O(n) \subset GL_n(\mathbb{R})$.

3. The map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is continuous (this is a homework problem). The same statement follows for the subgroups of $GL_n(\mathbb{R})$.

The Euclidean metric on $\mathbb{R}^n$ given by $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$ for $x, y \in \mathbb{R}^n$ is not the only reasonable metric on $\mathbb{R}^n$. Another metric on $\mathbb{R}^n$ is given by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|. \tag{2.7}$$

The question arises whether it can happen that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to one of these metrics, but not with respect to the other. To see that this doen't happen, it is useful to characterize continuity of a map $f\colon X \to Y$ between metric spaces X, Y in a way that involves the metrics on X and Y less directly than Definition 2.4 does. This alternative characterization will be based on the following notion of "open subsets" of a metric space.

Definition 2.8. Let X be a metric space. A subset $U \subset X$ is *open* if for every point $x \in U$ there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Here $B_\epsilon(x) = \{y \in X \mid d(y, x) < \epsilon\}$ is the *ball of radius ϵ around x*.

To illustrate this, lets look at examples of subsets of $\mathbb{R}^n$ equipped with the Euclidean metric. The subset $D^n_r = \{v \in \mathbb{R}^n \mid ||v|| \leq r\} \subset \mathbb{R}^n$ is not open, since for for a point $v \in D^n_r$ with $||v|| = r$ any open ball $B_\epsilon(v)$ with center v will contain points not in D^n_r. By contrast, the subset $B_r(0) \subset \mathbb{R}^n$ is open, since for any $x \in B_r(0)$ the ball $B_\delta(x)$ of radius $\delta = r - ||x||$ is contained in $B_r(0)$, since for $y \in B_\delta(x)$ by the triangle inequality we have

$$d(y, 0) \leq d(y, x) + d(x, 0) < \delta + ||x|| = (r - ||x||) + ||x|| = r.$$

Lemma 2.9. *A map $f: X \to Y$ between metric spaces is continuous if and only if $f^{-1}(V)$ is an open subset of X for every open subset $V \subset Y$.*

Corollary 2.10. *If $f: X \to Y$ and $g: Y \to Z$ are continuous maps, then so it their composition $g \circ f: X \to Z$.*

Exercise 2.11. (a) Prove Lemma 2.9

(b) Assume that d, d' are two metrics on a set X which are equivalent in the sense that there are constants $C, C' > 0$ such that $d(x, y) \leq C d_1(x, y)$ and $d_1(x, y) \leq C' d(x, y)$ for all $x, y \in X$. Show that a subset $U \subset X$ is open with respect to d if and only if it is open with respect to d'.

(c) Show that the Euclidean metric d and the metric (2.7) on $\mathbb{R}^n$ are equivalent. This shows in particular that a map $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous w.r.t. d if and only if it is continuous w.r.t. d_1.

2.2 Topological spaces

Lemma 2.9 and Exercise (b) above shows that it is better to *define* continuity of maps between metric spaces in terms of the *open subsets* of these metric space instead of the original ϵ-δ-definition. In fact, we can go one step further, forget about the metric on a set X altogether, and just consider a collection $\mathcal{T}$ of subsets of X that we declare to be "open". The next result summarizes the basic properties of open subsets of a metric space X, which then motivates the restrictions that we wish to put on such collections $\mathcal{T}$.

Lemma 2.12. *Open subsets of a metric space X have the following properties.*

(i) X and $\emptyset$ are open.

(ii) Any union of open sets is open.

(iii) The intersection of any finite number of open sets is open.

Definition 2.13. A *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X, called *open sets* which are required to satisfy conditions (i), (ii) and (iii) of the lemma above. The collection $\mathcal{T}$ is called a *topology* on X. The sets in $\mathcal{T}$ are called the *open sets*, and their complements in X are called *closed sets*. A subset of X may be neither closed nor open, either closed or open, or both.

A map $f\colon X \to Y$ between topological spaces X, Y is *continuous* if the inverse image $f^{-1}(V)$ of every open subset $V \subset Y$ is an open subset of X.

It is easy to see that the composition of continuous maps is again continuous.

Examples of topological spaces.

1. Let X be a metric space, and $\mathcal{T}$ the collection of those subsets of X that are unions of balls $B_\epsilon(x)$ in X (i.e., the subsets which are open in the sense of Definition 2.8). Then $\mathcal{T}$ is a topology on X, the *metric topology.*

2. Let X be a set. Then $\mathcal{T} = \{\text{all subsets of } X\}$ is a topology, the *discrete topology.* We note that *any* map $f\colon X \to Y$ to a topological space Y is continuous. We will see later that the only continuous maps $\mathbb{R}^n \to X$ are the constant maps.

3. Let X be a set. Then $\mathcal{T} = \{\emptyset, X\}$ is a topology, the *indiscrete topology.*

Sometimes it is convenient to define a topology $\mathcal{U}$ on a set X by first describing a smaller collection $\mathcal{B}$ of subsets of X, and then defining $\mathcal{U}$ to be those subsets of X that can be written as *unions* of subsets belonging to $\mathcal{B}$. We've done this already when defining the metric topology: Let X be a metric space and let $\mathcal{B}$ be the collection of subsets of X of the form $B_\epsilon(x) := \{y \in X \mid d(y,x) < \epsilon\}$ (the balls in X). Then the metric topology $\mathcal{U}$ on X consists of those subsets U which are unions of subsets belonging to $\mathcal{B}$.

Lemma 2.14. *Let $\mathcal{B}$ be a collection of subsets of a set X satisfying the following conditions*

1. *Every point $x \in X$ belongs to some subset $B \in \mathcal{B}$.*

2. *If $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 \cap B_2$ there is some $B \in \mathcal{B}$ with $x \in B$ and $B \subset B_1 \cap B_2$.*

Then $\mathcal{T} := \{\text{unions of subsets belonging to } \mathcal{B}\}$ is a topology on X.

Definition 2.15. If the above conditions are satisfied, we call the collection $\mathcal{B}$ is called a *basis for the topology* $\mathcal{T}$ or we say that $\mathcal{B}$ *generates the topology* $\mathcal{T}$.

It is easy to check that the collection of balls in a metric space satisfies the above conditions and hence the collection of open subsets is a topology as claimed by Lemma 2.12.

