Vaught's conjecture for superstable theories of finite rank

Steven Buechler*

February, 2006

Abstract

In [Vau61] Vaught conjectured that a countable first order theory has countably many or $2^{\aleph_0}$ many countable models. Here, the following special case is proved.

Theorem. *If T is a superstable theory of finite rank with $< 2^{\aleph_0}$ many countable models, then T has countably many countable models.*

The basic idea is to associate with a theory a $\bigwedge-$ definable group G (called the structure group) which controls the isomorphism types of countable models of the theory. The theory of modules is used to show that for $M \models T$, $G \cap M$ is, essentially, the direct sum of copies of finitely many finitely generated subgroups. This is the principle ingredient in the proof of the following main theorem, from which Vaught's conjecture follows immediately.

Structure Theorem. *Let T be a countable superstable theory of finite rank with $< 2^{\aleph_0}$ many countable models. Then for M a countable model of T there is a finite $A \subset M$ and a $J \subset M$ such that M is prime over $A \cup J$, J is $A-$ independent and $\{stp(a/A) : a \in J\}$ is finite.*

*Research supported by NSF. The author wishes to thank Ludomir Newelski and Elisabeth Bouscaren for comments on preliminary versions of the paper.

Contents

1 Introduction and Background

In [Vau61] Vaught conjectured that a countable first order theory has countably many or $2^{\aleph_0}$ many countable models. In this paper we prove Vaught's conjecture for superstable theories in which each complete type has finite $U-$ rank. The less common terminology is summarized in the next few paragraphs. The reader can find the fundamental results of stability theory in [She90], [Bue96] and [Pil96]. Deeper background is given in subsequent subsections. An outline of the proof is found in Subsection 1.9.

Unless stated otherwise, all theories are assumed to be superstable. Such a theory is said to be *of finite rank* if every complete type has finite U-rank ($U(-)$ denotes Lascar's rank, described in [Mak84] and [Bue96, §6.1]). A set is $\bigwedge-$ *definable over* A if it is the set of realizations in $\mathfrak{C}$ of a type over A. If H is $\bigwedge-$ definable and X is a set (for example a model) then $H(X)$ abbreviates $H \cap X$. Since we always work in T^{eq} we will shorten the terminology by simply writing T for T^{eq}. (Thus, $M \models T$ really means that M is a model of T^{eq}.) A *strong type over* A is a complete type over $acl(A)$. For a strong type p, if there is a strong type over B that is parallel to p, then $p|B$ denotes this strong type. A (possibly incomplete) type p over A is said to be *minimal* if p has exactly one complete nonalgebraic extension over any $B \supset A$. p is *minimal-by-finite* if it has finitely many such extensions over any $B \supset A$. A formula φ over A is *weakly minimal* if there is a cardinal λ such that for all $B \supset A$ there are $\leq \lambda$ nonalgebraic complete types over B containing φ. (Equivalently, $R^\infty(\varphi) = 1$, where $R^\infty(-)$ is Shelah's rank $R(-,L,\infty)$, called $\infty-$ rank.) A nonalgebraic type is weakly minimal if it contains a weakly minimal formula. The theory T is called weakly minimal if $T = T_0^{eq}$ for a weakly minimal theory T_0. The *multiplicity* of a type p, denoted $\mathrm{Mult}(p)$ is the number of complete non-forking extensions of p over the universe. (So, a complete type is minimal-by-finite if it has $U-$ rank 1 and finite multiplicity.) Two elements a and b in a model are *interalgebraic over* A if $acl(\{a\} \cup A) = acl(\{b\} \cup A)$. We say that T has *few models of cardinality* λ if it has fewer than 2^λ models of cardinality λ. T is *small* if it has countably many complete types over $\emptyset$. A stable theory is $1-$ *based* if for all a and A, $Cb(a/A) \subset acl(a)$. This is equivalent to the condition: any two sets A and B are independent over $acl(A) \cap acl(B)$.

For A a finite set we write $U(acl(A)) = k$ if $U(A) = U(tp(A)) = k$. A set $A = \{a,b,c\}$ is called an *algebraic triangle* if A is pairwise independent and $x \in A \implies x \in acl(A \setminus \{x\})$.

If T is any theory and A a set, $T(A)$ denotes the theory obtained by adding constants for the elements of A. If T is superstable of finite rank with few countable models, and A is a finite set, then $T(A)$ also satisfies these conditions. So, in subsequent sections, we will simultaneously prove facts about the family of theories, $T(A)$, as A ranges over finite sets, without explicitly referring to the parameters.

For sets A, B and C in any stable theory we say that A is *dominated by* B *over* C, and write $A \triangleleft B(C)$, if whenever $B' \underset{C}{\downarrow} B$, then $B' \underset{C}{\downarrow} A$. The following result governing the behavior of the relation under a change of base set is [Mak84, C.11(i) and (ii)]. Transitivity follows quickly from the transitivity of forking independence.

Lemma 1.1 *(i) Suppose that* $A_0 \subset A$ *and* $C \underset{A_0}{\downarrow} A$*. Then* B *dominated by* C *over* $A_0 \implies$ B *is dominated by* C *over* A*.*

(ii) Suppose that $A_0 \subset A$ *and* $BC \underset{A_0}{\downarrow} A$*. Then* B *dominated by* C *over* $A \implies B$ *is dominated by* C *over* A_0*.*

(iii) (Transitivity) *Given* $A \subset B \subset C \subset D$*, if* $D \triangleleft C(B)$ *and* $C \triangleleft B(A)$*, then* $D \triangleleft B(A)$*.*

The following result on the definability of rank is proved like the corresponding result for unidimensional theories. (See [She90, IX,1.11] or [Bue96, Proposition 7.2.1].)

Lemma 1.2 *Suppose that* T *is superstable and* $\varphi(x,a)$ *is a formula having finite* $\infty-$ *rank such that any nonalgebraic type containing* $\varphi(x,a)$ *is nonorthogonal to* $\emptyset$*. Then there is a formula* $\theta \in tp(a)$ *such that* $\models \theta(b) \implies R^{\infty}(\varphi(x,b)) = R^{\infty}(\varphi(x,a))$*.*

1.1 Minimal types

In the context of finite rank, minimal types play a central role because every regular type is nonorthogonal to a minimal type. We assume the reader is acquainted with the basic results about the geometry of minimal types, but state them here for ease of reference.

Lemma 1.3 *Let* T *be superstable and* p *a minimal strong type over* $\emptyset$*.*

(i) [Bue87c, Theorem A] If p *is nontrivial, then* p *is weakly minimal.*

(ii) [Bue85, Theorem 1] p *is locally modular or strongly minimal.*

(iii) [Bue85, Theorem 2] If p *and* q *are nonorthogonal modular minimal types, then they are not almost orthogonal.*

(iv) [Bue88, Lemma 1.3] If q *is a modular minimal type not almost orthogonal to* p*, then* p *is modular.*

(v) [Las88, Proposition 3.13] If p *is nonmodular and the nonalgebraic extension of* p *over* A *is modular, then there is an* $a \in acl(A)$ *realizing* p*.*

(vi) [Pil96, Proposition 2.1] If p *is locally modular and nonorthogonal to* $\emptyset$ *there is a modular minimal* $r \in S(acl(\emptyset))$ *nonorthogonal to* p*.*

(i)-(iv) are proved with arguments involving rank and canonical bases. (v) and (vi) require some of Hrushovski's results on the existence of groups (discussed below).

The effect of local modularity on the global properties of the theory is discussed in Subsection 1.3 of the paper. However, we assume the reader to be familiar with basic results such as: If T is superstable of finite rank and every minimal type is locally modular, then T is 1-based. (See [Bue86].)

Lemma 1.4 *Let D be a strongly minimal set over $\emptyset$.*

(i) If D is $\aleph_0-$ categorical then D is locally modular.

(ii) If D is locally modular, but nonmodular, then the unique nonalgebraic $p \in S(\emptyset)$ realized in D is isolated.

(iii) If $acl(\emptyset) \cap D$ is infinite, then any $a \in acl(D)$ is interalgebraic with a subset of D.

Proof: (i) is the Cherlin-Mills-Zil'ber Theorem, which is proved in [Zil84] and [CHL85]. The second part of the lemma is due to Zil'ber (see [Zil81]). (A complete proof is found in [Bue87b].) To prove (iii) suppose that $\bar{b}, \bar{c}$ are sequences from D, $a \downarrow \bar{c}$, and $\varphi(x, \bar{y}, \bar{z})$ is a formula such that $\models \varphi(a, \bar{b}, \bar{c})$, for all $\bar{y}$ and $\bar{z}$, $\models \exists^{<\omega} x \varphi(x, \bar{y}, \bar{z})$ and for all x and $\bar{z}$, $\models \exists^{<\omega} \bar{y} \varphi(x, \bar{y}, \bar{z})$. Since $acl(\emptyset) \cap D$ is infinite there are $\bar{c}' \in acl(\emptyset) \cap D$ and $\bar{b}'$ such that $\models \varphi(a, \bar{b}', \bar{c}')$. Thus, a and $\bar{b}'$ are interalgebraic. □

Recall that a strongly minimal set D is *pseudoprojective* if it is nontrivial and there is a $k < \omega$ such that for all a, $b \in D$ and closed $X \subset D$, $a \in acl(X \cup \{b\})$ implies that there is a $Y \subset X$ with $a \in acl(Y \cup \{b\})$ and $|Y| \leq k$. In [Bue91] the author proves

Lemma 1.5 *A pseudoprojective strongly minimal set is locally modular.*

A stationary type p over a finite set A is called *eventually nonisolated* (or e.n.i.) if there is some finite $B \supset A$ such that $p|B$ is nonisolated. It is not difficult to show that a minimal type over a finite set is e.n.i. if and only if it is not an $\aleph_0-$ categorical strongly minimal set.

Lemma 1.6 *If T is a superstable theory of finite rank with fewer than continuum many countable models, then every e.n.i. minimal type is nonorthogonal to the empty set. Furthermore, there are finitely many e.n.i. minimal types, up to nonorthogonality.*

Proof: This is proved very much like [Bou83, Proposition 4]. □

Lemma 1.7 *If T is a superstable theory with $< 2^{\aleph_0}$ many countable models and $stp(a)$ is minimal and modular, then $tp(a)$ is minimal-by-finite.*

Proof: This is proved in Proposition 3.1 of [Bue87a]. There we assumed $stp(a)$ to be weakly minimal and trivial instead of just modular, but this extra condition is not used in the proof. □

1.2 Semiminimal types

Definition 1.1 For q a minimal type and p a nonalgebraic strong type we say that p is *semiminimal with respect to q* (or $q-$ semiminimal) if there is some set A over which p and q are based and a realizing $p|A$ which is algebraic over A in a set of realizations of $q|A$. p is *semiminimal* if it is semiminimal with respect to some minimal type.

In Subsection 1.3 and Section 3 we extensively study the manner in which a model of a superstable theory of finite rank can be "constructed" in terms of the realizations of semiminimal types. In this subsection we only state the most basic properties.

Remark 1.1 It is clear that if p is $q-$ semiminimal, then any minimal type nonorthogonal to p is also nonorthogonal to q and any nonalgebraic strong type extending p is also $q-$ semiminimal. If q and q' are nonorthogonal minimal types, then a strong type is $q-$ semiminimal if and only if it is $q'-$ semiminimal. Also, if $stp(a/A)$ and $stp(b/A)$ are $q-$ semiminimal, then so is $stp(ab/A)$.

The main existence theorem for semiminimal types is

Lemma 1.8 *If T is superstable, q is minimal and $stp(a)$ is nonorthogonal to q, then there is a $b \in acl(a)$ such that $stp(b)$ is $q-$ semiminimal.*

Proof: This follows from [She90, V, Theorem 4.1] but a proof is included for convenience. Let B be a finite set over which there are elements a realizing $p|B$ and c realizing $q|B$ with $c \in acl(Ba)$. Let $C = Cb(cB/a)$ and $I = \{cB = c_0B_0, \ldots, c_nB_n\}$ a sufficiently long Morley sequence in $stp(cB/a)$ so that $C \subset acl(I)$. From $a \perp B_i$, $C \subset acl(a)$, $c_i \in acl(aB_i)$ and $stp(B_i) = stp(B_j)$, for all i, j, we obtain:

(1) $\{a, B_0, \ldots, B_n\}$ is independent, and subsequently,
(2) $stp(c_i/B_i)$ is nonorthogonal to $stp(c_j/B_j)$, for $i, j \leq n$, and
(3) C is independent from $\bar{B} = \cup_{i \leq n} B_i$.

We will now show that any $b \in C \setminus acl(\emptyset)$ realizes a $q-$ semiminimal strong type over $\emptyset$. Let $c = \{c_0, \ldots, c_n\}$ and let $M \supset \bar{B} \cup dom(q)$ be a large saturated model independent from bc over $\bar{B}$. The independence of b and M is guaranteed by (3). By (2) and the minimality of these strong types there is a Morley sequence J in $q|M$ such that $c \in acl(M \cup J)$. Thus, $b \in acl(M \cup J)$, completing the proof of the lemma. □

With respect to a strongly minimal type, being semiminimal is a property of a formula:

Lemma 1.9 *If T is superstable, q is a strongly minimal type and p is a $q-$ semiminimal strong type over $\emptyset$, then there is a $\varphi \in p$ such that every nonalgebraic strong type over $\emptyset$ containing φ is $q-$ semiminimal.*

The lemma is proved using the definability of types. The details are left to the reader.

Finding a "nice" basis for the strong types which are semiminimal with respect to strongly minimal sets is relatively easy:

Lemma 1.10 *Suppose that T is superstable, M is a model of T and q is a strongly minimal type over $a \in M$. Then, for any $b \in M$ such that $stp(b)$ is $q-$ semiminimal, $tp(b/q(M) \cup \{a\})$ is isolated.*

Proof: Let $\varphi \in q$ be strongly minimal. By the last lemma there is a formula $\psi \in stp(b)$ such that every strong type over $\emptyset$ containing ψ is $q-$ semiminimal. It follows that the restriction of the universe to $\varphi(\mathfrak{C}) \cup \psi(\mathfrak{C})$ is $\aleph_1-$ categorical. Hence, any element of $\psi(M)$ realizes an isolated type over $q(M) \cup \{a\}$. □

From this and Lemma 1.6 we obtain

Corollary 1.11 *Let T be superstable of finite rank with few countable models and M a model of T. Then there is some $a \in M$ and strongly minimal types $q_0, \dots, q_n \in S(a)$ such that, if $b \in M$ and $stp(b)$ is semiminimal with respect to an e.n.i. strongly minimal type, $tp(b/\{a\} \cup q_0(M) \cup \cdots \cup q_n(M))$ is isolated.*

1.3 Constructions and $1-$ based types

Here the manner in which a theory of finite rank is built from semiminimal sets is introduced. Deeper results will be developed in Section 3. For $C = \{c_i : i < \alpha\}$ an indexed set, $C_{<i}$ denotes $\{c_j : j < i\}$.

Definition 1.2 Given a set A, a sequence $C = \{c_i : i < \alpha\}$ is a *semiminimal construction over A* (or *sm-construction over A*) if for each $i < \alpha$, $stp(c_i/C_{<i} \cup A)$ is semiminimal or algebraic. If $A = \emptyset$ it is omitted, for brevity.

Remark 1.2 Since an extension of a semiminimal strong type is semiminimal or algebraic, a sequence C that is a sm-construction over A is also a sm-construction over $A' \supset A$.

The following is an immediate consequence of Lemma 1.8 and the finiteness of the theory's rank.

Lemma 1.12 *For any set B there is an enumeration of $acl(B)$ that is a sm-construction. If B is finite there is an $n \leq U(B)$ and $b = (b_1, \dots, b_n)$ with $stp(b_i/b_{<i})$ semiminimal for $1 \leq i \leq n$ and $acl(B) = acl(b)$.*

Notation. If $C = \{b_i : i < \alpha\}$ is an enumeration of $acl(B)$ that is a sm-construction as guaranteed in the preceding lemma, then C is called a *sm-construction of B*.

The properties collected in Theorem 2.1 depend heavily on the simplifications facilitated by locally modular minimal sets. These simplifications build on the following two results.

Lemma 1.13 *There is a sm-construction $\{b_i : i < \alpha\}$ of any given set B such that whenever $p = stp(b_i/b_{<i})$ is nonorthogonal to a locally modular minimal type, p is minimal.*

Proof: This is an immediate consequence of one of the basic properties of locally modular minimal types: If $stp(b/A)$ is non-orthogonal to a locally modular minimal type p, there is a $c \in acl(A \cup \{b\})$ such that $stp(c/A)$ is minimal and non-orthogonal to p. See [Bue86]. □

Definition 1.3 A type p is $1-$*based* if for all $A, B \subset p(\mathfrak{C})$, A and B are independent over $acl(A) \cap acl(B)$. A type-definable set is called $1-$based if the type that defines it is $1-$based.

An element a is called $1-$*based over* A if $stp(a/A)$ is $1-$based.

Remark 1.3 $1-$based is normally defined as a property of a theory. The definition here specializes to the standard one when $p = \{x = x\}$.

Lemma 1.14 *An element a is $1-$based if and only if there is an sm-construction $\{a_i : i < n\}$ of a such that $stp(a_i/a_{<i})$ is minimal and locally modular for each $i < n$.*

Proof: This follows from the proof of the main theorem in [Bue86] and Lemma 1.13. Also see the proof of Proposition 5.8 in [Pil96]. □

1.4 Multiplicity

Almost all that we know about multiplicity begins with

Lemma 1.15 (Transitivity) *If T is stable,* $\mathrm{Mult}(a/Ab) < \infty$ *and* $\mathrm{Mult}(b/A) < \infty$*, then* $\mathrm{Mult}(a/A) < \infty$.

(This was proved by Saffe in [Saf82, 1.5]. A published reference with an alternative proof is [PS85].)

Lemma 1.16 *If $stp(b)$ is properly weakly minimal and* $\mathrm{Mult}(b) < \infty$*, then $tp(b)$ is non-isolated. Furthermore, if $tp(a)$ is isolated then $a \perp b$ and $tp(a/b)$ is isolated.*

Proof: The first sentence follows from the fact that an isolated weakly minimal type of finite multiplicity has Morley rank 1. Turning to the second part, if $b \not\perp a$, then $b \in acl(a)$, contradicting that $tp(b)$ is nonisolated. Since $\mathrm{Mult}(b) < \infty$, $tp(b)$ has only finitely many extensions over a. Hence, there are only finitely many choices for $tp(a'/b)$, where a' realizes $tp(a)$, implying that $tp(a/b)$ is isolated. □

The key to Vaught's conjecture for weakly minimal theories, proved in [New90], is

Theorem 1.17 (Saffe's Condition) *Let T be superstable with few countable models, A finite and $p \in S(A)$ a nonisolated type of $U-$ rank 1. Then p is minimal-by-finite.*

(Technically, this is only proved for weakly minimal types in [New90]. If p is not weakly minimal, then Lemma 1.3(i) implies that every minimal type extending p is trivial. By Lemma 1.7 p has finite multiplicity.)

1.5 Almost atomic models

Definition 1.4 A model M is *almost atomic over* $A \subset M$ if for all $b \in M$ and all finite $A_0 \subset A$ there is a finite B, $A_0 \subset B \subset A$, $tp(a/B)$ is isolated.

Remark 1.4 If M is a model of a superstable theory, then M is almost atomic over A if M satisfies the property: there is a finite $A_0 \subset A$ such that for all finite A_1, $A_0 \subset A_1 \subset A$, $tp(b/A_1)$ is isolated. [That the later property implies that M is almost atomic over A is clear. Now let M be almost atomic over $A \subset M$ and $b \in M$. Let $B \subset A$ be a finite set such that b is independent from A over B. Since M is almost atomic over A there is a finite A_0, $A_0 \subset B \subset A$ such that $tp(b/A_0)$ is isolated. Suppose to the contrary that there is a finite $A_0 \subset A_1 \subset A$ such that $tp(b/A_1)$ is nonisolated. By the almost atomicity of M there is a finite B', $A_1 \subset B' \subset A$, with $tp(b/B')$ isolated, hence b depends on B' over A_1. This contradiction proves the equivalence.]

Lemma 1.18 *(i) If T has prime models over finite sets and A is countable then there is an almost atomic model over A.*

(ii) Let T be superstable with prime models over finite sets and $J = I_0 \cup \cdots \cup I_n$ be countable and independent over a finite set A, where I_j is indiscernible over A. Let M be an almost atomic model over $J \cup A$ with the property that for all $b \in M$ there is a finite $J_0 \subset J$ such that $tp(b/J_0 \cup A)$ has no forking extension over $J \cup A$. Then *M is prime over $J \cup A$. Furthermore, there is such an M.*

Proof: (i) is a straightforward construction. To prove (ii) it suffices to show that M is atomic over $J \cup A$. Let $b \in M$, and $J_0 \subset J$, a finite set such that $tp(b/J_0 \cup A)$ has no forking extension over $J \cup A$. Without loss of generality, $tp(b/J_0 \cup A)$ is isolated. Since J is the union of of finitely many independent sets of indiscernibles, it is easy to see that $tp(b/J_0 \cup A)$ has finitely many extensions over $J \cup A$. Thus, $tp(b/J \cup A)$ is isolated. That there is such an almost atomic model was first observed by Steinhorn [Ste80]. □

1.6 Groups

Geometrical stability deepened greatly with the work of Hrushovski on the prevalence of abelian structures in the 1-based context. Here is some background and terminology on 1-based groups which is used extensively in this paper.

The first bit of terminology is not standard but the concept is used so frequently it is worthy of a short name.

Definition 1.5 A group G that is connected and $\bigwedge -$ definable over $acl(A)$ is called *basic over* A. If $A = \emptyset$ it is omitted.

Let H be a basic group over A. Then H^- denotes the subgroup $acl(A) \cap H$. If A and B are subsets of H, or elements of H, we write $A =^* B$ if $A + H^- = B + H^-$. It follows from the proof of Proposition 2.1 in [Hru87] that if G is an $\bigwedge-$ definable group over A there is an $A-$ definable group $G_0 \supset G$ with $(G_0)^o = G$ (making use of the superstability assumption here).

Basic subgroups characterize strong types in $1-$ based groups in the following sense.

Lemma 1.19 ([HP87, Theorem 4.1]) *Let G be a $1-$ based basic group, $\bar{a} \in G^n$ and A any set. Then there is a basic group $H \subset G^n$ such that $stp(\bar{a}/A)$ is a translate of the generic in H. (That is, if $h \in H$ is generic over $A \cup \{\bar{a}\}$, then $\bar{a} + h$ is a realization of $stp(\bar{a}/A)$ that is independent from $\bar{a}$ over A.)*

It is worth emphasizing here that regardless of the set A, H is basic over $\emptyset$. In applications of this lemma it is often helpful to notice that, with notation as in the lemma, if $\bar{b}$ realizes $stp(\bar{a}/A)$ then $\bar{a} - \bar{b} \in H$. (Suppose that $\bar{b}$ realizes $stp(\bar{a}/A)$, and $\bar{c}$ is another realization of this strong type independent from $\bar{a}\bar{b}$ over A. Then, $\bar{c} - \bar{a} \in H$ and $\bar{c} - \bar{b} \in H$, so $\bar{a} - \bar{b} \in H$.) It follows from Lemma 1.19 that a basic group is abelian (see the Hrushovski-Pillay paper for details).

Algebraic dependence in $1-$ based groups is controlled by the following concept.

Definition 1.6 Let G_0 and G_1 be $\bigwedge-$ definable (over $\emptyset$) groups in an arbitrary stable theory. A subgroup H of $G_0 \times G_1$, $\bigwedge-$ definable over $acl(\emptyset)$, is called a $*-$ *homomorphism* (or *pseudo-homomorphism*) of G_0 into G_1 if the projection of H onto the first coordinate is all of G_0 and $\{a \in G_1 : (0,a) \in H\} = K$ is finite.

(Allowing H to be $\bigwedge-$ definable with parameters from $acl(\emptyset)$ is to ensure Lemma 1.20 holds. These additional parameters need to be taken into consideration when working with $*-$ homomorphisms.)

If H is a $*-$ homomorphism of G_0 into G_1 then for $K = \{a \in G_1 : (0,a) \in H\}$, H is the graph of a definable homomorphism σ_H from G_0 into G_1/K (and K is $\bigwedge-$ definable). We identify H and σ_H from hereon. With abuse of notation, we write $\sigma : G_0 \longrightarrow G_1$ if σ is a $*-$ homomorphism from G_0 into G_1.

Using that a $\bigwedge-$ definable group is contained in a definable one (in a superstable theory), when G and H are basic and $\sigma : G \longrightarrow H$ is a $*-$ homomorphism there are definable groups $G_0 \supset G$ and $H_0 \supset H$ and a definable $*-$ homomorphism $\tau : G_0 \longrightarrow H_0$ such that σ is the restriction of τ to G. Thus, as a relation on $\mathfrak{C}$ we will normally consider a $*-$ homomorphism to be a definable object.

Let G, H be basic groups and $\mathscr{A}$ the set of all $*-$ homomorphisms from G into H. For $\sigma \in \mathscr{A}$ and $a \in G$, $\sigma(a)$ denotes the obvious finite subset of H. Addition on $\mathscr{A}$ is defined by the usual rule: $\sigma + \tau(a) = \sigma(a) + \tau(a)$, where the operation on the right-hand side is on sets. An equivalence relation on $\mathscr{A}$ is defined by: $\sigma =^* \tau$ if $a \in G \implies \sigma(a) =^* \tau(a)$. Let $Hom^*(G,H) = \mathscr{A}/=^*$. It is easy to verify that $Hom^*(G,H)$ is an abelian group under $+$. We will frequently abuse the notation by writing $\sigma \in Hom^*(G,H)$, instead of

$\sigma/=^* \in Hom^*(G,H)$, for σ a $*-$ homomorphism. $Hom^*(G,G)$ is denoted $End^*(G)$, and its elements are called $*-$ *endomorphisms of* G. Two $*-$ endomorphisms of G can be multiplied by the rule: $\sigma\cdot\tau(a)=\sigma(\tau(a))$, where again the operation on the right is on sets. $End^*(G)$ is a ring under $+$ and $\cdot$.

The reader should observe that when G is basic and $R=End^*(G)$, R is a ring of endomorphisms of the group G/G^-. Normally, however, G^- is not $\bigwedge-$ definable, so G/G^- is not $\bigwedge-$ definable in $\mathfrak{C}$.

For G,H basic and $1-$ based, $\sigma\in Hom^*(G,H)$ is said to be surjective or onto H if every $d\in H$ is in $=^*\sigma(a)$ for some $a\in G$. σ is injective if $ker(\sigma)=\{a\in G: 0\in\sigma(a)\}$ is finite. If σ is injective and surjective then σ^{-1} (as a relation on $H\times G$) defines a $*-$ homomorphism from H onto G, $\sigma^{-1}\cdot\sigma=^* 1$ and σ is called a $*-$ *isomorphism.*

Algebraic dependence on generics in basic groups is characterized with the following consequence of Lemma 1.19. This is found within the proof of Theorem 3(b) in [Hru87].

Lemma 1.20 *(i) Suppose that T is superstable, A is a set, G,H are basic and $1-$ based, a is a generic of G over A and $b\in H\cap acl(\{a\}\cup A)$. Then there is a $\sigma\in Hom^*(G,H)$ such that $stp(ab/A)$ is a translate of theconnected component of the graph of σ. Moreover, $\sigma(a)=^* b'$, for some b' such that $b'\perp A$ and $U(b'/A)=U(b/A)$. If $A=\emptyset$, we may take b' to be b.*

(ii) Suppose, in addition, that $G=G_0\times\cdots\times G_n$, where G_i is a basic group, and $a=(a_0,\ldots,a_n)$, where $a_i\in G_i$, $i\le n$. Then, there are $\sigma_i\in Hom^*(G_i,H)$ such that $\sigma(a)=^*\sum_{i\le n}\sigma_i(a_i)$.

Proof: (i) Let $K\subset G\times H$ be the basic group such that $stp(ab/A)$ is a translate of the generic of K. Since $b\in acl(\{a\}\cup A)$, $\{y:\ (0,y)\in K\}=H_0$ is a finite subset of H^-. Thus, there is a $*-$ homomorphism σ from G into H whose graph is K. Since a is generic over A, the projection of K onto the first coordinate is all of G. Any $b'\in\sigma(a)$ is independent from A and has the same $U-$ rank over A as b (by the manner in which K determines $stp(ab/A)$).

Continuing, suppose that $A=\emptyset$. There is a c such that $(-a,c)\in K$, implying that $(0,b+c)=(a,b)+(-a,c)$ and (a,b) have the same coset modulo K. It follows that the difference of any two realizations of $stp((0,b+c))$ is in K. (Let $x=b+c$ and let $(0,y)$ be another realization of $stp((0,x))$. There is a tuple (a',b') realizing $stp((a,b))$ such that $(0,y)-(a',b')\in K$. As the difference of two realizations of $stp((a,b))$ is in K, $(0,x)-(0,y)+((a',b')-(a,b))$ being an element of K forces $(0,x)-(0,y)$ to be in K.) As the group H_0 above is finite, $b+c\in H^-$. Since $(a,-c)\in K$, $-c\in\sigma(a)$. Thus, $b=^*\sigma(a)$, as desired, to prove (i).

(ii) With these additional assumptions, K is a subgroup of $G_0\times\cdots\times G_n\times H$. For $i\le n$, let $K_i=\{(x,y):\ (0,\ldots,0,x,0,\ldots,0,y)\in K\}$, where the x is in the $i-th$ coordinate. As in (i), K_i is the graph of a $*-$ homomorphism σ_i from G_i into H. It is easy to verify that $\sum_{i\le n}\sigma_i(a_i)=^*\sigma(a)$, proving the lemma. □

In one later application of this lemma the following detailed information is needed. It is proved by inspecting the proof of (i) above.

Corollary 1.21 *Let G and H be basic $1-$based groups, $\psi : G \to H$ a $*-$homomorphism from G onto H and $\Psi \subset G \times H$ is the connected component of the graph of ψ. Let $a \in G$ and $d \in H$ be generics such that $\psi(a) =^* d$. Then, there is a $g \in G^-$ such that $(a+g,d) \in \Psi$.*

Corollary 1.22 *Suppose that G,H are basic and $1-$based, A is a set and $a \in G$, $b \in H$ generics over A that are interalgebraic over A. Then G and H are $*-$isomorphic.*

Proof: Let $K \subset G \times H$ be a basic subgroup such that $stp(ab/A)$ is a translate of the generic of K. Since a and b are generics, the projection of K on the first coordinate is G and the projection of K onto the second coordinate is H. Since a and b are interalgebraic over A, if $(g,h) \in K \implies g$ and h are interalgebraic. Thus, K is the graph of a $*-$isomorphism from G onto H. □

Corollary 1.23 ([Hru87]) *Suppose that $\{a_0,\ldots,a_n\}$ is an independent set of generics in the $1-$based basic group G, $b \in G$, and $b \in acl(a_0,\ldots,a_n)$. Then, there are $\alpha_i \in End^*(G)$, $i \leq n$, with $b =^* \sum_{i\leq n} \alpha_i(a_i)$.*

Corollary 1.24 *Let G be a $1-$based minimal group.*

(i) If $b,a_0,\ldots,a_n \in G$ and $b \in acl(a_0,\ldots,a_n)$, then there are $\alpha_i \in End^*(G)$ such that $b =^* \sum_{i\leq n} \alpha_i(a_i)$.

(ii) $End^*(G)$ is a division ring.

Proof: (i) Since G is minimal, there is $A \subset \{a_0,\ldots,a_n\}$, an independent set of generics such that $b \in acl(A)$. Now, apply Corollary 1.23 to b and A.

(ii) If $\alpha \in End^*(G)$ is nonzero and $a \in G$ is generic, then there is a generic $b =^* \alpha(a)$. Thus, α has finite kernel, implying that it is invertible. □

Hrushovski proves an existence theorem for such groups in [Hru87, Theorem 3(a)]:

Lemma 1.25 *If p is a nontrivial locally modular minimal type, then there are a finite set A and a minimal group G over A, such that a realization of $p|A$ is interalgebraic over A with a generic of G.*

The following lemma represents all algebraic closure in a $1-$based group in terms of $*-$homomorphisms.

Lemma 1.26 *Let c be a generic of a $1-$based basic group G and $d \in acl(c)$. Then there is a basic group H and $*-$homomorphism ψ from G onto H such that $\psi(c)$ and d are interalgebraic.*

Proof: Let $K \subset G$ be a basic group such that $stp(c/d)$ is a translate of the generic in K. Let $G_0 \supset K_0$ be definable groups with connected components G and K, respectively. Let $H_0 = G_0/K_0$, ψ the quotient map and $H = (H_0)^o$. The reader can verify that ψ is a $*-$ homomorphism from G onto H and $\psi(c)$ is interalgebraic with d (since $stp(c/d)$ is parallel to $stp(c/\psi(c))$). $\square$

We close the subsection with some terminology on actions. If $(G,\cdot)$ is a basic group over A, and P is a $\bigwedge-$ definable set we say that G *acts generically on* P if there is a definable binary operation $*$ such that (1) if $g \in G$ and $a \in P$ are $A-$ independent, then $g * a \in P$; and (2) if $g,h \in G,\ a \in P$, and $\{g,h,a\}$ is $A-$ independent, then $(g \cdot h) * a = g * (h * a)$. If $P = p(\mathfrak{C})$, for p a strong type, we say that G acts generically on p. We say that the generic action is *transitive* if for all $A-$ independent $a,b \in P$ there is a $g \in G$ such that $g * a = b$. A transitive generic action is *regular* if there is a unique such g for each independent pair a,b.

1.7 Sorted groups and abelian structures

Representation theory of modules and model theory of modules plays a significant role in our proof. We will work, however, not in the context of modules but in the more general abelian structures as defined by Fisher in [Fis77]. This term has come to mean something else over the years, so we adopt the following alternative terminology. We begin with many-sorted groups. For I a collection of sorts we call G an $I-$ *sorted group* (or *sorted group*, for short) if the universe of G consists of disjoint sets $G_i,\ i \in I$, where each G_i is a group. A sorted group G is *abelian* if each G_i is abelian. On an abelian sorted group G we let $+$ denote the group operation on any sort. When writing $a+b$ it is implicitly understood that a and b are from the same sort. Using sorted groups is a way to code into a single group some problem involving several groups; we simply interpret each group as a separate sort. Of course, we could instead form the product H of the groups in question, however, this introduces elements which are not in any of the original groups.

Fisher's definition of an abelian structure is equivalent to the following.

Definition 1.7 (i) A many-sorted language L is called *abelian* if each sort contains a constant symbol 0, a binary function $+$ and a unary function $-$. (L may contain additional function and relation symbols.)

(ii) For L an abelian language, an $L-$ structure A is called a *Fisher abelian structure* if each sort of A is an abelian group under the corresponding $0,+,-$; each additional function symbol is interpreted by a homomorphism and each additional relation symbol interpreted as a subgroup (of an appropriate product of sorts). Let $Ab(L)$ denote the class of abelian structures in L and notice that this class is elementary.

(iii) For L an abelian language, a class of $L-$ structures $\mathscr{K}$ is called an *abelian class* if the compactness theorem for systems of equations holds on $\mathscr{K}$, $\mathscr{K}$ is closed under substructures and products, and satisfies the Homomorphism Extension Property (HEP), defined as follows. $\mathscr{K}$ has the HEP if and only if, for any three models $A,B,C \in \mathscr{K}$,

$F : A \longrightarrow B$ an embedding and homomorphism $G : A \longrightarrow C$, there exists a $D \in \mathscr{K}$, $F' : C \longrightarrow D$ an embedding and homomorphism $G' : B \longrightarrow D$ such that $F' \circ G = G' \circ F$.

Remark 1.5 If L is an abelian language, $Ab(L)$ is an abelian class.

The notion of positive-primitive (pp) formula in an abelian language is defined in the obvious way.

Direct sum decompositions play a major role in the proof of the structure theorem. The concept is characterized for Fisher abelian structures as follows.

Definition 1.8 For A, B in $Ab(L)$ and $C = A + B$, C is the *direct sum* of A and B, written $A \oplus B$, if for all pp-formulas $\varphi(\bar{v}, \bar{w})$ and $\bar{a} \subset A$, $\bar{b} \subset B$, if $C \models \varphi(\bar{a}, \bar{b})$ then $C \models \varphi(\bar{a}, 0)$ and $C \models \varphi(0, \bar{b})$.

Direct sums exist in $Ab(L)$. If $Ab(L)$ is the class of $R-$ modules for some ring R, then direct sums in $Ab(L)$ are the same as direct sums as $R-$ modules.

As described in [Pre88, Section 3.2] much of the model theory of modules can be developed in an abelian class of Fisher abelian structures. In particular, pp-elimination of quantifiers holds, as well as the theory of pure-injectives, defined in the following result. Note that an embedding is *pure* if it preserves pp-formulas.