2.3 Constructions with topological spaces

2.3.1 Subspace topology

Definition 2.16. Let X be a topological space, and $A \subset X$ a subset. Then

$$\mathcal{T} = \{A \cap U \mid U \underset{open}{\subset} X\}$$

is a topology on A called the *subspace topology.*

Lemma 2.17. *Let X be a metric space and $A \subset X$. Then the metric topology on A agrees with the subspace topology on A (as a subset of X equipped with the metric topology).*

Lemma 2.18. *Let X, Y be topological spaces and let A be a subset of X equipped with the subspace topology. Then the inclusion map $i\colon A \to X$ is continuous and a map $f\colon Y \to A$ is continuous if and only if the composition $i \circ f\colon Y \to X$ is continuous.*

2.3.2 Product topology

Definition 2.19. The *product topology* on the Cartesian product $X \times Y = \{(x, y) \mid x \in X,\, y \in Y\}$ of topological spaces X, Y is the topology with basis

$$\mathcal{B} = \{U \times V \mid U \underset{open}{\subset} X, V \underset{open}{\subset} Y\}$$

The collection $\mathcal{B}$ obviously satisfies property (1) of a basis; property (2) holds since $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$. We note that the collection $\mathcal{B}$ is *not* a topology since the union of $U \times V$ and $U' \times V'$ is typically not a Cartesian product (e.g., draw a picture for the case where $X = Y = \mathbb{R}$ and U, U', V, V' are open intervals).

Lemma 2.20. *The product topology on $\mathbb{R}^m \times \mathbb{R}^n$ (with each factor equipped with the metric topology) agrees with the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.*

Proof: homework.

Lemma 2.21. *Let X, Y_1, Y_2 be topological spaces. Then the projection maps $p_i\colon Y_1 \times Y_2 \to Y_i$ is continuous and a map $f\colon X \to Y_1 \times Y_2$ is continuous if and only if the* component maps

$$X \xrightarrow{f} Y_1 \times Y_2 \xrightarrow{p_i} Y_i$$

are continuous for $i = 1, 2$.

Proof: homework

Lemma 2.22. *1. Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous maps. Then $f+g$ and $f \cdot g$ continuous maps from X to $\mathbb{R}$. If $g(x) \neq 0$ for all $x \in X$, then also f/g is continuous.*

2. Any polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous.

3. The multiplication map $\mu\colon GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is continuous.

Proof. To prove part (1) we note that the map $f+g\colon X \to \mathbb{R}$ can be factored in the form

$$X \xrightarrow{f\times g} \mathbb{R}\times\mathbb{R} \xrightarrow{a} \mathbb{R}$$

The map $f \times g$ is continuous by Lemma 2.21 since its component maps f, g are continuous; the map a is continuous by Example 2.2, and hence the composition $f+g$ is continuous. The argument for $f \cdot g$ is the same, with a replaced by m. To prove that f/g is continuous, we factor it in the form

$$X \xrightarrow{\quad f\times g \quad} \mathbb{R}\times\mathbb{R}^\times \xrightarrow{p_1\times(I\circ p_2)} \mathbb{R}\times\mathbb{R}^\times \xrightarrow{\quad m \quad} \mathbb{R},$$

where $\mathbb{R}^\times = \{t \in \mathbb{R} \mid t \neq 0\}$, p_1 (resp. p_2) is the projection to the first (resp. second) factor of $\mathbb{R}\times\mathbb{R}^\times$, and $I\colon \mathbb{R}^\times \to \mathbb{R}^\times$ is the inversion map $t \mapsto t^{-1}$. By Lemma 2.21 the p_i's are continuous, in calculus we learned that I is continuous, and hence again by Lemma 2.21 the map $p_1 \times (I \circ p_2)$ is continuous.

To prove part (2), we note that the constant map $\mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto a$ is obviously continuous, and that the projection map $p_i\colon \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto x_i$ is continuous by Lemma 2.21. Hence by part (1) of this lemma, the monomial function $x \mapsto a x_1^{i_1} \cdots x_n^{i_n}$ is continuous. Any polynomial function is a sum of monomial functions and hence continuous.

For the proof of (3), let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices and let

$$\mu\colon M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A,B) \mapsto AB$$

be the map given by matrix multiplication. By Lemma 2.21 the map μ is continuous if and only if the composition

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \xrightarrow{\mu} M_{n\times n}(\mathbb{R}) \xrightarrow{p_{ij}} \mathbb{R}$$

is continuous for all $1 \leq i, j \leq n$, where p_{ij} is the projection map that sends a matrix A to its entry $A_{ij} \in \mathbb{R}$. Since the $p_{ij}(\mu(A,B)) = (A \cdot B)_{ij}$ is a *polynomial* in the entries of the matrices A and B, this is a continuous map by part (2) and hence μ is continuous.

Restricting μ to invertible matrices, we obtain the multiplication map

$$\mu_| \colon GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

that we want to show is continuous. We will argue that in general if $f\colon X \to Y$ is a continuous map with $f(A) \subset B$ for subsets $A \subset X$, $B \subset Y$, then the *restriction* $f_{|A}\colon A \to B$ is continuous. To prove this, consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_{|A}"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where i, j are the obvious inclusion maps. These inclusion maps are continuous w.r.t. the subspace topology on A, B by Lemma 2.18. The continuity of f and i implies the continuity of $f \circ i = j \circ f_{|A}$ which again by Lemma 2.18 implies the continuity of $f_{|A}$. □

2.3.3 Quotient topology.

Definition 2.23. Let X be a topological space and let $\sim$ be an equivalence relation on X. We denote by $X/\sim$ be the set of equivalence classes and by

$$p\colon X \to X/\sim \qquad x \mapsto [x]$$

be the projection map that sends a point $x \in X$ to its equivalence class $[x]$. The *quotient topology* on $X/\sim$ is given by the collection of subsets

$$\mathcal{U} = \{U \subset X/\sim \mid p^{-1}(U) \text{ is an open subset of } X\}.$$

The set $X/\sim$ equipped with the quotient topology is called the *quotient space*.

The quotient topology is often used to construct a topology on a set Y which is not a subset of some Euclidean space $\mathbb{R}^n$, or for which it is not clear how to construct a metric. If there is a surjective map

$$p\colon X \longrightarrow Y$$

from a topological space X, then Y can be identified with the quotient space $X/\sim$, where the equivalence relation is given by $x \sim x'$ if and only if $p(x) = p(x')$. In particular, $Y = X/\sim$ can be equipped with the quotient topology. Here are important examples.