Lemma 1.27 ([Pre88, Theorem 2.8]) *Let L be an abelian language. The following conditions on $N \in Ab(L)$ are equivalent and any N satisfying them is called* pure-injective.

(i) *Every partial pp-type (in one variable) over N that is finitely satisfied in N is realized in N.*

(ii) *If N is purely embedded in $M \in Ab(L)$ then this embedding is split; i.e., $M = N \oplus M'$ for some M'.*

(iii) *For $M \in Ab(L)$ and $\bar{a}$ from M, if $\bar{b}$ in N is such that $pp^M(\bar{a}) \subset pp^N(\bar{b})$ then there is a morphism $f : M \longrightarrow N$ with $f\bar{a} = \bar{b}$.*

The theory of pure-injectives for Fisher abelian structures parallels that for modules. Notice that if M has the descending chain condition on pp-definable subgroups, then M is pure-injective.

Our approach to counting Fisher abelian structures is through the notion of representation type, defined as follows. (We call an abelian structure A finitely generated if there is a finite $\bar{a} \subset A$ such that every $b \in A$ is the image of $\bar{a}$ under a pp-definable homomorphism.)

Definition 1.9 A class $\mathscr{K}$ of Fisher abelian structures has *finite representation type* if $\mathscr{K}$ contains finitely many direct sum indecomposables, each finitely generated, and every element of $\mathscr{K}$ is a direct sum of copies of these indecomposables. Otherwise, $\mathscr{K}$ has *infinite representation type*.

Baldwin-McKenzie and, later, Prest connected the number of countable $R-$ modules with representation type (see [BM82] and [Pre84]). They prove

Lemma 1.28 *If $\mathscr{K}$ is the class of all $R-$ modules the following are equivalent:*

(1) *$\mathscr{K}$ has $< 2^{\aleph_0}$ many countable elements,*
(2) *$\mathscr{K}$ has countably many countable elements, and*
(3) *$\mathscr{K}$ has finite representation type.*

The first step in the proof is to show that every finitely generated element of $\mathscr{K}$ is pure-injective, hence every indecomposable such has local endomorphism ring. Next, we suppose that there are infinitely many finitely generated indecomposables A_i, $i < \omega$. For $X \subset \omega$ let $A_X = \bigoplus_{i \in X} A_i$. To show that $X \neq Y \implies A_X \ncong A_Y$ we use the fact that each A_i has local endomorphism ring and the following.

Lemma 1.29 *Let $M = \bigoplus_{i \in I} M_i$, where $End(M_i)$ is local for all $i \in I$, and $M = \bigoplus_{j \in J} N_j$, where $N_j \neq 0$ is indecomposable for all $j \in J$. Then a bijection $\beta : I \longrightarrow J$ exists with $M_i \cong N_{\beta(i)}$ for all $i \in I$.*

This contradiction shows that there are finitely many finitely generated indecomposables in $\mathscr{K}$. A further argument (using the pure-injectivity of each finitely generated element of $\mathscr{K}$ and Lemma 1.27) shows that each element of $\mathscr{K}$ is a direct sum of copies of these finitely generated indecomposables. Hence, $\mathscr{K}$ has finite representation type.

In our generalization of Lemma 1.28 to Fisher abelian structures we will not work in the full class $Ab(L)$, but in a sufficiently rich subclass. Moreover, the few models assumption is hard to apply outright because of a problem with defining parameters. So, we simply collect together a set of conditions sufficient to imply that a class has finite representation type, verifying the hypotheses when the time comes.

Lemma 1.30 *Let L be an abelian language and $\mathscr{K} \subset Ab(L)$ a class of structures satisfying the following conditions.*

(1) *$\mathscr{K}$ is closed under direct sums and pure submodules.*
(2) *If $A \subset M \in \mathscr{K}$ is finite there is a finitely generated $N \in \mathscr{K}$, $A \subset N \subset M$.*
(3) *Every finitely generated element of $\mathscr{K}$ is pure-injective.*
(4) *There are finitely many indecomposable elements of $\mathscr{K}$.*

Then *$\mathscr{K}$ has finite representation type.*

Proof: By (2), (3) and Lemma 1.27 every element of $\mathscr{K}$ is a direct sum of finitely generated substructures. Furthermore, each of these substructures in $\mathscr{K}$ can be written as a direct sum of indecomposable elements. $\square$

1.8 The rank 1 case

The main step in the proof of Vaught's conjecture for rank 1 theories was Newelski's proof of the Saffe Condition. It is proved in [Bue87a] that Vaught's conjecture for such theories follows from this. The result on which most of the work in [Bue87a] relies is Theorem 4.1, which we restate as

Lemma 1.31 *Suppose that T is superstable with few countable models, $tp(a)$ is isolated and properly weakly minimal, $stp(b/a)$ is properly weakly minimal and $tp(b/a)$ has finite multiplicity. Then there is a b' interalgebraic with b over a such that $tp(b')$ is minimal-by-finite.*

The following allied result is also needed in [Bue87a] and applications to this paper.

Lemma 1.32 *Suppose that T is superstable with few countable models, $tp(a)$ is properly weakly minimal, $b \downarrow a$, and $tp(a/b)$ is nonisolated. Then there is a b' interalgebraic with b over a such that $tp(b')$ is nonisolated.*

These lemmas allows us to work exclusively with minimal-by-finite types over $\emptyset$ or strongly minimal types; a model is at least almost atomic over the realizations of such types. If there are infinitely many such types and we have complete freedom to realize some types and omit others, then we can construct continuum many countable models. This is the principle behind Lemma 5.2(a) of [Bue87a], which we restate as the next lemma. (There we assumed T to be weakly minimal, however the proof is the same under these hypotheses.)

Lemma 1.33 *Suppose that T is superstable of finite rank with few countable models and $M \models T$ is countable. Let $F = \{p : p \in S(acl(\emptyset))$ is realized in M, properly weakly minimal, and its restriction to $\emptyset$ is minimal-by-finite$\}$. Then there is a finite set $A \subset \bigcup_{p \in F} p(M)$ such that every element of F is realized in $acl(A)$.*

Using Lemma 1.18(ii) and our results about minimal types we then get the following, from which Vaught's conjecture follows easily.

Theorem 1.34 (Weakly minimal structure theorem) *Let T be a weakly minimal theory with few countable models, and M a countable model of T. Then there is an e whose type over $\emptyset$ is isolated, minimal types $p_1, \ldots, p_n \in S(e)$ and bases I_j for p_j in M such that M is prime over $\{e\} \cup \bigcup_j I_j$.*

1.9 Outine of the proof

Assume throughout this subsection that the theory is superstable of finite rank with few countable models. The key to a model-theoretic structure theorem for countable models is to find a "nice" $X \subset M$ such that M is prime (i.e., atomic) over X. In this superstable context where prime models over infinite sets may not exist we settle for the almost atomic relation and later use Lemma 1.18 to get M prime over X after finding an appropriate X. The approach here is to find a set of elements X over which M is almost atomic whose elements realize types of the least possible rank with this property. To this end, define an element a to be *reducible* if there are elements $b_0, b_1 \in acl(a)$ such that $tp(a/b_0, b_1)$ is isolated and $a \notin acl(b_i)$, for $i = 0, 1$. a is *irreducible* if it is not reducible.

The majority of the paper (Sections 2–5) is devoted to proving the properties of irreducible elements collected as Theorem 2.1. Here are the fundamental ideas in proving these properties. First, we collect as $\mathbf{J}$ the irreducible elements that come from the "$\omega-$ stable part" of the theory, or purely trivial minimal sets; let $\mathbf{C}$ be the other irreducibles. It is easy to show that the structure theorem holds for $\mathbf{J}$, so all of our attention turns to $\mathbf{C}$.

That a countable model M is almost atomic over $X = \mathbf{J}(M) \cup \mathbf{C}(M)$ (Theorem 2.1(6)) is relatively easy to prove using sm-constructions and facts about minimal types found in Subsection 1.8. With similar methods it is shown that every element of $\mathbf{C}$ is a *bone*, meaning that $\text{Mult}(a/B) < \infty$ for $a \in \mathbf{C}$ and any B (Theorem 2.1(4)).

Our analysis of $\mathbf{C}$ flows through sm-constructions and the partition of these into so-called levels. We refer the reader to Definition 3.1. The results in Section 3 show how virtually all properties of elements are determined by dependence over fixed levels. If $a \in \mathbf{C}$ has m levels let $\underline{a} = \ell_{m-1}(a)$. Let $\mathbf{C}_m$ be the set of elements of $\mathbf{C}$ with $\leq m$ levels. It isn't too hard to show that $stp(a/\underline{a})$, called the *top strong type of* a, is minimal, locally modular, nontrivial, and $tp(a/\underline{a})$ is nonisolated. Since $\mathbf{C}_1$ consists of elements realizing minimal-by-finite types over $\emptyset$ (easily handled as with $\mathbf{J}$) we focus on $\mathbf{C}_m$ for $m > 1$. The elements of $\mathbf{C}_m$ realize minimal strong types over $\mathbf{C}_{m-1}$ so dependence on $\mathbf{C}_m$ over $\mathbf{C}_{m-1}$ is determined by $c\ell(-) = acl(- \cup C_{m-1})$. Using the local modularity of the top strong types we show that $c\ell$ is a projective pregeometry on $\mathbf{C}_m$ (Corollary 3.15).

With this result in place we turn from dependence relative to $\mathbf{C}_{m-1}$ to dependence between elements of $\mathbf{C}_m$ over $\emptyset$. One of the great advantages of working with irreducible elements is the Algebraic Dependence Property (Theorem 2.1): For $a \in \mathbf{C}$, $a \underset{\underline{a}}{\not\perp} b$ implies that $a \in acl(b)$. This is a primary tool in transferring dependence over $\mathbf{C}_{m-1}$ to algebraic dependence over $\emptyset$. In particular, a further argument shows that algebraic dependence on $\mathbf{C}$ is projective in the following sense: If $\{a\} \cup C \subset \mathbf{C}$ and $a \in acl(B \cup C) \setminus acl(B)$, there is $c \in acl(C) \cap \mathbf{C}$ such that $a \in acl(B \cup \{c\})$. The primary content of the projectivity property is two-fold. First, this is true without (yet) proving that the elements of $\mathbf{C}$ are $1-$based. Also, it is true restricted to $\mathbf{C}$, whereas being $1-$based only produces a $c \in \mathfrak{C}$.

Projectivity is a major property. It fuels virtually every subsequent proof. It is used in proving a "finite basis" property for $\{stp(a) : a \in \mathbf{C}_m\}$ (Proposition 4.18). It also proves (with a many-model argument) that there there is a bound on $\{U(a) : a \in \mathbf{C}\}$.

The most difficult property of $\mathbf{C}$ to prove is that each element is $1-$ based. Section 5 is devoted to proving this property. It would be surprising for a non-locally modular minimal set to support a projective closure operator. Indeed, projectivity and the few models assumption are used to prove that any minimal type "appearing" in $c \in \mathbf{C}$ is pseudomodular. Then apply Lemma 1.5 to prove that c is $1-$ based.

Reviewing the properties in Theorem 2.1, the only piece remaining is that there is a bound on the number of levels in $\mathbf{C}$ (implicit in the Bounded Rank property). This is deferred until Subsection 7.3, after we prove using groups that $\mathbf{C}$ is algebraic in the set of irreducibles having weight 1 strong types.

At this time, attention turns from properties of irreducibles, as reflected in Theorem 2.1, to generating a sorted group $\hat{H}$ such that $\mathbf{C} \subset acl(\hat{H})$ (modulo finitely many parameters). To begin, projectivity and the bound on rank shows that for $c \in \mathbf{C}_m$ with maximal rank relative to $\mathbf{C}_m$, there is $A = \{a,b,c\} \subset \mathbf{C}_m$ that is an algebraic triangle. From here, standard arguments in the $1-$ based context give rise to a basic group whose generic is interalgebraic with c (over some parameters). Continuing to analyze the relationships between elements of $\mathbf{C}$ and such groups eventually yields a sorted group $\hat{H}$, $\bigwedge-$ definable over a finite set A_0 with $tp(A_0)$ isolated such that for any model $M \supset A_0$ there is a finite $B \supset A_0$ with $\mathbf{C}(M) \subset acl(B \cup \hat{H}(M))$. This is known as the Structure Group Theorem (Theorem 7.17) and $\hat{H}$ is a *structure group* for T. This result reduces the structure theorem to proving that any $\hat{H}(M)$ is algebraic in an independent set J realizing finitely many strong types. This requires using abelian structure theory as described in Subsection 1.7.

Abelian structure theory cannot be applied to $\hat{H}$ itself since it may have $\emptyset-$ definable sets that are nonzero cosets of basic groups. This makes a smooth theory of direct sums impossible. This is remedied as follows. Using the finite basis proposition we show that after fixing a finite set, every coset $a + \hat{H}^-$ contains a generic of a basic group. We show in Proposition 6.1 that, in fact, for any finite sequence $\bar{a}$ from $\hat{H}$, $\bar{a} + \hat{H}^-$ contains a generic of a basic subgroup of $\hat{H}^n$ (some n). We factor by $\hat{H}^-$ to obtain a Fisher abelian structure as follows. For each basic subgroup K of $\hat{H}^n$ (some n) let P_K be a new relation symbol. Let L^+ contain 0, $+$ and each such P_K. Let $\hat{H}^+ = \hat{H}/\hat{H}^-$ as an L^+- structure with P_K interpreted by $(K + \hat{H}^-)/\hat{H}^-$. Then $\hat{H}^+$ is a Fisher abelian structure. Notice that $\hat{H}^+$ does not exist as structure in $\mathfrak{C}$ since $\hat{H}^-$ may not be definable.

For $A \subset \hat{H}$ let A^+ denote $(A + \hat{H}^-)/\hat{H}^-$. Let $\mathscr{C} = \{A^+ : A \subset \hat{H},\ acl(A) \cap \hat{H} = A\}$; i.e., the quotients of algebraically closed subgroups of $\hat{H}$. Using some of the fundamental properties of $\mathbf{C}$, not only is $\mathscr{C}_0 = \{\hat{H}(M)^+ : M \models T\}$ a subclass of $\mathscr{C}$, but $\mathscr{C}$ and $\mathscr{C}_0$ have the same countable elements. This certainly suggests that $\mathscr{C}$ has finite representation type. To prove this, however, additional properties of $\hat{H}^+$ are needed. Using the properties of $\hat{H}^-$ described in the preceding paragraph shows that $Th(\hat{H}^+)$ is quantifier-eliminable, totally transcendental and $\hat{H}^+$ is saturated. From here it is not difficult to show that every finitely generated element of $\mathscr{C}$ is pure-injective. A many-model argument using Azumaya's Lemma (Lemma 1.29) then shows that there are finitely many indecomposables, each finitely generated. Furthermore, every element of $\mathscr{C}$ is a direct sum of copies of these indecomposables.

Now given a countable model M there is a direct sum decomposition $\bigoplus_{i\in X} A_i^+ = \hat{H}(M)^+$ and there are finitely many A_i^+'s up to isomorphism as L^+-structures. Using that the sum is direct and that types in L^+ of a sequence $\bar{a}^+ = (\bar{a}+\hat{H}^-)/\hat{H}^-$ essentially determine $stp(\bar{a})$, there is $\bar{a}_i \subset A_i$ such that $\bar{a}_i^+$ generates A_i^+ and $A_i^+ \cong A_j^+ \implies stp(\bar{a}_i) = stp(\bar{a}_j)$, for $i, j \in X$. Then, take as J in the structure theorem $\{\bar{a}_i : i \in X\}$.

2 Irreducible Elements

The next few sections contain the analysis of dependence (under the few models assumption) needed to produce the structure group and show that it is essentially a module. While all of the theorems depend heavily on semiminimal constructions, the irreducible elements (defined below) are the key to understanding dependence on such constructions.

Definition 2.1 Let T be a superstable theory of finite rank. An element a is *reducible* if there are elements $b_0, b_1 \in acl(a)$ such that $tp(a/b_0, b_1)$ is isolated and $a \notin acl(b_i)$, for $i = 0, 1$. a is *irreducible* if it is not reducible.

Remark 2.1 Notice that if $b_0, b_1 \in acl(a)$ and $a \notin acl(b_i)$, then $U(b_i) < U(a)$, for $i = 0, 1$.

The results that drive virtually all uses of irreducible elements are collected into the following.

Theorem 2.1 (Properties of Irreducibles) *let T be a superstable theory of finite rank with few countable models. Let $\mathbf{J}$ be the set of a such that $stp(a)$ is semiminimal and has Morley rank, or $stp(a)$ is minimal and trivial. Let $\mathbf{C}$ be the set of irreducible a that are not in $\mathbf{J}$. For each $a \in \mathbf{C}$ there is $\underline{a} \in acl(a)$ such that $U(a/\underline{a}) = 1$; $\underline{a}$ is unique up to interalgebraicity.*

(1) (Algebraic Dependence Property) *For $a \in \mathbf{C}$, $a \not\perp_{\underline{a}} b$ implies that $a \in acl(b)$.*

(2) (Projectivity) *If $a \in \mathbf{C}$, $C \subset acl(\mathbf{C})$ and $a \in acl(B \cup C) \setminus acl(B)$, there is $c \in acl(C) \cap \mathbf{C}$ such that $a \in acl(B \cup \{c\})$.*

(3) (Bounded Rank) *There is a $k < \omega$ such that $a \in \mathbf{C} \implies U(a) < k$.*

(4) (Hereditary Finite Multiplicity Property) *For $a \in \mathbf{C}$ and B any set,* $\mathrm{Mult}(a/B) < \infty$.

(5) ($1-$based) *Every $a \in \mathbf{C}$ is $1-$based.*

(6) (Atomicity Property) *For M a countable model of T and $X = \mathbf{C}(M) \cup \mathbf{J}(M)$, M is almost atomic over X.*

The various properties in the theorem will be proved in subsequent sections. They are listed here not in the order in which they are proved but by a subjective prioritization of their importance.

3 Constructions and Levels

In this section the basic framework for decomposing a model of a superstable theory of finite rank with the realizations of semiminimal types is developed. **Unless stated otherwise it is assumed that the underlying theory is superstable of finite rank with few countable models.**

Notation. Given an indexed sequence $C = \{c_i : i < \alpha\}$ and $i \leq \alpha$, $C_{<i}$ denotes $\{c_j : j < i\}$.

3.1 Levels

Partitioning sm-constructions into levels gives a two-step characterization of dependence between sets: dependence between sets happens level-by-level and dependence on individual levels is on realizations of semiminimal types.

Definition 3.1 (i) Given sets A and B, the *first level of* A *over* B, $\ell_1(A/B)$ is $\{a \in acl(A) : stp(a/B)$ is semiminimal or algebraic$\}$.

(ii) Given a set C and $k \geq 0$, the k^{th} *level of* C, $\ell_k(C)$, is defined by recursion.

(1) $\ell_0(C) = \emptyset$;
(2) $\ell_{n+1}(C) = \ell_1(C/\ell_n(C)) \cup \ell_n(C)$.

We say that C *has* m *levels* if m is the least k such that $C \subset acl(\ell_k(C))$.

Remark 3.1 (i) $\ell_{k+1}(C) \supset acl(\ell_k(C))$.

(ii) Every element a has m levels for some $m \leq U(a)$. [This is straight-forward from the definition and the additivity of $U-$rank.]

Example 3.1 Let M be the direct sum of infinitely many copies of $\mathbb{Z}_4$. Recall that M has elimination of quantifiers, hence is totally categorical. Given $a \in M$ of order 4, $2a$ is an element of the strongly minimal set $2M$ and $stp(a/2a)$ is strongly minimal. Moreover, $\ell_1(a)$ is inter-algebraic with $2a$, and $\ell_2(a)$ is inter-algebraic with a over $2a$ (using elimination of quantifiers). Thus a is 2 levels.

Remark 3.2 Let a be a $1-$based element with m levels and $k \leq m$. There is a set $\bar{a} = \{a_0, \ldots, a_n\} \subset \ell_k(a)$ such that $stp(a_i/\ell_{k-1}(a))$ is minimal and $\ell_k(a) \subset acl(\bar{a} \cup \ell_{k-1}(a))$. [This follows from Lemma 1.13.]

The usefullness of the partitioning into levels lies in the following result. As stated in Proposition 3.1(ii), dependence between sets only happens level-by-level, while dependence on a given level is among realizations of semiminimal strong types.

Proposition 3.1 *Let B and C be sets and $k \geq 1$.*

(i) B is dominated by $\ell_k(B)$ over $\ell_{k-1}(B)$.

(ii) B is independent from $\ell_k(C)$ over $\ell_k(B)$.

The proposition is proved with the following lemma about the behavior of semiminimal sets vís-a-vís domination.

Lemma 3.2 *(i) Suppose that $A \subset B$, a is a set of elements realizing semiminimal strong types over B, $C \underset{A}{\perp} B$ and $C \underset{B}{\not\perp} a$. Then there is a $b \in acl(Ba)$ realizing a semiminimal strong type over A such that $a \underset{B}{\not\perp} b$.*

(ii) Suppose that $A \subset B$, aB is dominated by B over A and each element of the set b realizes a semiminimal type over A. Then, a and b are independent over B.

Proof: (i) By Remark 1.1, a can be partitioned as the union of $a_0, \ldots, a_n$, where each a_i realizes a semiminimal strong type over B and $stp(a_i/B) \perp stp(a_j/B)$, for $i \neq j$. The dependence of a and C over B forces some a_i to depend on C over B. So, replacing a by some a_i (if necessary), we may as well assume that $stp(a/B)$ is semiminimal. Also, without loss of generality, B is finite.

Let $r = stp(aB/A \cup C)$ and $C' = Cb(r)$. Let $\{a_0B_0, \ldots, a_mB_m\}$ be a Morley sequence in r in which C' is algebraic; $c' \in C'$ such that $C' \subset acl(c')$. Note:

$$a \underset{B}{\not\perp} c'. \tag{1}$$

For $\bar{a} = (a_0, \ldots, a_n)$, $stp(\bar{a}/\bigcup_{i \leq m} B_i)$ is semiminimal (by Remark 1.1), hence $stp(c'/\bigcup_{i \leq m} B_i)$ is semiminimal (or algebraic). The $A-$ independence of C and B implies the $A-$ independence of C and $\bigcup_{i \leq m} B_i$, hence, c' is $A-$ independent from $\bigcup_{i \leq m} B_i$. Thus, $stp(c'/A)$ is semiminimal. Now let b be an element of $Cb(c'/aB)$ in which it is algebraic. Combining (1) with the independence of c' and aB over b shows that a depends on b over B. Since b is algebraic over A in a sequence of realizations of $stp(c'/A)$, $stp(b/A)$ is semiminimal. Since $b \in acl(aB)$, we have proved (i).

(ii) As in the proof of (i) we may as well assume $stp(b/A)$ to be semiminimal. Let $C \supset A$ and $c_0, \ldots, c_m$ be such that b and C are $A-$ independent, $stp(c_i/C)$ is minimal and b is interalgebraic with $\bar{c} = \{c_0, \ldots, c_m\}$ over C. Without loss of generality, C is independent from $B \cup \{a, b\}$ over A. By Lemma 1.1(i),

$$\{a\} \cup B \cup C \text{ is dominated by } B \cup C \text{ over } C. \tag{2}$$

Let $\bar{c}' \subset \bar{c}$ be a maximal subset which is independent from B over C. Then, b is interalgebraic with $\bar{c}'$ over $B \cup C$, and a is independent from $\bar{c}'$ over $B \cup C$, by (2). Thus, a and b are independent over $B \cup C$. Since C is independent from $B \cup \{a, b\}$ over A, a and b are $B-$ independent, as required to prove (ii). $\square$

Proof of Proposition 3.1: (i) The bulk of the proof is contained in

Claim. $\ell_{k+1}(B)$ is dominated by $\ell_k(B)$ over $\ell_{k-1}(B)$.

Suppose towards a contradiction, that $\ell_{k+1}(B)$ is not dominated by $\ell_k(B)$ over $\ell_{k-1}(B)$. Since the elements of $\ell_{k+1}(B)$ realize semiminimal or algebraic strong types over $\ell_k(B)$, Lemma 3.2(i) yields a $c \in acl(\ell_{k+1}(B))$ realizing a semiminimal type over $\ell_{k-1}(B)$ such that c depends on $\ell_{k+1}(B)$ over $\ell_k(B)$. However, $stp(c/\ell_{k-1}(B))$ semiminimal and $c \in acl(B)$ implies that $c \in \ell_k(B)$, a contradiction that proves the claim.

An induction on $m \geq k+1$ and the transitivity of domination (Lemma 1.1(iii)) shows that $\ell_m(B)$ is dominated by $\ell_k(B)$ over $\ell_{k-1}(B)$. Since there is an m with $B \subset acl(\ell_m(B))$, (i) is proved.

(ii) This can be proved by induction on k. Assuming that B is independent from $\ell_{k-1}(C)$ over $\ell_{k-1}(B)$, B is dominated by $\ell_k(B)$ over $\ell_{k-1}(B) \cup \ell_{k-1}(C)$, by Lemma 1.1(i). Since each element of $\ell_k(C)$ realizes a semiminimal or algebraic strong type over $\ell_{k-1}(B) \cup \ell_{k-1}(C)$, Lemma 3.2(ii) implies that B is independent from $\ell_k(C)$ over $\ell_k(B)$. □

That nonisolated types with Morley rank do not “appear” above the first level in our proof is the following consequence of Lemma 3.2(ii).

Lemma 3.3 *Suppose that T has few countable models, $stp(a/A)$ is semiminimal with respect to a strongly minimal set and $tp(a/A)$ is nonisolated. Then $\{c\} \cup A$ is not dominated by A over $\emptyset$.*

Proof: Without loss of generality, A is finite. Let r be a strongly minimal type such that $stp(c/A)$ is semiminimal with respect to r. As $tp(c/A)$ is nonisolated, r is e.n.i., hence, nonorthogonal to $\emptyset$, by Lemma 1.6. Thus, by Lemma 1.8, there is a strong type p over $\emptyset$ which is semiminimal with respect to r. By Lemma 1.9, there are formulas $\varphi \in stp(c/A)$ and $\psi \in p$ such that any nonalgebraic strong type containing φ or ψ is semiminimal with respect to r. Now restrict to $\varphi \vee \psi$. More specifically, add constants to the language for A, let $\mathfrak{C}_0 = \varphi(\mathfrak{C}) \cup \psi(\mathfrak{C})$ and $T_0 = Th(\mathfrak{C}_0)$. Notice that $\mathfrak{C}_0$ is $\aleph_1-$categorical.

Let N be the prime model over c in T_0. Let ψ_0 be a strongly minimal formula extending ψ. Without loss of generality, the parameters b used in defining ψ_0 are in $\psi(N)$. By the $\aleph_1-$categoricity of the theory, N is prime over $\psi_0(N) \cup \{b\}$. As $tp(c)$ is nonisolated, $c \not\perp \psi(N)$. Returning to the original theory, c and $\psi(\mathfrak{C})$ are dependent over A. Since an element of $\psi(\mathfrak{C})$ has semiminimal strong type over $\emptyset$, $\{c\} \cup A$ is not dominated by A over $\emptyset$ by Lemma 3.2(ii). □

In a few instances we need to work over a finite set beyond $\emptyset$. The level partition does not collapse when this is done independently, as the following corollary atests.

Corollary 3.4 *Let A be a finite set and $T(A)$ the theory obtained by adding constants for the elements of A. If c has k levels and $c \perp A$ then c has k levels in $T(A)$.*

Proof: It is not hard to see that c cannot have more than k levels in $T(A)$. Arguing inductively $\ell_{k-1}(c)$ has $k-1$ levels in $T(A)$. By Proposition 3.1 and Lemma 1.1 c is independent from $\ell_{k-2}(c) \cup A$ over $\ell_{k-1}(c) \cup A$. It follows that c cannot have $< k$ levels over A, proving the corollary. $\square$

Recall the property *weight* of a strong type as defined in [She90, Ch V, Definition 3.2] (see also [Bue96, Definition 5.6.6]). Let $wt(p)$ denote the weight of p and $wt(a/B) = wt(stp(a/B))$. If $B = \emptyset$ it is omitted as usual.

The following is not used below but it does help describe levels in terms of familiar concepts.

Corollary 3.5 *For any element c, $wt(c) = wt(\ell_1(c))$.*

Proof: This is a straightforward consequence of Proposition 3.1, which is left to the reader. $\square$

The partitioning of the universe into levels is further developed in the remainder of the subsection.

Definition 3.2 Let $X \subset \mathfrak{C}$, perhaps a class with $|X| = |\mathfrak{C}|$ instead of a set. For any class Y let $\ell_1(Y/X)$ be the collection of $a \in acl(Y)$ such that $stp(a/B)$ is semiminimal or algebraic for all sufficiently large $B \subset X$.

With this notation, $\ell_1(\mathfrak{C}) = \ell_1(\mathfrak{C}/\emptyset)$ is the collection of a such that $stp(a)$ is semiminimal or algebraic. Equivalently, $\ell_1(\mathfrak{C})$ is the collection of elements with ≤ 1 level. Continuing by recursion let $\ell_{m+1}(\mathfrak{C}) = \ell_1(\mathfrak{C}/\ell_m(\mathfrak{C}))$, the collection of elements with $\leq m+1$ levels.

Corollary 3.6 *For any set A and $m < \omega$, $\ell_m(A) = acl(A) \cap \ell_m(\mathfrak{C})$. Moreover, A is independent from $\ell_m(\mathfrak{C})$ over $\ell_m(A)$.*

Proof: From the definitions it is clear that $\ell_m(A) \subset acl(A) \cap \ell_m(\mathfrak{C})$. Since every subset of $\ell_m(\mathfrak{C})$ has $\leq m$ levels, Proposition 3.1 implies that A is independent from $\ell_m(\mathfrak{C})$ over $\ell_m(A)$, from which the corollary follows. $\square$

One step in using levels to analyze dependence on a model is the characterization of dependence on $\ell_m(\mathfrak{C})$ relative to $\ell_{m-1}(\mathfrak{C})$. The next series of results contributes to our understanding.

Remark 3.3 The following principle drives much of the work in this section. Remember, we are operating under the few countable models assumption here.

Focal Point. The number of countable models is greatly controlled by properties of algebraic closure on $\ell_k(\mathfrak{C})$ over $\ell_{k-1}(\mathfrak{C})$. More specifically, we need to understand $acl(-)$ over $\ell_{k-1}(\mathfrak{C})$ restricted to the set of $a \in \ell_k(\mathfrak{C})$ such that $stp(a/\ell_{k-1}(\mathfrak{C}))$ is minimal and locally modular.

Structure theorems in superstable theories of finite rank often involve calculating the dimensions of minimal types in a model. In the weakly minimal case, e.g., every countable model contains an e, with $tp(e)$ isolated, and minimal types $p_1, \ldots, p_n \in S(e)$, each nonforking over $\emptyset$, such that M is prime over e and bases over p_j, $1 \leq j \leq n$. More generally, suppose that $p \in S(a)$ is minimal, p forks over $\emptyset$ and let p_b denote the conjugate of p over b realizing $q = tp(a)$, which may be nonisolated. Since we are studying the number of countable models we can assume that p is eventually nonisolated; i.e., not $\omega-$ categorical. Then for $\mathscr{P} = \{ p_b : b \in q(\mathfrak{C}) \}$ the possible isomorphism types for $\mathscr{P}(M) = \bigcup \{ p_b(M) : b \in q(M) \}$, for M a model, may be difficult to characterize. We must determine both the isomorphism type of $q(M)$ and $\mathscr{P}(M)$ over $q(M)$. Remember that each p_b is minimal so has a unique nonalgebraic extension over $q(M)$, in fact any set containing b. Thus, the structure of $\mathscr{P}(M)$ is determined by the size of a basis for $p_b(M)$ over $q(M)$ and dependence between this basis and bases for other $p_c(M)$'s. Dependence among bases for $p_c(M)$'s can be further clarified under the following possibilities for nonorthogonality on $\mathscr{P}$.

Case 1. p is orthogonal to $\emptyset$. Then there are unboundedly many nonorthogonality classes in $\mathscr{P}$. In fact, since p is eventually nonisolated Lemma 1.6 implies that the theory has continuum many countable models. Assume from hereon that p is nonorthogonal to $\emptyset$.

Case 2. There is a minimal or even semiminimal type r over some $e \in acl(\emptyset)$ such that $r \not\perp^{a} p$. Then $p(M) \subset acl(r(M) \cup \{a\})$, for any model. Replacing r by a union of its conjugates over $\emptyset$ we can assume that $\mathscr{P}(M) \subset acl(q(M) \cup r(M))$. Thus, the complexity created by $\mathscr{P}$ has been eliminated in favor of r, a union of semiminimal types. Assume from hereon that there is no such r. In particular, p is not modular.

Case 3. p is strongly minimal. Using that p is nonorthogonal to $\emptyset$ and eventually nonisolated it is shown in Lemma 3.3 that there is an r satisfying the condition under Case 2. This contradiction eliminates this case, hence p is properly weakly minimal.

We are left with the case of p locally modular (since it is properly weakly minimal), nonmodular, and nonorthogonal to $\emptyset$. There is a modular strong type r in $S(acl(\emptyset))$ nonorthogonal to p (Lemma 1.3(vi)). By Lemma 1.7 we can assume r is over some $e \in acl(\emptyset)$. Replacing q by one of the finitely many extensions over e we can assume that $r \not\perp p_b$, for every b realizing q. Fixing e we can require that $r \in S(\emptyset)$. In this case, for any model M and $b \in q(M)$, if p_b is realized in M by some c, $p_b(M) \subset acl(\{c,b\} \cup r(M))$. It follows that $\mathscr{P}(M)$ is determined by the dimension of $r(M)$ and $\{ b \in q(M) :$ p_b is realized in $M \}$. Of course, if subsets of $\mathscr{P}$ can be realized independently there will be continuum many countable models. Relevant to our problem, then, is the relation $CL(-)$ on $q(\mathfrak{C})$ defined by: for $\{b\} \cup X \subset q(\mathfrak{C})$, $b \in CL(X)$ if for all models M containing $\{b\} \cup X$ and realizing p_c, for $c \in X$, M also realizes p_b. The fact that the elements of $\mathscr{P}$ are minimal and nonisolated allow the following characterization of CL. Given $X \cup \{b\} \subset q(\mathfrak{C})$, $b \in CL(X)$ if and only if for any model $M \supset X \cup \{b\}$ such that M contains an e_c realizing p_c, for all $c \in X$, there is an $e_b \in acl(X \cup \{b\} \cup \{e_c : c \in X\})$.

In conclusion, the contribution of $\mathscr{P}$ to the number of countable models is controlled by $acl(-)$ on $\mathscr{P}(\mathfrak{C})$ over $q(\mathfrak{C}) \cup r(\mathfrak{C})$.

Our breakdown of the universe into levels allows us to focus on the case when p is realized in $\ell_k(\mathfrak{C})$ for some $k > 1$, in which case, $q(\mathfrak{C}) \subset \ell_{k-1}(\mathfrak{C})$. Since $k > 1$ and r

is minimal $r(\mathfrak{C})$ is also contained in $\ell_{k-1}(\mathfrak{C})$. This finally justifes the focal point given above.

In this setting of few countable models, the semiminimal strong types realized in levels >1 are nonorthogonal to locally modular minimal sets.

Lemma 3.7 *For $m>1$ and $a \in \ell_m(\mathfrak{C}) \setminus acl(\ell_{m-1}(\mathfrak{C}))$ there is $\bar{b} = \{b_0, \ldots, b_k\} \subset \ell_m(\mathfrak{C})$ such that a is interalgebraic with $\bar{b}$ over $\ell_{m-1}(\mathfrak{C})$ and $stp(b_i/\ell_{m-1}(\mathfrak{C}))$ is properly weakly minimal and nontrivial or strongly minimal and $\omega-$ categorical.*

Proof: Let a be independent from $\ell_{m-1}(\mathfrak{C})$ over $A \subset \ell_{m-1}(\mathfrak{C})$. $stp(a/A)$ is semiminimal with respect to a minimal type q. If q is not e.n.i. it is $\omega-$ categorical, hence strongly minimal and locally modular. By Lemma 1.13 there is a set $\bar{b}$ as required.

Assuming that q is e.n.i. Lemma 1.6 states that q is nonorthogonal to $\emptyset$. If q is strongly minimal Lemma 3.3 implies that a is not dominated by A over $\emptyset$. Similarly if q is trivial. This contradicts that $a \in \ell_m(\mathfrak{C}) \setminus acl(\ell_{m-1}(\mathfrak{C}))$ and $m>1$, so q is properly weakly minimal and nontrivial. Since this implies that q is locally modular there is the required set $\bar{b}$ by Lemma 1.13 again. □

In our setting, where locally modular minimal sets will dominate, the following result helps to clarify algebraic closure on $\ell_k(\mathfrak{C})$ over $\ell_{k-1}(\mathfrak{C})$, for $k>1$.