Example 2.24. 1. The *real projective space of dimension* n is the set

$$\mathbb{RP}^n := \{\text{1-dimensional subspaces of } \mathbb{R}^{n+1}\}.$$

The map

$$S^n \longrightarrow \mathbb{RP}^n \qquad \mathbb{R}^{n+1} \ni v \mapsto \text{subspace generated by } v$$

is surjective, leading to the identification

$$\mathbb{RP}^n = S^n/(v \sim \pm v),$$

and the quotient topology on $\mathbb{RP}^n$.

2. Similarly, working with complex vector spaces, we obtain a quotient topology on the the *complex projective space*
$$\mathbb{CP}^n := \{\text{1-dimensional subspaces of } \mathbb{C}^{n+1}\} = S^{2n+1}/(v \sim zv), \qquad z \in S^1$$
3. Generalizing, we can consider the *Grassmann manifold*
$$G_k(\mathbb{R}^{n+k}) := \{k\text{-dimensional subspaces of } \mathbb{R}^{n+k}\}.$$
There is a surjective map
$$V_k(\mathbb{R}^{n+k}) = \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^{n+k},\ v_i\text{'s are orthonormal}\} \quad \twoheadrightarrow \quad G_k(\mathbb{R}^{n+k})$$
given by sending $(v_1, \dots, v_k) \in V_k(\mathbb{R}^{n+k})$ to the k-dimensional subspace of $\mathbb{R}^{n+k}$ spanned by the v_i's. Hence the subspace topology on the Stiefel manifold $V_k(\mathbb{R}^{n+k}) \subset \mathbb{R}^{(n+k)k}$ gives a quotient topology on the Grassmann manifold $G_k(\mathbb{R}^{n+k}) = V_k(\mathbb{R}^{n+k})/\sim$. The same construction works for the complex Grassmann manifold $G_k(\mathbb{C}^{n+k})$.

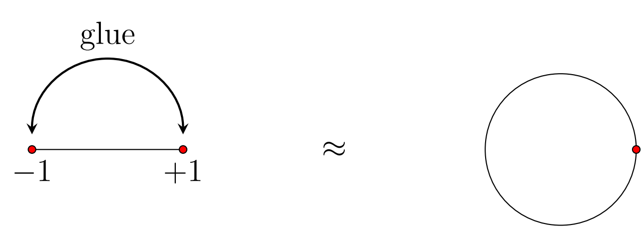
As the examples below will show, sometimes a quotient space $X/\sim$ is homeomorphic to a topological space Z constructed in a different way. To establish the homeomorphism between $X/\sim$ and Z, we need to construct continuous maps
$$f\colon X/\sim \longrightarrow Z \qquad g\colon Z \to X/\sim$$
that are inverse to each other. The next lemma shows that it is easy to check continuity of the map f, the map *out of the quotient space*.

Lemma 2.25. *The projection map $p\colon X \to X/\sim$ is continuous and a map $f\colon X/\sim \to Z$ to a topological space Z is continuous if and only if the composition $f \circ p\colon X \to Z$ is continuous.*

As we will see in the next section, there are many situations where the continuity of the inverse map for a continuous bijection f is automatic. So in the examples below, and for the exercises in this section, we will defer checking the continuity of f^{-1} to that section.

Notation. Let A be a subset of a topological space X. Define a equivalence relation $\sim$ on X by $x \sim y$ if $x = y$ or $x, y \in A$. We use the notation X/A for the quotient space $X/\sim$.

Example 2.26. (1) We claim that the quotient space $[-1, +1]/\{\pm 1\}$ is homeomorphic to S^1 via the map $f\colon [-1, +1]/\{\pm 1\} \to S^1$ given by $[t] \mapsto e^{\pi i t}$. Geometrically speaking, the map f wraps the interval $[-1, +1]$ once around the circle. Here is a picture.



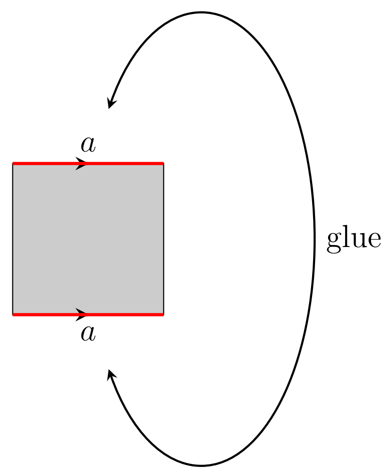
It is easy to check that the map f is a bijection. To see that f is continuous, consider the composition

$$[-1,+1] \xrightarrow{p} [-1,+1]/\{\pm 1\} \xrightarrow{f} S^1 \xrightarrow{i} \mathbb{C} = \mathbb{R}^2,$$

where p is the projection map and i the inclusion map. This composition sends $t \in [-1,+1]$ to $e^{\pi i t} = (\sin \pi t, \cos \pi t) \in \mathbb{R}^2$. By Lemma 2.21 it is a continuous function, since its component functions $\sin \pi t$ and $\cos \pi t$ are continuous functions. By Lemma 2.25 the continuity of $i \circ f \circ p$ implies the continuity of $i \circ f$, which by Lemma 2.18 implies the continuity of f. As mentioned above, we'll postpone the proof of the continuity of the inverse map f^{-1} to the next section.

(2) More generally, D^n/S^{n-1} is homeomorphic to S^n. (proof: homework)

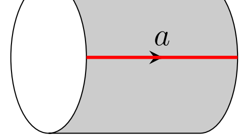
(3) Consider the quotient space of the square $[-1,+1]\times[-1,+1]$ given by identifying $(s,-1)$ with $(s,1)$ for all $s \in [-1,1]$. It can be visualized as a square whose top edge is to be glued with its bottom edge. In the picture below we indicate that identification by labeling those two edges by the same letter.



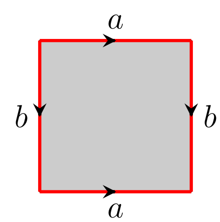
The quotient $([-1,+1]\times[-1,+1])/(s,-1) \sim (s,+1)$ is homeomorphic to the cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [-1,+1], y^2 + z^2 = 1\}.$$

The proof is essentially the same as in (1). A homeomorphism from the quotient space to C is given by $f([s,t]) = (s, \sin \pi t, \cos \pi t)$. The picture below shows the cylinder C with the image of the edge a indicated.