Proposition 3.8 *Let A be a class such that $\ell_1(\mathfrak{C}/A)$ contains an a such that for some set $A_0 \subset A$, $p = stp(a/A_0)$ is minimal, nontrivial, locally modular with associated division ring F. Let $X = \{b \in \ell_1(\mathfrak{C}/A) : stp(b/A) \text{ is minimal and } \not\perp p\}$, $c\ell(-) =$ algebraic closure on X over A and further assume that $(X, c\ell)$ has dimension ≥ 3. Then $(X, c\ell)$ is a projective pregeometry and the associated division ring is F.*

Proof: This is Lemma 0.6 in [New94]. A proof is included for completeness.

Claim. $(X, c\ell)$ is a projective pregeometry.

Suppose that $\{c,b\} \cup \bar{x} \subset X$ is finite and $c \in c\ell(\{b\} \cup \bar{x}) \setminus c\ell(b)$. Let B be a subset of A such that $\{c,b\} \cup \bar{x}$ is independent from A over B. Let d be the canonical base of cb over $\bar{x} \cup B$. Since c and b are in locally modular minimal sets over B, $d \in acl(\{c,b\} \cup B) \cap acl(\bar{x} \cup B)$. It follows directly that $U(d/B) = U(d/A) = 1$, hence $d \in X$. Thus, $c \in c\ell(b,d)$ and $d \in c\ell(\bar{x})$, proving the claim.

The completion of the proof and our general understanding of $(X, c\ell)$ will be aided by

Claim. Let $\bar{x} \subset X$ be finite, $\bar{x}$ independent from A over the subset B and $X_0 = X \cap \ell_1(\mathfrak{C}/B)$. If $d \in c\ell(\bar{x})$ there is an $e \in X_0$, $c\ell(e) = c\ell(d)$.

Let $C \subset A$ be finite such that $\{d\} \cup \bar{x}$ is independent from A over $B \cup C$. Then $U(d/C \cup B) = 1$ and $d \in acl(\bar{x} \cup B \cup C)$. Let e be the canonical base of $stp(dC/\bar{x} \cup B)$. Then e is interalgebraic with d over C and $e \in acl(\bar{x} \cup B)$. Since $\bar{x}$ is independent from C over B, e is independent from C over B. Thus, $e \in X_0$, completing the proof of the claim.

The standard theory of coordinatizing a projective space implies that the associated division ring can be calculated from any subspace of dimension ≥ 3. Let $\{b,c,d\} \subset X$ have dimension 3 and $B \subset A$ a set with bcd independent from A over B. Without loss of generality $B \supset A_0$ (defined in the statement of the proposition.) Let $Y = X \cap \ell_1(\mathfrak{C}/B)$ and $c\ell' =$ algebraic closure over B restricted to Y. Since B is a set $p(\mathfrak{C}) \subset Y$ and has infinite dimension in $(Y, c\ell')$. The division ring calculated in $(Y, c\ell')$ with a $3-$dimensional subset of $p(\mathfrak{C})$ is F. By the first claim $(Y, c\ell')$ is projective so F is also the division ring calculated with $\{b,c,d\}$ in $(Y, c\ell')$. By the second claim, after factoring by the closures of singletons, the subspace generated by $\{b,c,d\}$ in $(Y, c\ell')$ equals the subspace generated by $\{b,c,d\}$ in $(X, c\ell)$. Thus, F is the division ring associated to $(X, c\ell)$. $\square$

3.2 Ubiquity of Irreducibles

Here we prove Theorem 2.1(6); i.e., for M a countable model of T and $X = \mathbf{C}(M) \cup \mathbf{J}(M)$, M is almost atomic over X. We also show that most elements of $\ell_m(\mathfrak{C})$, for $m > 1$, of interest in our proof are represented by irreducible elements (i.e., interalgebraic with an irreducible over $\ell_{m-1}(\mathfrak{C})$). This is an important aspect of understanding dependence between irreducibles.

Definition 3.3 For a an element with m levels, $\underline{a}$ denotes some $d \in acl(a) \cap \ell_{m-1}(a)$ such that $\ell_{m-1}(a) \subset acl(d)$. The *top strong type of* a refers to $stp(a/\underline{a})$.

Remark 3.4 Since a in the definition has m levels, $a \subset acl(\ell_m(a) \cup \underline{a})$. Thus, a is interalgebraic over $\underline{a}$ with a sequence of elements from semiminimal sets. Moreover, $\underline{a}$ is unique up to interalgebraicity (Corollary 3.6).

Lemma 3.9 *Let a be an element such that $tp(a/\underline{a})$ is nonisolated and has $U-$ rank 1. Then there is an irreducible b such that a and b are interalgebraic over $\underline{a}$.*

Proof: If a has one level then it is easy to see that a itself is irreducible, so assume that a has more than one level. By Lemma 3.3 and the assumption that $tp(a/\underline{a})$ is nonisolated, $stp(a/\underline{a})$ is properly weakly minimal.

Let A be the set of $c \in acl(a)$ such that a depends on c over $\underline{a}$. Choose $b \in A$ such that $U(b)$ is minimal among all elements of A. We will show that b is irreducible. By Proposition 3.1 and the fact that $U(a/\underline{a}) = 1$, b and a are interalgebraic over $\underline{a}$, a and b have the same number of levels, and b is independent from $\underline{a}$ over $\underline{b}$.

Claim. There is such a b with $tp(b/\underline{b})$ nonisolated.

First, $tp(b/\underline{a})$ is nonisolated since a and b are interalgebraic over $\underline{a}$. Since b is independent from $\underline{a}$ over $\underline{b}$ and $stp(b/\underline{b})$ is properly weakly minimal, Lemma 1.32 applies to yield a b' interalgebraic with b over $\underline{a}$ such that $tp(b'/\underline{b})$ is nonisolated and has $U-$ rank 1. Using that b' is interalgebraic with b over $\underline{a}$ shows that b' has the same number of levels as b, hence $\underline{b}' \in acl(\underline{b})$. If $\underline{b}'$ and $\underline{b}$ are not interalgebraic, then $U(b') < U(b)$ contradicting the minimal rank assumption on $U(b)$. Replacing the original b by b' proves the claim.

Suppose towards a contradiction that b is reducible, $c_0, c_1 \in acl(b)$, $tp(b/c_0c_1)$ is isolated and $b \notin acl(c_i)$ for $i = 0, 1$. If b depends on c_i over $\underline{b}$ then a and c_i are interalgebraic over $\underline{a}$ since $U(b/\underline{b}) = U(a/\underline{a}) = 1$. This would contradict the minimal rank assumption on $U(b)$, so c_i, for $i = 0, 1$, are independent from b over $\underline{b}$, implying that $c_0, c_1 \in acl(\underline{b})$. Since $tp(b/c_0, c_1)$ is isolated and b is independent from c_0c_1 over $\underline{b}$, we contradict that $tp(b/\underline{b})$ is nonisolated. $\square$

Lemma 3.10 *Suppose that B is finite and $tp(c/B)$ is nonisolated. Then there is an irreducible b, $b \in acl(\{c\} \cup B)$ such that b depends on c over B.*

Proof: Let c have k levels. Arguing inductively we can assume that $tp(\underline{c}/B)$ is isolated and $tp(c/B\underline{c})$ is nonisolated. Moreover, assume that the lemma is true for c' with k levels and $U(c') < U(c)$. If $k = 1$ and some $b \in \ell_1(c)$ is semiminimal and has Morley rank then $b \in \mathbf{J}$, so we're done. So, we can assume that $k > 1$ or $stp(c)$ is orthogonal to any strong type with Morley rank. In any case there are $c_0, \ldots, c_m \in \ell_k(c)$ such that $c \in acl(c_0, \ldots, c_m, \underline{c})$ and, for $i \leq m$, $stp(c_i/\underline{c})$ is minimal and locally modular. By induction, we can assume that for $c' = \{c_0, \ldots, c_{m-1}\}$, $tp(c'/\{\underline{c}\} \cup B)$ is isolated and $tp(c_m/\{c', \underline{c}\} \cup B)$ is nonisolated. If $stp(c_m/\underline{c})$ is trivial then $k = 1$ by Lemma 3.7 and $tp(c_m)$ is nonisolated by Lemma 1.7, hence $c_m \in \mathbf{J}$. Thus, we can assume that $stp(c_m/\underline{c})$ is properly weakly minimal. Since $tp(c_m/c', \underline{c}, B)$ is nonisolated Lemma 1.32 applies to yield d such that c_m and d are interalgebraic over $\{c', \underline{c}\} \cup B$ and $tp(d/\underline{c})$ is nonisolated and has $U-$rank 1. By Lemma 3.9 there is an irreducible $b \in acl(d, \underline{c})$ with d dependent on b over $\underline{c}$. This completes the proof. $\square$

Elements can be decomposed in terms of irreducible elements of in the following sense.

Corollary 3.11 *For any element $a \notin acl(\emptyset)$ there are $b_0, \ldots, b_n \in acl(a) \cap (\mathbf{C} \cup \mathbf{J})$ such that $tp(a/b_0, \ldots, b_n)$ is isolated.*

Proof: Simply iterate Lemma 3.10. $\square$

We now restate and prove the atomicity property of the main theorem on irreducibles.

Proposition 3.12 (Theorem 2.1(6)) *Let T, $\mathbf{C}$ and $\mathbf{J}$ be as specified in the theorem. For M a countable model of T and $X = \mathbf{C}(M) \cup \mathbf{J}(M)$, M is almost atomic over X.*

Proof: Given $a \in M$ let $B \subset X$ be a finite set such that a is independent from X over B. Let A, $B \subset A \subset X$, be finite. If $tp(a/A)$ were nonisolated there would be $b \in acl(\{a\} \cup A) \cap (\mathbf{C} \cup \mathbf{J})$ such that a depends on b over A. This contradicts that a is independent from X over B, so $tp(a/A)$ is isolated, proving the proposition. $\square$

Remark 3.5 In the finalo proof of the Structure Theorem we will apply Lemma 1.18(ii), which requires an additional assumption. The extra condition will not be hard to satisfy when the time comes.

3.2.1 Levels of irreducible

The level partition of an irreducible element takes on a particular form that is especially useful in analyzing dependence among them.

Lemma 3.13 *If a is an irreducible element with > 1 level then the top strong type of a is properly weakly minimal, locally modular and nonmodular. Moreover, $tp(a/\underline{a})$ is nonisolated and minimal-by-finite.*

Proof: Let p denote the top strong type of a. That p is locally modular if it is properly weakly minimal is simply by Lemma 1.3(ii).

The lemma is proved by establishing the following properties in order.

(i) p is semiminimal.

(ii) $tp(a/\underline{a})$ is nonisolated.

(iii) A minimal type q nonorthogonal to p is nonorthogonal to $\emptyset$.

(iv) q is nontrivial.

(v) q does not have Morley rank.

(vi) p is properly weakly minimal.

(vii) $tp(a/\underline{a})$ is minimal-by-finite.

(i) Suppose to the contrary that p is not semiminimal. Since $\ell_m(a)$ is algebraic over $\underline{a}$ in a sequence of elements of semiminimal sets, there are $a_0, a_1 \subset \ell_m(a)$ such that $a \in acl(\{a_0,a_1\} \cup \underline{a})$ and $a \notin acl(\{a_i\} \cup \underline{a})$, for $i = 0,1$. Letting $b_i = \{a_i\} \cup \underline{a}$, for $i = 0,1$, and noting that $tp(a/\{b_0,b_1\})$ is isolated exhibits a contradiction to the irreducibility of a.

(ii) Since $\underline{a} \in acl(a)$ and $a \notin acl(\underline{a})$ the irreducibility of a requires that $tp(a/\underline{a})$ is nonisolated.

(iii) Let q be a minimal type nonorthogonal to p. If q is not e.n.i. then q is $\omega-$categorical. It is straightforward to show that for any finite A and b such that $stp(b/A)$ is semiminimal with respect to q, $tp(b/A)$ is isolated. Thus p nonisolated $\Longrightarrow$ q is e.n.i. By Lemma 1.6, q is nonorthogonal to $\emptyset$.

(iv) Suppose to the contrary that q is trivial. Since q is nonorthogonal to $\emptyset$ we can assume that $dom(q) = B$ is independent from a. Since q is modular and nonorthogonal to p, there is a b realizing q such that a and b are dependent over $\underline{a} \cup B$. This contradicts that a is dominated by $\underline{a}$ over $\emptyset$ to prove that q is nontrivial.

(v) This is immediate by Lemma 3.7.

(vi) Since q is nontrivial and not strongly minimal it is properly weakly minimal by Lemma 1.3(i). Moreover, by Lemma 1.3(ii), q is locally modular. Lemma 1.13 yields $(a_0,\ldots,a_k)$ interalgebraic with a over $\underline{a}$ such that $stp(a_i/\underline{a})$ is weakly minimal for $i \leq k$. The irreducibility of a implies that a is interalgebraic with a_0 over $\underline{a}$. Thus p is properly weakly minimal, completing the proof.

(vii) By (ii), (vi) and Saffe's Condition (Theorem 1.17). $\square$

Corollary 3.14 *Let* $k > 1$ *and* $X = \ell_{k-1}(\mathfrak{C})$. *There are sets* $Y_0, \ldots, Y_m \subset \ell_k(\mathfrak{C})$ *such that*

(1) $\ell_k(\mathfrak{C})$ *is almost atomic over* $Y_0 \cup \cdots \cup Y_m \cup X$.
(2) $a, b \in Y_i$ *implies* a *is irreducible and* $stp(a/\underline{a})$ *is nonorthogonal to* $stp(b/\underline{b})$.
(3) *algebraic closure on* Y_i *relative to* X *is a projective pregeometry.*

Proof: Let Z be the set of irreducible $b \in \ell_k(\mathfrak{C}) \setminus X$. Let $c \subset \ell_k(\mathfrak{C})$ be finite and $B \subset acl(c)$ a finite set of irreducibles such that $tp(c/B)$ is isolated (Corollary 3.11). Since B consists of elements of X and Z this shows that $\ell_k(\mathfrak{C})$ is almost atomic over $X \cup Z$.

By Lemma 1.6 there are finitely many e.n.i. minimal types up to nonorthogonality. For $b \in Z$, $stp(b/\underline{b})$ is minimal and e.n.i., so Z can be partitioned into $Y_0 \cup \cdots \cup Y_m$ such that (2) in the statement of the corollary holds.

It follows from Proposition 3.8 that for $W =$ the elements a of $\ell_k(\mathfrak{C})$ such that $stp(a/\underline{a})$ is minimal and nonorthogonal to the top strong type of some element of Y_i, $acl(-)$ on W over X is a projective pregeometry. To show that $acl(-)$ on Y_i is a projective pregeometry we will show that every $c \in W$ is interalgebraic over X with an element of Y_i. First, $acl(-)$ on $W \cap acl(Y_i)$ over X is a projective pregeometry. Let $c \in W \cap acl(Y_i)$. Fpr $a \in Y_i$, $tp(a/\underline{a})$ is nonisolated, so there must be a finite set A, $\underline{c} \subset A \subset X$ such that $tp(c/A)$ is nonisolated. We an incorporate A into $\underline{c}$ and apply Lemma 3.9 to find an irreducible d interalgebraic with c over $\underline{c}$. Clearly, $d \in Y_i$. That is, every element of $W \cap acl(Y_i)$ is interalgebraic with an element of Y_i over X. Thus, $acl(-)$ on Y_i over X is a projective pregeometry. This proves the corollary. □

Definition 3.4 Let $\mathbf{C}_l$ denote the elements of $\mathbf{C}$ with $\leq l$ levels.

Corollary 3.15 *For* $l > 1$ *algebraic closure on* $\mathbf{C}_l$ *over* $\ell_{l-1}(\mathfrak{C})$ *is the union of finitely many projective pregeometries.*

Proof: This is just a restatement of (3) in Corollary 3.14. □

Remark 3.6 Proposition 3.12 focuses our attention on irreducible elements. In the next section we show how to extend ths dependence relation over X to algebraic dependence over $\emptyset$.

4 Algebraic Dependence Property, Projectivity and Consequences

Here the algebraic dependence property on irreducibles Theorem 2.1(1), restated as Proposition 4.4, is proved. This is the key tool used to prove hereditary finite multiplicity and bound on rank for the elements of $\mathbf{C}$ (Theorem 2.1(4) and (3)). The first of these properties will also be proved in this section, along with a restricted version of the bound on rank. Projectivity (Theorem 2.1(2)), proved in Subsection 4.3, will follow rather quickly from the algebraic dependence property and hereditary finite multiplicity.

4.1 Algebraic Dependence Property

Most of the proof of algebraic dependence property is contained in the following lemma.

Lemma 4.1 *Let a be irreducible with $m > 1$ levels, $X = \ell_{m-1}(\mathfrak{C})$ and b such that a depends on $X \cup \{b\}$ over $\underline{a}$. Then $acl(\underline{a}) \cap acl(b) \neq acl(\emptyset)$.*

Proof: Without loss of generality, $U(b)$ is minimal among the elements $d \in acl(b)$ such that a depends on $X \cup \{d\}$ over $\underline{a}$. Since $U(a/\underline{a}) = 1$, $a \in acl(X \cup \{b\})$. If b has $< m$ levels then Proposition 3.1 implies that a is independent from $X \cup \{b\}$ over $\underline{a}$; a contradiction. Similarly, if b has $> m$ levels, a depends on $X \cup \ell_m(b)$ over $\underline{a}$, contradicting the minimality assumption on $U(b)$. Thus, b has m levels. Since $stp(a/\underline{a})$ is minimal and locally modular we can conclude that $stp(b/\underline{b})$ is minimal and locally modular. Hence, $b \in acl(X \cup \{a\})$.

Among the elements d with m levels such that $acl(X \cup \{a\}) = acl(X \cup \{d\})$ we can require that b has the additional property that $U(\underline{a}) - U(\underline{a}/b) = k$ is minimal.

Claim. $k > 0$.

Suppose to the contrary that $\underline{a} \mathop{\smile\!\!\!\!\!\mid} b$. Since a is dominated by $\underline{a}$ over $\emptyset$, a is independent from b. By Lemma 1.1 and Proposition 3.1 it follows that ab has m levels and $\ell_{m-1}(ab) = \underline{ab}$. Thus, ab is independent from X over $\underline{ab}$, contradicting that $a \in acl(X \cup \{b\})$ and proving the claim.

Let $e = Cb(\underline{a}/b)$, which, by the claim, is in $acl(b) \setminus acl(\emptyset)$. We will show that $e \in acl(\underline{a})$, which is sufficient to prove the lemma. Let a' be a realization of $stp(a/b)$, $a \mathop{\smile\!\!\!\!\!\mid}_b a'$.