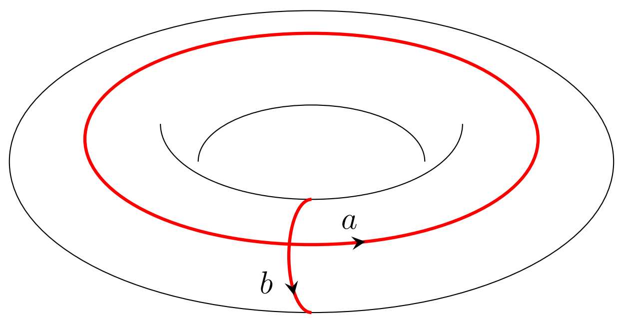
(4) Consider again the square, but this time using an equivalence relations that identifies more points than the one in the previous example. As before we identify $(s,-1)$ and $(s,1)$ for $s \in [-1,1]$, and in addition we identify $(-1,t)$ with $(1,t)$ for $t \in [-1,1]$. Here is the picture, where again corresponding points of edges labeled by the same letter are to be identified.



We claim that the quotient space is homeomorphic to the torus

$$T := \{x \in \mathbb{R}^3 \mid d(x,K) = d\},$$

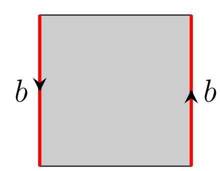
where $K = \{(x_1,x_2,0) \mid x_1^2 + x_2^2 = 1\}$ is the unit circle in the xy-plane and $0 < d < 1$ is a real number (see) via a homeomorphism that maps the edges of the square to the loops in T indicated in the following picture below.



Exercise: prove this by writing down an explicity map from the quotient space to T, and arguing that this map is a continuous bijection (as always in this section, we defer the proof of the continuity of the inverse to the next section).

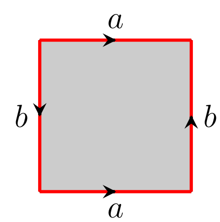
(5) We claim that the quotient space $D^n/\sim$ with equivalence relation generated by $v \sim -v$ for $v \in S^{n-1} \subset D^n$ is homeomorphic to the real projective space $\mathbb{RP}^n$. Proof: exercise. In particular, $\mathbb{RP}^1 = S^1/v \sim -v$ is homeomorphic to $D^1/\sim = [-1,1]/-1 \sim 1$, which by example (1) is homeomorphic to S^1.

(6) The quotient space $[-1,1] \times [-1,1]/\sim$ with the equivalence relation generated by $(-1,t) \sim (1,-t)$ is represented graphically by the following picture.



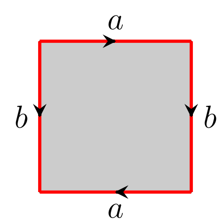
This topological space is called the *Möbius band*. It is homeomorphic to a subspace of $\mathbb{R}^3$ shown by the following picture

(7) The quotient space of the square by edge identifications given by the picture



is the *Klein bottle*. It is harder to visualize, since it is not homeomorphic to a subspace of $\mathbb{R}^3$ (which can be proved by the methods of algebraic topology).

(8) The quotient space of the square given by the picture



is homeomorphic to the real projective plane $\mathbb{RP}^2$. Exercise: prove this (hint: use the statement of example (5)). Like the Klein bottle, it is challenging to visualize the real projective plane, since it is not homeomorphic to a subspace of $\mathbb{R}^3$.

2.4 Properties of topological spaces

In the previous subsection we described a number of examples of topological spaces X, Y that we claimed to be homeomorphic. We typically constructed a bijection $f\colon X \to Y$ and argued that f is continuous. However, we did not finish the proof that f is a homeomorphism, since we defered the argument that the inverse map $f^{-1}\colon Y \to X$ is continuous. We note that not every continuous bijection is a homeomorphism. For example if X is a set, X_δ (resp. X_{ind}) is the topological space given by equipping the set X with the discrete (resp. indiscrete) topology, then the identity map is a continuous bijection from X_δ to X_{ind}. However its inverse, the identity map $X_{\mathrm{ind}} \to X_\delta$ is not continuous if X contains at least two points.

Fortunately, there are situations where the continuity of the inverse map is automatic as the following proposition shows.

Proposition 2.27. *Let $f\colon X \to Y$ be a continuous bijection. Then f is a homeomorphism provided X is compact and Y is Hausdorff.*

The goal of this section is to define these notions, prove the proposition above, and to give a tools to recognize that a topological space is compact and/or Hausdorff.

2.4.1 Hausdorff spaces

Definition 2.28. Let X be a topological space, $x_i \in X$, $i = 1, 2, \dots$ a sequence in X and $x \in X$. Then *x is the limit of the x_i's* if for any open subset $U \subset X$ containing x there is some N such that $x_i \in U$ for all $i \geq N$.

Caveat: If X is a topological space with the indiscrete topology, *every point* is the limit of every sequence. The limit is *unique* if the topological space has the following property:

Definition 2.29. A topological space X is *Hausdorff* if for every $x, y \in X$, $x \neq y$, there are disjoint open subsets $U, V \subset X$ with $x \in U$, $y \in V$.

Note: if X is a metric space, then the metric topology on X is Hausdorff (since for $x \neq y$ and $\epsilon = d(x,y)/2$, the balls $B_\epsilon(x)$, $B_\epsilon(y)$ are disjoint open subsets). In particular, any subset of $\mathbb{R}^n$, equipped with the subspace topology, is Hausdorff.

Warning: The notion of *Cauchy sequences* can be defined in metric spaces, but not in general for topological spaces (even when they are Hausdorff).

Lemma 2.30. *Let X be a topological space and A a closed subspace of X. If $x_n \in A$ is a sequence with limit x, then $x \in A$.*

Proof. Assume $x \notin A$. Then x is a point in the open subset $X \setminus A$ and hence by the definition of limit, all but finitely many elements x_n must belong to $X \setminus A$, contradicting our assumptions. $\square$

2.4.2 Compact spaces

Definition 2.31. An *open cover* of a topological space X is a collection of open subsets of X whose union is X. If for every open cover of X there is a finite subcollection which also covers X, then X is called *compact*.

Some books (like Munkres' *Topology*) refer to open covers as *open coverings*, while newer books (and wikipedia) seem to prefer to above terminology, probably for the same reasons as me: to avoid confusions with *covering spaces*, a notion we'll introduce soon.

Now we'll prove some useful properties of compact spaces and maps between them, which will lead to the important Corollaries 2.36 and 2.34.

Lemma 2.32. *If $f\colon X \to Y$ is a continuous map and X is compact, then the image $f(X)$ is compact.*

In particular, if X is compact, then any quotient space $X/\sim$ is compact, since the projection map $X \to X/\sim$ is continuous with image $X/\sim$.

Proof. To show that $f(X)$ is compact assume that $\{U_a\}$, $a \in A$ is an open cover of the subspace $f(X)$. Then each U_a is of the form $U_a = V_a \cap f(X)$ for some open subset $V_a \in Y$. Then $\{f^{-1}(V_a)\}$, $a \in A$ is an open cover of X. Since X is compact, there is a finite subset A' of A such that $\{f^{-1}(V_a)\}$, $a \in A'$ is a cover of X. This implies that $\{U_a\}$, $a \in A'$ is a finite cover of $f(X)$, and hence $f(X)$ is compact. □

Lemma 2.33. *1. If K is a closed subspace of a compact space X, then K is compact.*

2. If K is compact subspace of a Hausdorff space X, then K is closed.

Proof. To prove (1), assume that $\{U_a\}$, $a \in A$ is an open covering of K. Since the U_a's are open w.r.t. the subspace topology of K, there are open subsets V_a of X such that $U_a = V_a \cap K$. Then the V_a's together with the open subset $X \setminus K$ form an open covering of X. The compactness of X implies that there is a finite subset $A' \subset A$ such that the subsets V_a for $a \in A'$, together with $X \setminus K$ still cover X. It follows that U_a, $a \in A'$ is a finite cover of K, showing that K is compact.

The proof of part (2) is a homework problem. □

Corollary 2.34. *If $f\colon X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.*

Proof. We need to show that the map $g\colon Y \to X$ inverse to f is continuous, i.e., that $g^{-1}(U) = f(U)$ is an open subset of Y for any open subset U of X. Equivalently (by passing to complements), it suffices to show that $g^{-1}(C) = f(C)$ is a closed subset of Y for any closed subset C of C.

Now the assumption that X is compact implies that the closed subset $C \subset X$ is compact by part (1) of Lemma 2.33 and hence $f(C) \subset Y$ is compact by Lemma 2.32. The assumption that Y is Hausdorff then implies by part (2) of Lemma 2.33 that $f(C)$ is closed. □

Lemma 2.35. *Let K be a compact subset of $\mathbb{R}^n$. Then K is* bounded*, meaning that there is some $r > 0$ such that K is contained in the open ball $B_r(0) := \{x \in \mathbb{R}^n \mid d(x,0) < r\}$.*

Proof. The collection $B_r(0) \cap K$, $r \in (0,\infty)$, is an open cover of K. By compactness, K is covered by a *finite* number of these balls; if R is the maximum of the radii of these finitely many balls, this implies $K \subset B_R(0)$ as desired. □

Corollary 2.36. *If $f\colon X \to \mathbb{R}$ is a continuous function on a compact space X, then f has a maximum and a minimum.*

Proof. $K = f(X)$ is a compact subset of $\mathbb{R}$. Hence K is bounded, and thus K has an infimum $a := \inf K \in \mathbb{R}$ and a supremum $b := \sup K \in \mathbb{R}$. The infimum (resp. supremum) of K is the limit of a sequence of elements in K; since K is closed (by Lemma 2.33 (2)), the limit points a and b belong to K by Lemma 2.30. In other words, there are elements $x_{min}, x_{max} \in X$ with $f(x_{min}) = a \leq f(x)$ for all $x \in X$ and $f(x_{max}) = b \geq f(x)$ for all $x \in X$. □

In order to use Corollaries 2.34 and 2.36, we need to be able to show that topological spaces we are interested in, are in fact compact. Note that this is *quite difficult* just working from the definition of compactness: you need to ensure that *every* open cover has a finite subcover. That sounds like a lot of work...

Fortunately, there is a very simple classical characterization of compact subspaces of Euclidean spaces:

Theorem 2.37. (Heine-Borel Theorem) *A subspace $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.*

We note that we've already proved that if $K \subset \mathbb{R}^n$ is compact, then K is a closed subset of $\mathbb{R}^n$ (Lemma 2.33(2)), and K is bounded (Lemma 2.35).

There two important ingredients to the proof of the converse, namely the following two results:

Lemma 2.38. *A closed interval $[a, b]$ is compact.*

This lemma has a short proof that can be found in any pointset topology book, e.g., [Mu].

Theorem 2.39. *If $X_1, \dots, X_n$ are compact topological spaces, then their product $X_1 \times \cdots \times X_n$ is compact.*

For a proof see e.g. [Mu, Ch. 3, Thm. 5.7]. The statement is true more generally for a product of *infinitely many* compact space (as discussed in [Mu, p. 113], the correct definition of the product topology for infinite products requires some care), and this result is called *Tychonoff's Theorem*, see [Mu, Ch. 5, Thm. 1.1].

Proof of the Heine-Borel Theorem. Let $K \subset \mathbb{R}^n$ be closed and bounded, say $K \subset B_r(0)$. We note that $B_r(0)$ is contained in the n-fold product

$$P := [-r, r] \times \cdots \times [-r, r] \subset \mathbb{R}^n$$

which is compact by Theorem 2.39. So K is a closed subset of P and hence compact by Lemma 2.33(1). □

2.4.3 Connected spaces

Definition 2.40. A topological space X is *connected* if it can't be written as decomposed in the form $X = U \cup V$, where U, V are two non-empty disjoint open subsets of X.

For example, if a, b, c, d are real numbers with $a < b < c < d$, consider the subspace $X = (a,b) \amalg (c,d) \subset \mathbb{R}$. The topological space X is not connected, since $U = (a,b)$, $V = (c,d)$ are open disjoint subsets of X whose union is X. This remains true if we replace the open intervals by closed intervals. The space $X' = [a,b] \amalg [c,d]$ is not connected, since it is the disjoint union of the subsets $U' = [a,b]$, $V' = [c,d]$. We want to emphasize that while U' and V' are not open as subsets of $\mathbb{R}$, they are *open subsets of* X', since they can be written as

$$U' = (-\infty, c) \cap X' \qquad V' = (b, \infty) \cap X',$$

showing that they are open subsets for the subspace topology of $X' \subset \mathbb{R}$.