Claim. $\underline{a}' \mathop{\smile\!\!\!\!\!\mid}_a b$.

By the conjugacy of a and a' over b, $acl(X \cup \{a'\}) = acl(X \cup \{b\})$ as well as $acl(X \cup \{a\}) = acl(X \cup \{b\})$. Thus, $acl(X \cup \{a\}) = acl(X \cup \{a'\})$. By the minimal rank definition of k, $U(\underline{a}') - U(\underline{a}'/a) \geq k$. That is, $U(\underline{a}'/a) \leq U(\underline{a}) - k = U(\underline{a}'/b)$. Since $\underline{a}'$ is independent from a over b, $U(\underline{a}'/a) \geq U(\underline{a}'/b)$, hence $U(\underline{a}'/a) = U(\underline{a}'/b)$. Thus, $U(\underline{a}'/ab) = U(\underline{a}'/b) = U(\underline{a}'/a)$, proving the claim.

Since $e = Cb(\underline{a}/b) = Cb(\underline{a}'/b)$ the claim implies that $e \in acl(a)$. □

A further preliminary lemma is needed before turning to the main body of the proof.

Lemma 4.2 *Let $a \in \mathbf{C}$ and $e \in acl(\underline{a})$. Then a is irreducible over e. Moreover, if a has n levels over e, $\underline{a}$ is interalgebraic with $\ell_{n-1}(a)$ over e.*

Proof: Suppose that there are $b_0, \ldots, b_n \in acl(a)$ such that $tp(a/b_0, \ldots, b_n, e)$ is isolated and $a \notin acl(b_i, e)$, for $i \leq n$. Then, letting $c_i = b_i e$ for each i, produces elements that contradict the irreducibility of a over $\emptyset$.

Let $\underline{a}_e$ denote $\ell_{n-1}(a)$ over e. If $\underline{a}_e \notin acl(\underline{a})$, then a depends on $\underline{a}_e$ over $\underline{a}$; i.e., $a \in acl(\underline{a}, \underline{a}_e)$. This contradicts the irreducibility of a. Using that a is interalgebraic with a sequence realizing semiminimal strong types over $\underline{a}_e$, and $stp(a/\underline{a}_e)$ must be nonorthogonal to $stp(a/\underline{a})$, we quickly contradict the irreducibility of a unless $\underline{a} \in acl(\underline{a}_e)$. $\square$

Corollary 4.3 *If $a \in \mathbf{C}$ and $e \in acl(\underline{a})$ then a is in $\mathbf{C}$ computed over e.*

Proposition 4.4 (Algebraic Dependence Property) *If $a \in \mathbf{C}$ and a depends on b over $\underline{a}$, then $a \in acl(b)$.*

Proof: This is proved by induction on $U(a)$. If $U(a) = 1$, equivalently a has one level, then $\underline{a} = \emptyset$ and $a \in acl(b)$. Now suppose that a has more than one level. By Lemma 4.1 there is an $e \in acl(\underline{a}) \cap acl(b) \setminus acl(\emptyset)$. By Lemma 4.2 and the fact that the top strong type of a is minimal and locally modular, a is in the class $\mathbf{C}$ as computed over e. Suppose a has n levels over e. Then, since $U(a/\underline{a}) = 1$, and a is irreducible over e, $\ell_{n-1}(a)$ over e is $\underline{a}$. Thus, a depends on b over $\ell_{n-1}(a)$, as computed over e. By induction, $a \in acl(b,e) = acl(b)$, completing the proof. $\square$

The algebraic dependence property is used in many-model arguments through the follow consequence.

Corollary 4.5 *Suppose that $a \in \mathbf{C}$ and $tp(a/b)$ is isolated. Then $a \in acl(b)$.*

Proof: Since $tp(a/\underline{a})$ is nonisolated and minimal, $a \in acl(\underline{a}, b)$. By the algebraic dependence property, $a \in acl(b)$. $\square$

Corollary 4.6 *If $A \subset \mathbf{C}$ is countable and $acl(A) \cap \mathbf{C} = A$, then there is a countable model M such that $\mathbf{C}(M) = A$.*

Proof: Apply the preceding corollary with Lemma 1.18(i). $\square$

4.2 Hereditary Finite Multiplicity Property

Using the algebraic dependence property we prove in this short subsection the Hereditary Finite Multiplicity Property. New terminology is adopted to make subsequent proofs easier to read.

Definition 4.1 An element a is called a *bone* if for all b, $\text{Mult}(a/b) < \infty$.

So, the Hereditary Finite Multiplicity Property says that every element of $\mathbf{C}$ is a bone.

Proposition 4.7 (Theorem 2.1(4)) *Every $a \in \mathbf{C}$ is a bone.*

Proof: Suppose, towards a contradiction, that $a \in \mathbf{C}$ is not a bone and $U(a)$ is minimal among such counterexamples in any theory satisfying our hypotheses. Notice that an element of $\mathbf{C}$ with one level is a bone, so a has more than one level. [If a has one level then $stp(a)$ is properly weakly minimal and $tp(a)$ is nonisolated, so $tp(a)$ is minimal-by-finite.] Also,

$(*)$ if $e \in \ell_1(a) \setminus acl(\emptyset)$, then a is a bone over e.

[By Corollary 4.3 a is in $\mathbf{C}$ computed over e, so a is a bone over e by the minimality of $U(a)$.]

Let $a' \subset \ell_1(a)$ be a maximal subset such that $b \in a' \implies \text{Mult}(b) < \infty$. Let $\tilde{a} = \{b \in \ell_1(a) : stp(b)$ is minimal and $\text{Mult}(b) = \infty\}$ (which is nonempty by $(*)$ and the transitivity of finite multiplicity Lemma 1.15). Among the counterexamples to the proposition we further assume that $U(\tilde{a}/a')$ is minimal.

Claim. $U(\ell_1(a)) = 1$ and $\text{Mult}(\ell_1(a)) = \infty$.

Let $a_0 \in \tilde{a}$ and notice that $a_0 \perp a'$. Let $a'' \subset \ell_1(a)$, $a'' \supset a'$, be independent of a_0 and such that $a_0 a''$ is interalgebraic with $\ell_1(a)$. Assume towards a contradiction that $a'' \notin acl(\emptyset)$. Then a is a bone over a'' by $(*)$, so $\text{Mult}(a_0/a'') < \infty$. By Lemma 1.32 there is a b interalgebraic with a_0 over a'' such that $b \perp a''$ and $tp(b)$ is minimal-by-finite. This contradicts the maximality of a', to prove the claim.

Let $c \in \ell_1(a)$ such that $acl(c) = \ell_1(a)$. Clearly, $stp(c)$ is properly weakly minimal and $tp(c)$ is isolated (by Saffe's Conjecture). To motivate our proof notice that if $U(a/c) = 1$ then Lemma 1.31 yields a b interalgebraic with a over c with $tp(b)$ minimal-by-finite. This would contradict that a is irreducible with more than one level. In this setting (where $U(a/c)$ may be > 1) we need a more general result, which is proved by very much the same argument using the algebraic dependence property.

Let $C = \{x : tp(x) = tp(c)\}$. The next property, which lays the foundation for the many-model argument, follows quickly from [Bue87a, Lemma 3.5]. The details are left to the reader.

(§) For $b \in C$ there are types $p_i \in S(b)$, for $i < \omega$, such that if b_i realizes p_i, $i < \omega$, then for any $j < \omega$ and finite $B \subset \{b_k : j \neq k < \omega\}$, $\text{Mult}(b_j/B \cup \{b\}) = \infty$. (A fortiori, $B \cup \{b, b_j\}$ is independent.)

Fix $b \in C$ and $p_i \in S(b)$ as in (§). For $i < \omega$ let b_i realize p_i and let a_i be such that $tp(a_i b_i) = tp(ac)$. Let $q_i = tp(a_i/b)$. For $X \subset \omega$ let M_X be a countable model almost atomic over $\{b\} \cup \{a_i : i \in X\}$. We claim that M_X realizes q_i if and only if $i \in X$. Suppose to the contrary that $i \notin X$ and $e \in M_X$ realizes q_i. Then, for some finite $Y \subset X$ and $A = \{a_j : j \in Y\}$, $tp(e/A \cup \{b\})$ is isolated. By Corollary 4.5, $e \in acl(A \cup \{b\})$. Let $f \in acl(e)$ realize p_i. By (§), $\text{Mult}(f/\{b_j : j \in Y\} \cup \{b\}) = \infty$. Since each a_j is a bone over b_j, $\text{Mult}(A/\{b_j : j \in Y\} \cup \{b\}) < \infty$. By the transitivity of finite multiplicity, Lemma 1.15, $\text{Mult}(f/A \cup \{b\}) = \infty$, contradicting that $f \in acl(A \cup \{b\})$.

This final contradiction proves the proposition. $\square$

Remark 4.1 Clearly, if A is a set of bones and $a \in acl(A)$, then a is a bone. We will use this fact when dealing with subsets of $acl(\mathbf{C})$.

The following application will see only minor uses below, but it illustrates some of the properties of $\mathbf{C}$ being developed.

Corollary 4.8 *Let* $\{c\} \cup D \subset \mathbf{C}$*,* B *and* A *finite sets with* $tp(A/B)$ *isolated and* $c \in acl(D \cup B \cup A)$*. Then,* $c \in acl(D \cup B)$*.*

Proof: By the algebraic dependence property (Theorem 2.1(1)) it suffices to show that $d \in acl(B \cup D \cup \underline{d})$. To this end we assume that d is independent from $B \cup D$ over $\underline{d}$ and show, towards a contradiction, that d is independent from A over $B \cup D \cup \underline{d}$. Specifically, we prove by induction on i that d is independent from $\ell_i(A)$ over $B \cup D \cup \underline{d} \cup \ell_{i-1}(A)$. Since $tp(A/B)$ is isolated, each element of $\ell_i(A)$ realizes an isolated semiminimal strong type over $\ell_{i-1}(A) \cup B$. If this semiminimal strong type has Morley rank it is orthogonal to $stp(d/B \cup D \cup \underline{d})$. If $\bar{b} \subset \ell_i(A)$ is a sequence of elements realizing properly weakly minimal types over $\ell_{i-1}(A)$, the fact that $D \cup \{d\}$ is a bone and Lemma 1.16 implies that $\bar{b}$ is independent from $B \cup D \cup \{d\}$ over $\ell_i(A) \cup B$. This completes the proof of the lemma. □

4.3 Projectivity

Using the Algebraic Dependence Property and Hereditary Finite Multiplicity we prove the following critical characteristic of dependence on $\mathbf{C}$.

Proposition 4.9 (Theorem 2.1(2) Projectivity) *Let* $a \in \mathbf{C}$*,* $C \subset acl(\mathbf{C})$*, and* $a \in acl(B \cup C) \setminus acl(B)$*. Then there is* $c \in acl(C) \cap \mathbf{C}$ *such that* $a \in acl(B \cup \{c\})$*.*

Proof: Let $c \in acl(C)$ have a type of least rank among all $d \in acl(C)$ such that a depends on d over $\underline{a}B$. Notice that c is a bone by Remark 4.1. By the minimal rank assumption, a is independent from $\underline{c}$ over $\underline{a}B$. Note that no matter how many levels a has, $stp(a/\underline{a})$ is properly weakly minimal and locally modular. So, c is interalgebraic over $\underline{c}$ with a set $\{c_0, \ldots, c_n\}$ such that $stp(c_i/\underline{c})$ is semiminimal and, for $c' = \{c_0, \ldots, c_{n-1}\}$, a is independent from $c'\underline{c}$ over $\underline{a}B$ and a depends on c_n over $\underline{a}Bc'\underline{c}$. This says that $stp(c_n/\underline{c})$ is nonorthogonal to $stp(a/\underline{a})$, so we can require that that strong type is properly weakly minimal and locally modular. Moreover, since c is a bone, $\text{Mult}(c_n/\underline{c}) < \infty$; i.e., $tp(c_n/\underline{c})$ is minimal-by-finite.

By Lemma 3.9 there is an irreducible d interalgebraic with c_n over $\underline{c}$. It remains to show that c is algebraic in d. Assuming that $c \notin acl(d)$, c' must be nonempty, and $d \underset{\underline{c}}{\perp} c'$ (since $c_n \underset{\underline{c}}{\perp} c'$). By the minimal rank assumption c, a depends on dc' over $B \cup \{\underline{c}, \underline{a}\}$ and a is independent from both d and c' over $B \cup \{\underline{c}, \underline{a}\}$. The fact that $stp(a/\underline{a})$ is locally modular now yields $f \in acl(dc'\underline{c})$ such that $U(f/\underline{c}) = 1$ and a depends on f

over $B\cup\{\underline{c},\underline{a}\}$. Since $U(fe)<U(dc'\underline{c})=U(c)$ we contradict the minimality assumption on $U(c)$. Thus, $a\in acl(\{d\}\cup B)$, proving the proposition. $\square$

Largely as a property of the top strong types of irreducibles, algebraic closure on $\mathbf{C}_l$ over $\ell_{l-1}(\mathfrak{C})$ is the union of finitely many projective pregeometries (Corollary 3.15). The projectivity property of irreducibles allows the transfer of this property to one about $\mathbf{C}$ itself as follows.

Corollary 4.10 *For $l>1$ algebraic closure on $\mathbf{C}_l$ over $\mathbf{C}_{l-1}$ is the union of finitely many projective pregeometries (or $\mathbf{C}_l\subset acl(\mathbf{C}_{l-1})$).*

Proof: Suppose that $\{a\}\cup A\subset \mathbf{C}_l\setminus acl(\mathbf{C}_{l-1})$ and $a\in acl(A\cup\ell_l(\mathfrak{C}))$. Let $B\subset\ell_l(\mathfrak{C})$ be such that $a\in acl(A\cup B)$. By the standard canonical base argument we can assume that B is algebraic in a set of conjugates of $\{a\}\cup A$. In other words, $B\subset acl(\mathbf{C})$. Now apply projectivity to yield $b\in acl(B)\cap\mathbf{C}$ such that $a\in acl(\{b\}\cup A)$. Since $B\subset\ell_l(\mathfrak{C})$, $b\in\mathbf{C}_{l-1}$, proving the corollary. $\square$

4.4 Bounded rank property, part I, and applications

One of the conditions in the main theorem on irreducibles is a bound on the ranks of the elements of $\mathbf{C}$ (Theorem 2.1(5)). The following proposition proves this for elements of $\mathbf{C}$ with fewer than l levels. Later, we will prove that there is a bound on the number of levels of elements of $\mathbf{C}$. The proposition is proved here because it has significant consequences on its own that will be useful in proofs resulting in the $1-$ based property. For instance, a key consequence is the existence of algebraic triangles (Corollary 4.13) that will prove the existence of definable groups with irreducible generics. They key ingredient in the proof is the projectivity of algebraic dependence on $\mathbf{C}$.

Proposition 4.11 *For each l there is a bound on $U(a)$ for $a\in\mathbf{C}$ with $\leq l$ levels.*

Proof: Every element of $\mathbf{C}$ with one level realizes a minimal strong type over $\emptyset$, so we can assume $l>1$. Suppose towards a contradiction that the proposition fails for irreducible elements with l levels, and l is the least such. Let $k^*<\omega$ be such that if $b\in\mathbf{C}$ has $<l$ levels then $U(b)\leq k^*$. Let $\Omega_0=\{U(a): a\in\mathbf{C}$ has l levels$\}$, an infinite set by hypothesis. Let $\Omega=\{k_i: i<\omega\}\subset\Omega_0$ be such that $k_0>k^*$ and, for each i, $k_i>\sum_{j<i}k_j+k^*$. Let $A=\{a_i: i\in\Omega\}$ be an independent set with $a_i\in\mathbf{C}$ having l levels and $U(a_i)=i$. For $X\subset\Omega$ let M_X be a countable model almost atomic over A_X, where A_X denotes $\{a_i: i\in X\}$.

Since the theory has few countable models there are $X,Y\subset\Omega$ with some $m\in X\setminus Y$ for which there is an isomorphism $f: M_Y\longrightarrow M_X$. Let $b_i=f(a_i)$, for $i\in Y$ and $B=\{b_i: i\in Y\}$. Then M_X is almost atomic over the independent set B as well as A_X. Thus, $a_m\in acl(B)$ (by Corollary 4.5). Let $Y_0\subset Y$ be minimal with $a_m\in acl(B_{Y_0})$. Let $Y_1\subset Y_0$ be a maximal set such that with $Y_2=Y_0\setminus Y_1$ and $B'=\{\underline{b}_i: i\in Y_1\}$, $a_m\in acl(B'\cup B_{Y_2})$. By projectivity (Proposition 4.9) there is $c \in acl(B') \cap \mathbf{C}$ such that c and a_m are interalgebraic over B_{Y_2}. Since $B' \subset \ell_{l-1}(\mathfrak{C})$, c has $< l$ levels. Thus, $U(c) \leq k^*$. If Y_2 doesn't contain an integer greater than m, the rank of $B_{Y_2} \cup \{c\}$, which is $\leq \sum Y_2 + k^*$, is $< m$ by the definition of Ω and the fact that $m \notin Y$. This contradicts that $a_m \in acl(B_{Y_2} \cup \{c\})$, so there is a maximal $n \in Y_2$ which is greater than m. Let $Y_3 = Y_2 \setminus \{n\}$. Then $a_m \in acl(B_{Y_3} \cup \{c, b_n\})$ and the maximality of Y_1 imply that b_n and $B_{Y_3} \cup \{c, a_m\}$ are dependent over $\underline{b}_n$. By the algebraic dependence property $b_n \in acl(B_{Y_3} \cup \{c, a_m\})$. Since $U(a_m) = m$ and $U(c) \leq k^*$, $n = U(b_n) \leq \sum Y_3 + m + k^*$. This contradicts the definition of Ω to prove the proposition. □

This bound is a principle ingredient in the existence of definable groups with irreducible generics, leading to the structure group. The corollary required for the group existence theorem (Corollary 4.13) is given here because of another consequence (Corollary 4.14) needed in this section.

Definition 4.2 (i) For $c \in \mathbf{C}$, a minimal type p *appears in* c if for some set B, p is nonorthogonal to $stp(c/B)$.

(ii) For $c, d \in \mathbf{C}$, d is *relevant to* c, written $d \prec c$, if d has no more levels than c and the top strong type of d is nonorthogonal to the top strong type of c.

(iii) For $c, d \in \mathbf{C}$ we write $c \approx d$ if $c \prec d$ and $d \prec c$.

(iv) $c \in \mathbf{C}$ with l levels is *maximally irreducible* if $d \prec c$ and d has l levels implies that $U(d) \leq U(c)$.