Lemma 2.41. *Any interval I in $\mathbb{R}$ (open, closed, half-open, bounded or not) is connected.*

Proof. Using proof by contradiction, let us assume that I has a decomposition $I = U \cup V$ as the union of two non-empty disjoint open subsets. Pick points $u \in U$ and $v \in V$, and let us assume $u < v$ without loss of generality. Then

$$[u,v] = U' \cup V' \qquad \text{with} \qquad U' := U \cap [u,v] \quad V' := U \cap [u,v]$$

is a decomposition of $[u,v]$ as the disjoint union of non-empty disjoint open subsets U', V' of $[u,v]$. We claim that the supremum $c := \sup U'$ belongs to both, U' and V', thus leading to the desired contradiction. Here is the argument.

- Assuming that c doesn't belong to U', for any $\epsilon > 0$, there must be some element of U' belonging to the interval $(c - \epsilon, c)$, allowing us to construct a sequence of elements $u_i \in U'$ converging to c. This implies $c \in U'$ by Lemma 2.30, since U' is a *closed* subspace of $[u,v]$ (its complement V' is open).

- By construction, every $x \in [u,v]$ with $x > c = \sup U'$ belongs to V'. So we can construct a sequence $v_i \in V'$ converging to c. Since V' is a closed subset of $[u,v]$, we conclude $c \in V'$.

□

Theorem 2.42. (Intermediate Value Theorem) *Let X be a connected topological space, and $f\colon X \to \mathbb{R}$ a continuous map. If elements $a, b \in \mathbb{R}$ belong to the image of f, then also any real number c between a and b belongs to the image of f.*

Proof. Assume that c is not in the image of f. Then $X = f^{-1}(-\infty, c) \cup f^{-1}(c, \infty)$ is a decomposion of X as a union of non-empty disjoint open subsets. □

There is another notion, closely related to the notion of connected topological space, which might be easier to think of geometrically.

Definition 2.43. A topological space X is *path connected* if for any points $x, y \in X$ there is a path connecting them. In other words, there is a continuous map $\gamma\colon [a,b] \to X$ from some interval to X with $\gamma(a) = x$, $\gamma(b) = y$.

Lemma 2.44. *Any path connected topological space is connected.*

Proof. Using proof by contradiction, let us assume that the topological space X is path connected, but not connected. So there is a decomposition $X = U \cup V$ of X as the union of non-empty open subsets $U, V \subset X$. The assumption that X is path connected allows us to find a path $\gamma\colon [a,b] \to X$ with $\gamma(a) \in U$ and $\gamma(b) \in V$. Then we obtain the decomposition

$$[a,b] = f^{-1}(U) \cup f^{-1}(V)$$

of the interval $[a,b]$ as the disjoint union of open subsets. These are non-empty since $a \in f^{-1}(U)$ and $b \in f^{-1}(V)$. This implies that $[a,b]$ is not connected, the desired contradiction. $\square$

For typical topological spaces we will consider, the properties "connected" and "path connected" are equivalent. But here is an example known as the *topologist's sine curve* which is connected, but not path connected, see [Mu, Example 7, p. 156]. It is the following subspace of $\mathbb{R}^2$:

$$X = \{(x, \sin\frac{1}{x}) \in \mathbb{R}^2 \mid 0 < x < 1\} \cup \{(0,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}.$$

3 Appendix: Manifolds

The purpose of this section is to provide interesting examples of topological spaces and homeomorphisms between them. There are many examples of "weird" topological spaces. There are non-Hausdorff spaces (they don't have well-defined limits) or the topologist's sine curve, which is connected, but not path connected. While there is a huge literature concering pathological topological spaces, I must admit that I find those examples most interesting that "show up in nature". For example, topological spaces that appear as "configuration spaces" or "phase spaces" of physical systems. Often these are a particularly nice kind of topological space known as *manifold.*

There is much to say about manifolds. For example, you can find the text books *Introduction to topological manifolds* and *Introduction to smooth manifolds* by John Lee. For this section, our focus is to discuss manifolds of dimension 2. Unlike higher dimensional manifolds, we can represent manifolds of dimension 2 by pictures, which greatly helps the intuition about these objects.

Definition 3.1. A *manifold of dimension* n or n*-manifold* is a topological space X which is locally homeomorphic to $\mathbb{R}^n$, that is, every point $x \in X$ has an open neighborhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$. Moreover, it is useful and customary to require that X is Hausdorff (see Definition 2.29) and *second countable*, which means that the topology of X has a countable basis.

In most examples, the technical conditions of being Hausdorff and second countable are easy to check, since these properties are inherited by subspaces.

Homework 3.2. Show that a subspace of a Hausdorff space is Hausdorff. Show that a subspace of a second countable space is second countable.

Examples of manifolds.

1. Any open subset $U \subset \mathbb{R}^n$ is an n-manifold. The technical condition of being a second countable Hausdorff space is satisfied for U as a subspace of the second countable Hausdorff space $\mathbb{R}^n$.

2. The n-sphere $S^n := \{x \in \mathbb{R}^n \mid ||x|| = 1\}$ is an n-manifold. To prove this, let us look at the subsets

$$\begin{aligned} U_i^+ &:= \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i > 0\} \subset S^n \\ U_i^- &:= \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i < 0\} \subset S^n \end{aligned}$$

   We want to argue that the map

$$\phi_i^\pm \colon U_i^\pm \longrightarrow \mathring{D}^n \qquad \text{given by} \qquad \phi_i^\pm(x_0, \dots, x_n) := (x_0, \dots, x_{i-1}, \widehat{x}_i, x_{i+1}, \dots, x_n)$$

   is a homeomorphism, where $\mathring{D}^n := \{(v_1, \dots, v_n) \in D^n \mid v_1^2 + \cdots + v_n^2 < 1\}$ is the open n-disk. It is easy to verify that the map

$$\mathring{D}^n \longrightarrow U_i^\pm \qquad v = (v_1, \dots, v_n) \mapsto (v_1, \dots, v_i, \pm\sqrt{1 - ||v||^2}, v_{i+1}, \dots, v_n)$$

   is in fact the inverse to $\phi_i^\pm$. Here $||v||^2 = v_1^2 + \cdots + v_n^2$ is norm squared of $v \in \mathring{D}^n$. Both maps, $\phi_i^\pm$ and its inverse, are continuous since all their components are continuous. This shows that $\phi_i^\pm$ is in fact a homeomorphism, and hence the n-sphere S^n is a manifold of dimension n.

Homework 3.3. Show that the product $X \times Y$ of manifold X of dimension m and a manifold Y of dimension n is a manifold of dimension $m+n$. Make sure to prove that $X \times Y$ is second countable and Hausdorff.

Homework 3.4. Show that the real projective space $\mathbb{RP}^n$ is manifold of dimension n. Make sure to prove that $\mathbb{RP}^2$ is second countable and Hausdorff.

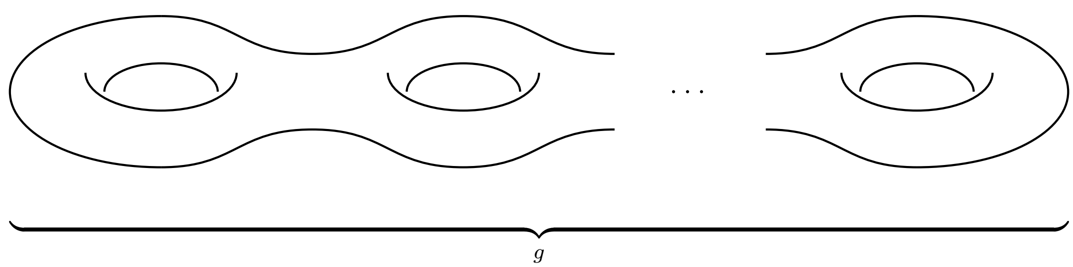
Examples of manifolds of dimension 2.

1. The 2-torus T. We recall that there are various ways of defining the torus, one being as the product $S^1 \times S^1$ which is a manifold of dimension 2 by Excercise 3.3, since S^1 is a manifold of dimension 1.

2. The real projective plane $\mathbb{RP}^2$.

3. The Klein bottle K. It is not hard to verify directly that K is a manifold of dimension 2. Alternatively, we will see in Lemma **??** that the Klein bottle is homeomorphic to the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ of two copies of the projective plane $\mathbb{RP}^2$, which implies in particular that K is a 2-manifold.

4. The surface Σ_g of genus g is the subspace of $\mathbb{R}^3$ given by the following picture:



   Here g is the number of "holes" of Σ_g. In particular Σ_1, the surface of genus 1, is the torus. By convention, the surface Σ_0 of genus 0 is the 2-sphere S^2. Since we have described the surface of genus g as a subspace of $\mathbb{R}^3$ given by a picture rather than a formula, it is impossible to give a precise argument that this subspace is locally homeomorphic to $\mathbb{R}^2$, but hopefully the picture makes this obvious at a heuristic level.

The connected sum construction. This construction produces a new manifold $M\#N$ of dimension n from two given manifolds M and N of dimension n. The manifold $M\#N$ is called the *connected sum* of M and N. The construction proceeds as follows. First we make some choices:

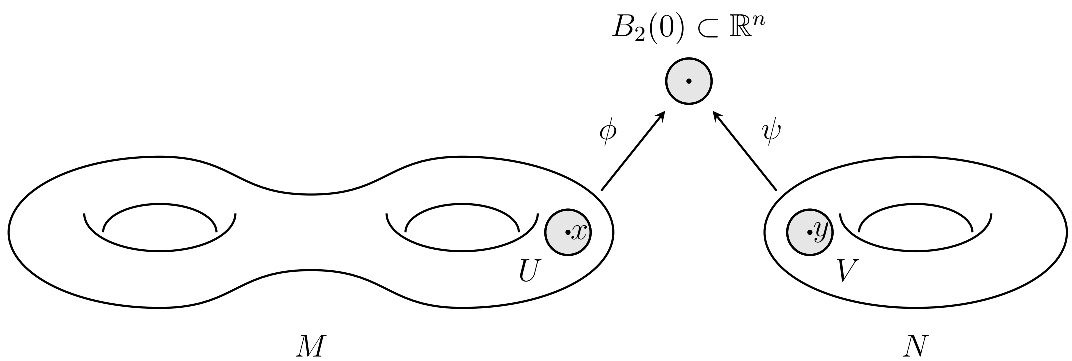
- We pick points $x \in M$ and $y \in N$.

- We pick a homeomorphism ϕ between an open neighborhood U of x and the open ball $B_2(0)$ of radius 2 around the origin $0 \in \mathbb{R}^n$. Similarly, we pick a homeomorphism $\psi\colon V \xrightarrow{\approx} B_2(0)$ where $V \subset N$ is an open neighborhood of $y \in N$.

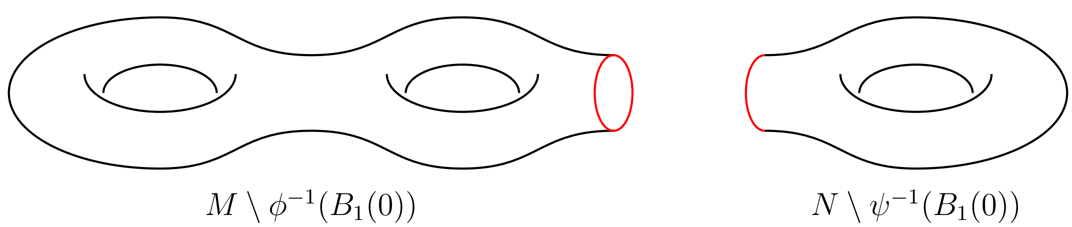
The existence of homeomorphisms ϕ, ψ with these properties follows from the assumption that M, N are manifolds of dimension n. This implies that there is an open neighborhood $U' \subset M$ of x and a homeomorphism ϕ' between U' and an open subset $V' \subset \mathbb{R}^n$. Composing ϕ by a translation in $\mathbb{R}^n$ we can assume that $\phi(x) = 0 \in \mathbb{R}^n$. Since V' is open, there is some $\epsilon > 0$ such that the open ball $B_\epsilon(0)$ of radius ϵ around $0 \in \mathbb{R}^n$ is contained in V'. Then restricting ϕ' to $U := (\phi')^{-1}(B_\epsilon(0)) \subset M$ gives a homeomorphism between U and $B_\epsilon(0)$. Then the composition

$$U \xrightarrow[\approx]{\phi'_U} B_\epsilon(0) \xrightarrow[\approx]{\text{multiplication by } 2/\epsilon} B_2(0)$$

is the desired homeomorphism ϕ between a neighborhood U of $x \in M$ and $B_2(0) \subset \mathbb{R}^n$. Analogously, we construct the homeomorphism ψ. Here is a picture illustrating the situation.