Remark 4.2 For some $c \in \mathbf{C}$ let C be the elements of $\mathbf{C}$ relevant to c. Then $C = acl(C) \cap \mathbf{C}$. It follows that $acl(-)$ restricted to C is projective.

The irreducibles relevant to c are the ones needed to understand dependence on c and its conjugates. Recall that a set A is an *algebraic triangle* if $x \in A = \{b, c, d\} \implies x \in acl(A \setminus \{x\})$ and A is pairwise independent.

Lemma 4.12 *Let $c \in \mathbf{C}$ have l levels and let $\mathbf{C}_0$ be the irreducibles relevant to c. Let d be an element of $\mathbf{C}_0$ with maximal $U-$ rank among those with l and $d \downarrow c$. Then there is a $b \in \mathbf{C}_0$ which is independent from c with $U(b) = U(d)$ and $x \in A = \{b, c, d\} \implies x \in acl(A \setminus \{x\})$.*

Proof: Since the top strong types of c and d are locally modular and nontrivial (Lemma 3.13) and $c \underset{\underline{cd}}{\downarrow} d$ there is a $b' \in acl(\{c, d\})$ with $U(b/\underline{cd}) = 1$ and $\{b', c, d\}$ an algebraic triangle over $\underline{cd}$. It's clear that $tp(b'/\underline{cd})$ is nonisolated so Lemma 3.10 implies that b' is interalgebraic over $\underline{cd}$ with an irreducible b. Then $b \in acl(c, d)$ and $\{b, c, d\}$ is an algebraic triangle over $\underline{cd}$. Since c depends on bd over $\underline{c}$, $c \in acl(b, d)$ by the algebraic dependence property. Similarly, $d \in acl(b, c)$, so $x \in A = \{b, c, d\} \implies x \in acl(A \setminus \{x\})$ $\{b, c, d\}$.

Certainly, b has l levels and is relevant to c. Since $U(d)$ is maximal among such irreducibles, $U(b) \leq U(d)$. Since $c \downarrow d$ and b is interalgebraic with d over c, $U(b) \geq U(b/c) = U(d/c) = U(d)$. Thus, $U(b) = U(b/c) = U(d)$, completing the proof. □

Corollary 4.13 *Let $c \in \mathbf{C}$ be maximally irreducible and $d \prec c$ maximally irreducible and $d \downarrow c$. Then there is a $b \in \mathbf{C}$ such that $\{b,c,d\}$ is an algebraic triangle.*

Proof: Find b as in the preceding lemma. Calculating ranks as in the last paragraph in the proof of the lemma shows that b, c and d all have maximal rank and $\{b,c,d\}$ is pairwise independent. □

Some additional corollaries play a role in the construction of the structure group later in the paper. They help characterize the complexity of $\mathbf{C}$ with respect to nonorthogonality.

Corollary 4.14 *Suppose that $c \in \mathbf{C}$ has more than one level and $\mathbf{C}_0$ is the class of irreducibles relevant to c. Then there are finitely many minimal types appearing in some element of $\mathbf{C}_0$.*

Proof: Examining the proof of Lemma 4.12 shows that if $e \in \mathbf{C}_0$ and p is a minimal type appearing in e, then p is nonorthogonal to a minimal type appearing in a $d \in \mathbf{C}_0$ of maximal rank. Furthermore, the preceding corollary implies that the same minimal types appear in the elements of $\mathbf{C}_0$ of maximal rank. That is, the minimal types appearing in some element of $\mathbf{C}_0$ with l levels are the minimal types appearing in any element of $\mathbf{C}_0$ of maximal rank with l levels. There are finitely many such minimal types. □

Corollary 4.15 *Any minimal type appearing in $c \in \mathbf{C}$ is nonorthogonal to $\emptyset$.*

Proof: By Lemma 4.12 there are irreducible b,d such that $c \downarrow d$, $c \downarrow b$ and $c \in acl(b,d)$. If $c' \in \ell_i(c)$, $c' \in acl(b,d,\ell_{i-1}(c))$, implying that $stp(c'/\ell_{i-1}(c))$ is nonorthogonal to a minimal type appearing in d. Since $c \downarrow d$ this minimal type is nonorthogonal to the empty set. □

Remark 4.3 Every e.n.i. minimal type is nonorthogonal to $\emptyset$ by Lemma 1.6, so we don't need Corollary 4.15 for these types. The corollary is needed to handel the $\omega-$ categorical strongly minimal sets that may appear in c.

Corollary 4.16 *There are finitely many nonorthogonality classes among the top strong types of elements of $\mathbf{C}$.*

Proof: Top strong types are properly weakly minimal hence e.n.i. By Lemma 1.6 there are finitely many nonorthogonality classes among them. □

Remark 4.4 If $c \in \mathbf{C}$ and d realizes $stp(c)$ then $d \approx c$. [Clearly, c and d have the same number of levels. The top strong type of any element of $\mathbf{C}$ is nonorthogonal to $\emptyset$. Since $\underline{d}$ and $\underline{c}$ realize the same strong type over $\emptyset$, the top strong types of c and d are nonorthogonal. Thus, $d \approx c$.]

The following technical property will be used the proofs of a few propositions.

Lemma 4.17 *If $c \in \mathbf{C}$, $A \subset \mathbf{C}$ is finite and $c \perp A$ then c is irreducible over A.*

Proof: Suppose c has k levels. Without loss of generality, $k > 1$. Because $c \perp A$, $\ell_i(c/A) = acl(\ell_i(c) \cup A)$ for each $i \leq k$. Thus, c has k levels over A, $\ell_{k-1}(c/A) = acl(\underline{c} \cup A)$, $U(c/\ell_{k-1}(c/A)) = 1$ and $tp(c/\underline{c} \cup A)$ is nonisolated. By Lemma 3.9 there is a b that is irreducible over A and interalgebraic with c over $\underline{c} \cup A$. By Corollary 3.11 there are $\bar{b} = \{b_0, \ldots, b_n\} \subset (\mathbf{C} \cup \mathbf{J}) \cap acl(\{b\} \cup A)$ such that $tp(b/A \cup \bar{b})$ is isolated. Since b is irreducible over A, $b \in acl(A \cup \bar{b})$ (by the algebraic dependence property). Thus, c depends on $\bar{b}$ over $\underline{c} \cup A$ and $\bar{b} \in acl(\{c\} \cup A)$. Since $U(c/\underline{c}) = 1$ there is a some i such that c depends on b_i over $\underline{c} \cup A$. Since $\{c, b_i\} \cup A \subset \mathbf{C}$, projectivity (Proposition 4.9) now applies to yield a $d \in acl(c) \cap \mathbf{C}$ interalgebraic with b_i over A. Then $U(d) = U(d/A) = U(b_i/A) < U(c)$. Since c depends on $\underline{c}$ over d, $c \in acl(d)$, contradicting the calculation of $U(d)$. □

4.5 A finite basis lemma

One of the most useful propositions in the proof of Vaught's conjecture for weakly minimal theories is the following finite basis property: if $tp(a)$ is minimal-by-finite for each $a \in A$ there is a finite $B \subset A$ such that for each $a \in A$, $stp(a)$ is realized in $acl(B)$. Proposition 4.18 generalizes this to sets of irreducibles. A weaker and easier to prove version of the result is adequate to proving the $1-$based property of $\mathbf{C}$, however the version here is used later. The main ingredients are the algebraic dependence property, the hereditary finite multiplicity property and the few models assumption.

Definition 4.3 With $\mathbf{C}$ as defined in Theorem 2.1 let $\mathbf{C}^*$ be the set of finite sequences of elements of $\mathbf{C}$.

Remark 4.5 If $c \in \mathbf{C}^*$ and $tp(c/A)$ is isolated then $c \in acl(A)$. Moreover, if $A, C \subset \mathbf{C}^*$ and $A \subset acl(C \cup B)$ there is an $A' \subset acl(C) \cap \mathbf{C}^*$ such that A and A' are interalgebraic over B. Moreover, c is a bone (by Remark 4.1). [$c = (c_0, \ldots, c_n) \subset \mathbf{C}$. $tp(c/A)$ isolated implies that for $i \leq n$, $tp(c_i/A)$ is isolated, hence $c_i \in acl(A)$ (Corollary 4.5). The moreover clause follows from the projectivity of algebraic closure on $\mathbf{C}$.]

Proposition 4.18 *Let A_0 be a finite set, A be a subset of $\mathbf{C}^*$ such that for some k, $U(c/A_0) \leq k$ for each $c \in A$. Then there is a finite $B \subset A$ such that for each $c \in A$, $stp(c/A_0)$ is realized in $acl(B \cup A_0)$.*

Proof: In the proof we use the following concept. For sets X and B we call X *minimal-like over* B if for all $Y \subset X$ and $x \in X$, $x \not\perp_B Y \implies x \in acl(Y \cup B)$. A set which is minimal-like over B is algebraic in any maximal $B-$ independent subset.

Let $P = \{ stp(c/A_0) : c \in A \}$. Arguing inductively we assume that A is a counter-example to the proposition with k minimal (ranging over all theories). We can also require that ($\sharp$) A is relatively algebraically closed in the sense that $c \in acl(A) \cap \mathbf{C}^*$ and $U(a/A_0) < k$ implies $a \in A$.

Claim 1. There is a finite set B, $A_0 \subset B \subset A$, such that $(*)$ if $a \in A$ and $U(a/A_0) < k$ then $stp(a/A_0)$ is realized in $acl(B)$. If, in addition, $a \in A$ is independent from B over A_0, $acl(B)$ contains an independent sequence of realizations of $stp(a/A_0)$ of length $k+1$.

The set B is constructed in $k+1$ steps using that the proposition is true for finite A'_0 and $a \in A$ with $U(a/A'_0) < k$. Let $B_{-1} = A_0$ and suppose that B_{i-1} has been defined. Let $B_i \supset B_{i-1}$ be a finite subset of A such that if $a \in A$ and $U(a/B_{i-1}) < k$ then $stp(a/B_{i-1})$ is realized in $acl(B_i)$. Let $B = B_k$. To check the additional clause, if $a \in A$ is independent from B over A_0 and $U(a/A_0) < k$, then for each $i \leq k$, $acl(B_i)$ contains a_i realizing $stp(a/B_{i-1})$. Since a is independent from B over A_0, $\{a_0, \ldots, a_k\} \subset B$ is a Morley sequence in $stp(a/A_0)$, proving the claim.

Since A is a counter-example with k minimal there is a sequence of elements

$(*)$ $\{ a_i : i < \omega \} \subset A$ such that for all i there is no realization of $stp(a_i/A_0)$ in A which depends on $\{a_0, \ldots, a_{i-1}\} \cup B$ over A_0.

Let $P_0 = \{ stp(a_i/A_0) : i < \omega \}$ and note that $U(a_i/B) = U(a_i/A_0) = k$. A relatively easy many-model argument would yield a contradiction if for each i there would be no realization of $tp(a_i/A_0)$ dependent on $\{a_0, \ldots, a_{i-1}\} \cup B$ over A_0. The rest of the proof is focused on finding a sequence of elements of A with this property. The problem is that $tp(a_i/B)$ is not stationary and conjugates of $stp(a_i/B)$ may not be realized in A. (Actually, it will be enough to find a sequence $\{ c_i : i < \omega \} \subset \{ a_i : i < \omega \}$ such that $tp(c_i/B)$ is not realized in $acl(\{ c_j : j \neq i \} \cup B)$.) It is tempting to replace A_0 by B, but the proof requires the relationship between A_0 and B revealed in the next claim.

Let $P' = \{ stp(a_i/B) : i < \omega \}$. Let Q be the set of strong types over B extending $tp(a_i/B)$ for some $i < \omega$.

Claim 2. Suppose that $\bar{b} = \{b_0, \ldots, b_n\} \subset A$, $\bar{b} \cup \{B\}$ is independent over A_0, $c \in A$, $stp(b_i/B), stp(c/B) \in Q$, for $i \leq n$, and $c \in acl(\bar{b} \cup B)$. Then for any $\bar{b}' \subset \bar{b}$ such that c depends on $\bar{b}'$ over B, $stp(c/A_0)$ is realized in $acl(\bar{b}' \cup B)$.

Let $\bar{b}'' = \bar{b} \setminus \bar{b}'$. By Remark 4.5 there is $e \in acl(c\bar{b}'') \cap acl(\bar{b}'B) \cap \mathbf{C}^*$ such that $c \in acl(\bar{b}''e)$. It follows that $c\bar{b}''$ is independent from $\bar{b}'B$ over eA_0, so c depends on e over A_0. Let $f \in acl(\bar{b}'') \cap \mathbf{C}^*$ be such that f and c are interalgebraic over e. Since $c \not\perp_{A_0} e$ and $\bar{b}' \perp_{A_0} e$, $U(f/A_0) = U(f/eA_0) = U(c/eA_0) < k$. Since $f \in acl(\bar{b}'')$, f is independent from B over A_0, and $f \in A$ by ($\sharp$). By $(*)$ $acl(B)$ contains a Morley sequence F in $stp(f/A_0)$ of length $k+1$. Using that e is interalgebraic with c over $\bar{b}''$ and independent from $\bar{b}''$ over A_0, shows that $U(e/A_0) \leq k$. Thus, there is an element

$f' \in F$ independent from e over A_0. Since $stp(f'/eA_0) = stp(f/eA_0)$ there is $c' \in acl(ef')$ realizing $stp(c/A_0)$. Since $ef' \in acl(\bar{b}'B)$ the claim is proved.

Claim 3. For $\bar{b}$ as in the preceding claim there are finitely many elements of Q realized in $acl(\bar{b}B)$.

Suppose that the claim is false and let $\bar{b}$ have minimal length for which it fails. Let $\{c_i : i < \omega\}$ be a subset of $acl(\bar{b}B)$ where $q_i = stp(c_i/B) \in Q$ and $i \neq j \implies q_i \neq q_j$. By the definition of Q there is, for each i, $\bar{b}_i^* d_i$ realizing $tp(\bar{b}c_i/B)$ with $stp(d_i/B) \in P'$. Since $\mathrm{Mult}(\bar{b}/B) < \infty$ (Remark 4.5) we may assume that $\bar{b}_i^* =$ a fixed $\bar{b}^*$, for all i.

Since $D = \{d_i : i < \omega\} \subset acl(\bar{b}^*B)$, D cannot be $B-$ independent. If D were a subset of A we would have a direct contradiction to the properties of $\{a_i : i < \omega\}$. While this may not be true, a $B-$ independent subset of D realizes the same strong type over B as some subset of $\{a_i : i < \omega\}$. This will be coupled with the following subclaim to prove Claim 3.

Subclaim. There is a finite $B-$ independent $D' \subset D$ and an infinite $D_0 \subset D$ containing D' such that $D_0 \subset acl(D' \cup B)$.

In the terminology at the start of the proof the target set D_0 will be minimal-like over B. We first claim there is an infinite subset of $D \cup \bar{b}^*$ that is minimal-like over B. Suppose that this is not the case. Let f be an automorphism fixing B and taking $\bar{b}^*$ to $\bar{b}$. Letting $e_i = f(d_i)$, $stp(e_i/B) \in Q$ and $e_i \in acl(\bar{b} \cup B)$, for each i. Let $E = \{e_i : i < \omega\}$. By the assumption on D, $E \cup \bar{b}$ does not contain an infinite set which is minimal-like over B. Let E' be a maximal minimal-like subset of $E \cup \bar{b}$ that contains $\bar{b}$. By this maximality every element of $E \setminus E'$ depends on a subset of E' over B but is not algebraic in that subset over B. Since E is infinite and E' is finite we may as well assume there is a fixed subset E'' of E' such that each element of $E \setminus E'$ depends on E'' over B but is not in $acl(E'' \cup B)$. Because E' is minimal-like over B we can assume that E'' is $B-$ independent. Moreover, there is a $B-$ independent E^*, $E'' \subset E^* \subset E'$, with $E' \subset acl(E^* \cup B)$. Since $E \subset acl(\bar{b} \cup B) = acl(E' \cup B) = acl(E^* \cup B)$, the second claim implies that for each $i < \omega$, $stp(e_i/B)$ is realized in $acl(E'' \cup B)$. A calculation using $U-$ rank shows that $|E''| < |\bar{b}|$. [Since $E \subset acl(\bar{b} \cup B)$ every subset of $E \cup \bar{b}$ has $U-$ rank $\leq k \cdot |\bar{b}|$. Every $q \in Q$ has $U-$ rank k, so $U(E''/B) = k \cdot |E''|$. For $e_i \in E \setminus E'$, $U(e_i/E'' \cup B) > 0$, so $U(E'' \cup \{e_i\}/B) > k \cdot |E''|$. Thus $|E''|$ must be $< |\bar{b}|$.] This contradicts that $\bar{b}$ is a counter-example to the claim of minimal length. This shows that $E \cup \bar{b}$, hence $D \cup \bar{b}^*$, contains an infinite minimal-like set F, the goal of this paragraph.

Let $D_0 = F \cap D$ and $\bar{b}^{**} = F \cap \bar{b}^*$. Since F is minimal-like over B and is algebraic over B in a finite set, a maximal $B-$ independent set $D' \subset D_0$ is finite. Since F is minimal-like, $D_0 \subset acl(D' \cup B)$. This proves the subclaim.

Since D' is $B-$ independent there is $A' \subset \{a_i : i < \omega\}$ realizing $stp(D'/B)$. By the subclaim there are infinitely many j such that $stp(a_j/B)$ is realized in $acl(A' \cup B)$. This contradicts $(*)$ to prove the claim.

Let $R = \{tp(c/B) : stp(c/B) \in Q\}$ and $A' = \{c \in A : stp(c/B) \in Q\}$. Notice that the third claim implies that there are only finitely many elements of R realized in $acl(\bar{b} \cup B)$, where $\bar{b} \subset A'$ is a $B-$ independent sequence.

Claim 4. There is an infinite subset $R' = \{r_i : i < \omega\}$ of R such that for $q_0, \dots, q_n$ distinct elements of R' and $c_i \in A'$ realizing q_i, if $\{c_0, \dots, c_{n-1}\}$ is $B-$ independent, then c_n is not in $acl(\{c_0, \dots, c_{n-1}\} \cup B)$.

The r_i's are found by recursion. Suppose that $r_0, \dots, r_i$ have been found with the desired property. For any fixed $B-$ independent sequence $d_0, \dots, d_i \in A'$, where d_j realizes r_j, there are finitely many elements of R realized in $acl(\{d_0, \dots, d_i\} \cup B)$ (Claim 3). The set of types realized is controlled by $stp(d_0, \dots, d_i/B)$. Since elements of R have finite multiplicity there are finitely many elements of R realized in $acl(\{d_0, \dots, d_i\} \cup B)$, for all such d_j. Let $r_{i+1} \in R$ be an element not realized in $acl(\{d_0, \dots, d_i\} \cup B)$, for any $B-$ independent sequence $d_0, \dots, d_i \in A'$, where d_j realizes r_j.