The next step is to remove the open disc $\phi^{-1}(B_1(0))$ from the manifold M and the open disc $\psi^{-1}(B_1(0))$ from the manifold N. The following picture shows the resulting topological spaces $M \setminus \phi^{-1}(B_1(0))$ and $N \setminus \psi^{-1}(B_1(0))$. Here the red circles mark the points corresponding to the sphere $S^{n-1} \subset B_2(0)$ via the homeomorphisms ϕ and ψ, respectively.



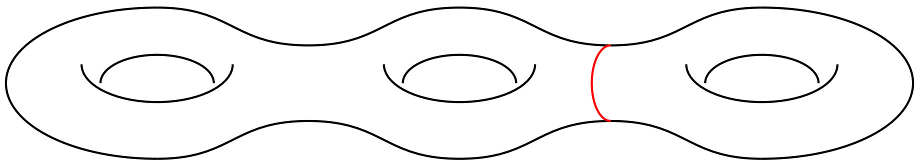
The final step is to pass to a quotient space of the union

$$M \setminus \phi^{-1}(B_1(0)) \quad \cup \quad N \setminus \psi^{-1}(B_1(0))$$

given by identifying points in $\phi^{-1}(S^{n-1})$ with their images under the homeomorphism

$$\phi^{-1}(S^{n-1}) \xrightarrow{\approx} \psi^{-1}(S^{n-1}) \qquad z \mapsto \psi^{-1}(\phi(z)).$$

The connected sum $M\#N$ is this quotient space. In terms of our pictures, the manifold $M\#N$ is obtained by gluing the two red circles, and is given by the following picture.



$M\#N$

Theorem 3.5. (Classification Theorem for compact connected 2-manifolds.) *Every compact connected manifold of dimension 2 is homeomorphic to exactly one of the following manifolds:*

- *The connected sum* $\underbrace{T\#\ldots\#T}_{g}$ *of g copies of the torus T, $g \geq 0$;*
- *The connected sum* $\underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{k}$ *of k copies of the real projective plane $\mathbb{RP}^2$, $k \geq 1$;*

3.1 Categories and functors

Before giving the formal definition of categories, let us recall examples of mathematical objects that are quite familiar.

mathematical objects	appropriate maps between these objects
sets	maps
groups	group homomorphisms
vector spaces over a fixed field	linear maps
topological spaces	continuous maps

There are obvious similarities between these four cases of mathematical objects, suggesting to destill their commonality into a definition.

Definition 3.6. A *category* C consists of the following data:

- A class $\operatorname{ob}\mathsf{C}$ of *objects of* C.

- For any two objects $A, B \in \operatorname{ob} \mathsf{C}$ a set $\mathsf{C}(A,B)$ of *morphisms from A to B*. It is common to use the notation $A \xrightarrow{f} B$ to indicate that f is a morphism from A to B, and to call A the *domain* or *source* of f, and B its *codomain* or *target*.

- Morphisms $f \in \mathsf{C}(A,B)$ and $g \in \mathsf{C}(B,C)$ can be composed to obtain a morphism $g \circ f \in \mathsf{C}(A,C)$. In other words, there is a *composition map*

$$\begin{aligned} \circ\colon \mathsf{C}(B,C) \times \mathsf{C}(A,B) &\longrightarrow \mathsf{C}(A,C) \\ (g,f) &\mapsto g \circ f \end{aligned}$$

  These are required to satisfy the following properties:

  (associativity) If $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ are morphisms, then $(h \circ g) \circ f = h \circ (g \circ f)$.

  (identity) For every object B there exists a morphism $\mathrm{id}_B\colon B \to B$, called *identity morphism* such that for all morphism $f\colon A \to B$ and $g\colon B \to C$ we have $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$.

Next we want to define what a *functor* is. As usual, before giving the formal definition, we want to give at least one example of the to be defined notion as a motivation for the definition. Our motivating example of a functor is the fundamental group:

- For each topological space X equipped with a base point $x_0 \in X$, we have its fundamental group $\pi_1(X, x_0)$.

- A continuous map $f\colon X \to Y$ with $f(x_0) = y_0 \in Y$ leads to a group homomorphism $f_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$.

From an abstract point of view, the fundamental group takes an object (X, x_0) of one category (the category of pointed topological spaces) and produces an object $\pi_1(X, x_0)$ of another category (the category of groups). Moreover, it takes a morphism $f\colon (X, x_0) \to (Y, y_0)$ in the category of pointed topological spaces and produces a morphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ in the category of groups.

Definition 3.7. A *functor* $F\colon \mathsf{C} \to \mathsf{D}$ from a category C to a category D consists of the following data:

- An assignment that maps each object $A \in \operatorname{ob} \mathsf{C}$ to an object $F(A) \in \operatorname{ob} \mathsf{D}$.

- An assignment that maps each morphism $g\colon A \to B$ in C to a morphism $F(g)\colon F(A) \to F(B)$ in D.

We require:

(Compatibility with composition) For morphisms $f\colon A \to B$ and $g\colon B \to C$ in C

$$F(g \circ f) = F(g) \circ F(f) \in \mathsf{D}(A, C).$$

(Compatibility with identities) For any object $A \in \mathsf{C}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

Examples of functors.

functors	on objects	on morphisms
$\pi_1\colon \mathsf{Top}_* \to \mathcal{G}\mathrm{p}$	$(X, x_0) \mapsto \pi_1(X, x_0)$	$(X, x_0) \xrightarrow{f} (Y, y_0) \mapsto \pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$
$-\otimes W\colon \mathsf{Vect}_k \to \mathsf{Vect}_k$	$V \mapsto V \otimes W$	$V \xrightarrow{f} V' \mapsto V \otimes W \xrightarrow{f \otimes \mathrm{id}} V' \otimes W$
$F\colon \mathsf{Set} \longrightarrow \mathsf{Vect}_k$	$S \mapsto k[S]$	$S \xrightarrow{f} T \mapsto k[S] \xrightarrow{k[f]} k[T]$

Here $k[S]$ is the k-vector space of finite linear combinations $\sum_{s \in S} k_s s$ of elements of s with coefficients $k_s \in k$. The adjective *finite* means that we require $k_s = 0$ for all but finitely many $s \in S$. The map $k[f]\colon k[S] \to k[T]$ sends $\sum_{s \in S} k_s s$ to $\sum_{s \in S} k_s f(s)$, which is a finite linear combination of elements of T.

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