To verify that $R' = \{r_i : i < \omega\}$ satisfies the conditions of the claim assume there are $r_{i_0}, \dots, r_{i_m}, r_l, r_{j_0}, \dots, r_{j_n} \in R'$ and $C = \{c_{i_0}, \dots, c_{i_n}, c_{j_0}, \dots, c_{j_n}\}$ $B-$ independent, $c_j \in A'$ realizing r_j, $i_0 < \cdots < i_m < l < j_0 < \cdots < j_n$ and $c_l \in acl(C \cup B)$. Suppose $m+n$ is minimal for which there is such a sequence. Let $C' = C \setminus \{c_{j_n}\}$. If c_l depends on C' over B then Claim 2 implies that $stp(c_i/B)$ is realized in $acl(C' \cup B)$. This contradicts the minimality assumption on $m+n$, so $U(c_l/C' \cup B) = k$. Thus, $c_{j_n} \in acl(\{c_l\} \cup C' \cup B)$ (by a $U-$ rank calculation). This contradicts the construction of the r_j's to prove the claim.

This final claim enables a many-model argument that yields the final contradiction. Fix a set $\{c_i : i < \omega\} \subset A$ where c_i realizes r_i. For $X \subset \omega$ let $C_X = \{c_i : i \in X\}$ and $M_X \supset C_X \cup B$ a countable model almost atomic over $C_X \cup B$. The few models condition on the theory yields $X, Y \subset \omega$ with $i \in Y \setminus X$ such that r_i is realized by $d \in M_X$. By Remark 4.5, $d \in acl(C_X \cup B)$. Since A is relatively algebraically closed $(\sharp)$, $d \in A$. This contradicts Claim 4 to complete the proof of the proposition. $\square$

5 $1-$based Property

Here the technical properties of irreducible elements are combined to prove the $1-$based property. The main tool is the projectivity of algebraic closure on $\mathbf{C}$ but the finite basis lemma (Proposition 4.18) and the bound on rank (Proposition 4.11) play an essential role. Several new terms will be defined here but they are only used in this section.

Assuming the $1-$based property fails we will find an element $c^* \in \mathbf{C}$ with $\ell_1(c)$ contained in a non-locally modular strongly minimal set D. Projectivity of algebraic closure on $\mathbf{C}$ induces a projective closure operator on conjugates of $\ell_1(c)$. This is unusual since D is non-locally modular, but to obtain a contradiction deeper information on closure on the conjugates of c^* is needed. This will be motivated further as the proof develops.

Definition 5.1 For any c, $\tilde{c}$ denotes the maximal subset of $\ell_1(c)$ that is orthogonal to any locally modular minimal set.

$\tilde{c}$ consists of realizations of strong types semiminimal with respect to non-locally modular strongly minimal sets, and $\ell_1(c) \setminus \tilde{c}$ is $1-$based (or in $acl(\emptyset)$).

Definition 5.2 $c \in \mathbf{C}$ is called *fictional* if it is not $1-$based.

We assume until the final contradiction that there is a fictional irreducible element in some theory of finite rank with few countable models.

Lemma 5.1 *There is a fictional irreducible c (in some theory) such that*

(1) *c has the minimum number of levels among all fictional irreducibles,*
(2) *c is $1-$based over $\tilde{c}$, and*
(3) *$\tilde{c} = \ell_1(c) \subset D$ for some strongly minimal set D over $\emptyset$ with $D \cap acl(\emptyset)$ infinite.*

Proof: To begin let c be a fictional irreducible with the minimum number l of levels among fictional irreducibles in all relevant theories. For $\bar{d} = \ell_1(c)$, c is irreducible over $\bar{d}$ (Lemma 4.2) and has $l-1$ levels over $\bar{d}$. Since c is fictional with a minimum number of levels, c is $1-$based over $\bar{d}$. In fact, c is $1-$based over $\tilde{c}$.

Partition $\tilde{c}$ as $c_0, \ldots, c_n$, where $stp(c_i)$ is semiminimal and these strong types are pairwise orthogonal. Let $d \in acl(c)$ be a maximal element such that $\tilde{d} = \{c_1, \ldots, c_n\}$. In the theory T' with a constant for d, $\tilde{c} = c_0$. Moreover, if $e \in \ell_i(c)$ (over d) and $e \notin acl(c_0)$ then $ed \in acl(c)$, $\tilde{ed} = \{c_1, \ldots, c_n\}$ and $ed \notin acl(d)$, contradicting the maximality of d. Thus, in T', c is fictional, $\tilde{c} = \ell_1(c) = c_0$.

By the semiminimality of $stp(c_0)$ there is an e independent from c such that c_0 is interalgebraic over e with elements from a strongly minimal set D over e. Since D is not locally modular it is not $\omega-$categorical. Thus, by picking e large enough we can require that $acl(e) \cap D$ is infinite. By Lemma 4.17, c is irreducible over e and c has the same levels over e as over $\emptyset$. Adding constants for the elements of e produces the fictional irreducible c as required for the lemma. $\square$

Fix a fictional irreducible c^* and a strongly minimal set D as in the lemma, and let l be the number of levels of c^*. Let $\mathscr{C}$ be the class of irreducibles relevant to c^*. Let $\mathscr{C}'$ be the elements of $\mathscr{C}$ with $< l$ levels and $\hat{\mathscr{C}}$ the $1-$based elements of $\mathscr{C}$. Since c^* is a fictional with the fewest number of levels, $\hat{\mathscr{C}} \supset \mathscr{C}'$. Remark 4.2 states that $\mathscr{C}$ is algebraically closed relative to $\mathbf{C}$. It follows from Corollary 4.10 that $acl(-)$ on $\mathscr{C}$ relative to $\mathscr{C}'$ is a projective pregeometry. By the same reasoning $\hat{\mathscr{C}}$ is algebraically closed in $\mathbf{C}$. Since $\hat{\mathscr{C}} \supset \mathscr{C}'$,

$$c\ell(-) = acl(- \cup \hat{C}) \text{ on } \mathscr{C} \text{ is a projective pregeometry.} \tag{3}$$

Let $DIM(-)$ denote $c\ell-$dimension. Notice that $a \in c\ell(B)$ if there is $e \in \hat{\mathscr{C}}$ interalgebraic with a over B.

For $A \subset \mathscr{C}$ and $X \subset D$, the *support of A is X* if X is interalgebraic with $\tilde{A}$ and A is *supportable* if $\tilde{A} \subset D$. It isn't immediately clear that every $A \subset \mathscr{C}$ is supportable. (Since every element of $\mathscr{C}$ is relevant to c^*, $\tilde{A}$ is semiminimal with respect to D but it may not be a subset of D.) However, the class of supportable elements of $\mathscr{C}$ is algebraically closed within $\mathscr{C}$ (using Lemma 1.4(iii)), so there is no harm in using $DIM(-)$ in reference to $c\ell-$ closures of supportable sets. For X a closed subset of D let $\mathscr{C}_X$ denote the set of elements of $\mathscr{C}$ with support $\subset X$. Of course, $\mathscr{C}_X \supset \hat{\mathscr{C}}$.

The following lemma is the principle technical tool in the proof. It facilitates a many-model argument later in the section. The fact that D is a non-locally modular strongly minimal set has little to do with the proof. An expanded argument shows that when $c \in \mathbf{C}$ has l levels and $Y = \ell_1(c)$, $\{d : stp(d/Y) = stp(c/Y)\}$ has finite dimension relative to the irreducibles with $< l$ levels. We won't prove this generalization since it isn't needed in the paper.

Being able to vary $DIM(\mathscr{C}_X)$ in a many-model argument requires the following.

Lemma 5.2 *If X is a finite dimensional closed subset of D then $DIM(\mathscr{C}_X)$ is finite.*

Proof: Let X be a set of least dimension in D for which the lemma fails. It is clear then that every element of X is in the support of some element of $\mathscr{C}_X$.

Claim 1. For some $c \in \mathscr{C}_X$, $DIM(\{d : stp(d/X) = stp(c/X)\})$ is infinite.

Let $P = \{stp(c/X) : c \in \mathscr{C}_X\}$. Each element of $\mathscr{C}$ has $\leq l$ levels so there is a bound k on $U(c)$, for $c \in \mathscr{C}$ (by Proposition 4.11). By Proposition 4.18 there are $c_0, \ldots, c_m \in \mathscr{C}_X$ such that each element of P is realized in $acl(\{c_0, \ldots, c_m\} \cup X)$. In fact,

$$d \in acl(\{c_0, \ldots, c_m\} \cup X) \implies d \in acl(\{c_0, \ldots, c_m\}) \text{ (by projectivity).}$$

Since each element of X is in the support of some element of $\mathscr{C}_X$, $X \subset acl(\{c_0, \ldots, c_m\})$. Thus, letting $p_i = stp(c_i/X)$, for $i \leq m$, $\mathscr{C}_X \subset acl(p_0(\mathfrak{C}) \cup \cdots \cup p_m(\mathfrak{C}))$. If $DIM(p_i(\mathfrak{C}))$ were finite for each i, $DIM(\mathscr{C}_X)$ would be finite. Thus, $DIM(p_i(\mathfrak{C}))$ is infinite for some i, proving Claim 1.

By the claim there is a $c^\sharp \in \mathscr{C}_X$ such that for $p = stp(c^\sharp/X)$, $DIM(p(\mathfrak{C}))$ is infinite. The minimality assumption on the dimension of X implies that X is the support of $c^\sharp$. Let C be the supportable elements of $\mathscr{C}$ whose support has dimension $< \dim(X)$. Then $C \supset \hat{\mathscr{C}}$, but C may not be algebraically closed. However, if $e \in acl(C) \cap \mathbf{C}$, and Y is the support of e, it is easy to show that $\{f : stp(f/Y) = stp(e/Y)\}$ has finite $c\ell-$ dimension.

Claim 2. Any Morley sequence in p is $c\ell-$ independent and $c\ell-$ independent from C.

Assume $\{c_0, \ldots, c_n\}$ is a Morley sequence in p of shortest length that is $c\ell-$ dependent on C. Then there is $B \subset C$ such that $c_n \in acl(\{c_0, \ldots, c_{n-1}\} \cup B)$. Since all of these elements are in $\mathbf{C}$ projectivity applies to yield an $e \in acl(B) \cap \mathbf{C}$ such that c_n and e are interalgebraic over $\bar{c} = \{c_0, \ldots, c_{n-1}\}$. Let I be a maximal Morley sequence in $p(\mathfrak{C})$ containing $\bar{c}$. Then, $I \subset acl(\bar{c} \cup E)$, where E is the set of realizations of $stp(e/X)$ (noting that the support of e must be algebraic in X). In fact, since E is invariant under automorphisms fixing $acl(X)$, any maximal Morley sequence J in $p(\mathfrak{C})$ is algebraic in E and any subset of J of n elements. Since any realization of p is contained in a maximal Morley sequence containing n elements of I, $p(\mathfrak{C}) \subset acl(I \cup E) = acl(\bar{c} \cup E)$. From our comment immediately preceding the claim, $DIM(E)$ is finite, hence $DIM(p(\mathfrak{C}))$ is finite; the contradiction that proves Claim 2.

Any conjugate over $\emptyset$ of a Morley sequence in p is called a *fiber*. The *length* of a fiber is its length as a Morley sequence over the relevant support. Notice that this length is also its $c\ell-$dimension. Before setting up the many-model argument we clarify the behavior of fibers. Let $\bar{c} = \{c_0, \ldots, c_n\}$ be a fiber consisting of a Morley sequence in p. While the support of $\bar{c}$ is X, $\ell_1(\bar{c})$ will contain elements of locally modular minimal sets. Said another way, $\bar{c}$ will not be independent from $\hat{\mathscr{C}}$ over X. All we know is that $c_n \notin acl(\hat{\mathscr{C}} \cup \bar{c} \setminus \{c_n\})$. This is why we use $c\ell$ and work relative to $\hat{\mathscr{C}}$ in the following claim. Note that if $\bar{c}$ is $c\ell-$dependent on B over A there is an $e \in acl(B) \cap acl(\bar{c}) \cap \mathscr{C} \setminus c\ell(A)$. Also,

$$A_0, \ldots, A_n \text{ fibers independent over } \emptyset \implies A_n \text{ is } c\ell-\text{independent from } A_0, \ldots, A_{n-1}. \quad (4)$$

(If $e \in acl(A_0, \ldots, A_{n-1}) \cap acl(A_n) \cap \mathscr{C} \setminus c\ell(\emptyset)$, then the support of e is not in $acl(\emptyset$ (by Claim 2). This contradicts the independence A_n from $A_0, \ldots, A_{n-1}$.)

Claim 3. Let $A_0, \ldots, A_n$ independent fibers, B a fiber in $c\ell(A_0 \cup \cdots \cup A_n)$ which is $c\ell-$dependent on $A^\sharp = A_0 \cup \cdots \cup A_{n-1}$. Then $B \subset c\ell(A^\sharp)$.

Let Y be the support of B. To see that $Y \subset acl(A^\sharp)$ let $e \in acl(A^\sharp) \cap acl(B) \cap \mathscr{C}$ such that B is $c\ell-$dependent on e (projectivity). By Claim 2, the support of e cannot have dimension less than $\dim(X) = \dim(Y)$. Thus, $Y \subset acl(e) \subset acl(A^\sharp)$. In other words, the support of $B \cup A^\sharp$ is algebraic in $A^\sharp$, hence independent from A_n. Assuming that $B \not\subset c\ell(A^\sharp)$ there is an $e \in acl(A_n) \cap acl(B \cup A^\sharp) \cap \mathscr{C}$, $e \notin c\ell(\emptyset)$ such that B is $c\ell-$dependent on e over $A^\sharp$. Since e is algebraic in both A_n and $B \cup A^\sharp$, the support of e is algebraic in the support of each of these sets, hence is in $acl(\emptyset)$. That is, $e \in \hat{\mathscr{C}}$. This contradicts the choice of e to prove the claim.

With this claim in hand we execute the many-model argument that proves the lemma. Let $\Omega = \{k_i : i < \omega\}$ be an infinite subset of ω such that for each $i > 0$, $k_i > \sum_{j<i} k_j$ and $B_\Omega = \{B_i : i \in \Omega\}$ an independent set of fibers with $DIM(B_i) = i$. For $\Delta \subset \Omega$ let $B_\Delta = \{B_i : i \in \Delta\}$ and $M_\Delta \supset B_\Delta$ a countable model that is almost atomic over B_Δ. Notice that $\mathscr{C} \cap M_\Delta$ is contained in $acl(B_\Delta)$ (by Corollary 4.6). Since the theory has few countable models there are $\Delta, \Gamma \subset \Omega$ with some $m \in \Delta \setminus \Gamma$ with $M_\Delta \cong M_\Gamma$. Thus, M_Δ is almost atomic over a set $A_\Gamma = \{A_i : i \in \Gamma\}$ which is conjugate to B_Γ. Thus, $B_m \subset acl(A_\Gamma)$. Let $\Gamma' \subset Y$ be a minimal set with $B_m \subset c\ell(A_{\Gamma'})$. Let $\Delta' \subset \Delta$ be a finite set containing m such that $A_{\Gamma'} \subset acl(B_{\Delta'})$. Let $\Delta'' = \Delta' \setminus \{m\}$. Partition Γ' as $\Gamma_0 \cup \Gamma_1$ where $j \in \Gamma_0$ implies that A_j is $c\ell-$dependent on $B_{\Delta''}$ and $j \in \Gamma_1 \implies A_j$ is $c\ell-$independent from $B_{\Delta''}$. By Claim 3, $j \in \Gamma_0 \implies A_j \subset c\ell(B_{\Delta''})$, hence B_m is $c\ell-$independent from A_{Γ_0}. Turning to other fibers, since $j \in \Gamma_1 \implies A_j$ is $c\ell-$independent from $B_{\Delta''}$ and

in $c\ell(B_{\Delta'})$, $DIM(A_j) = j \leq DIM(B_m) = m$. Since $m \notin \Gamma$, $j \in \Gamma_1 \implies j < m$. By the definition of Ω, $\sum\Gamma_1 < m$. However, $B_m \subset c\ell(A_{\Gamma_0} \cup A_{\Gamma_1})$, $m = DIM(B_m/A_{\Gamma_0})$ implies that $\sum\Gamma_1 = DIM(A_{\Gamma_1}/A_{\Gamma_0}) \geq m$. This final contradiction proves the lemma. $\square$

Remark 5.1 As $X \subset D$ increases we expect $DIM(\mathscr{C}_X)$ to increase. However, by the lemma this will be finite whenever $\dim(X)$ is finite.

The sets $\mathscr{C}_X$ are proper classes since they contain all realizations of some strong types. To use properties like the preceding lemma in a many-model argument we must work with countable approximations of such sets. To this end, let M be a countable saturated model. For X a closed subset of $D(M)$ of finite dimension such that $DIM(\mathscr{C}_X) > 0$ the *component over* X is $\mathscr{C}_X(M)$. Notice that $DIM(\mathscr{C}_X) > 0$ whenever $\dim(X)$ is large enough to contain the support of some element of $\mathscr{C} \setminus \hat{\mathscr{C}}$. The reader can verify that, since M is saturated, if $DIM(\mathscr{C}_X) = k$ then $\mathscr{C}_X(M)$ contains a sequence $\bar{a}$ with $DIM(\bar{a}) = k$, and, in this case, $\mathscr{C}_X \subset c\ell(\bar{a})$. The component $\mathscr{C}_X(M)$ is *decomposable* if there are components $C_0, C_1 \subset \mathscr{C}_X(M)$ such that $\mathscr{C}_X \subset c\ell(C_0 \cup C_1)$ and the supports of C_0 and C_1 are independent. A component is *indecomposable* if it isn't decomposable. In the proof of the next lemma we use the fact that components with independent support are $c\ell-$ independent. [If $DIM(C_0/C_1) < DIM(C_0)$ there is an $e \in acl(C_0) \cap acl(C_1) \cap \mathscr{C}$ such that $e \notin c\ell(\emptyset)$. Thus, the support Y of e is in $acl(C_0) \cap acl(C_1) \setminus acl(\emptyset)$, contradicting that the C_i are independent.] Since every component contains $\hat{\mathscr{C}}(M)$ components with independent support are still dependent over $\emptyset$.

In the sequel we will apply $c\ell(-)$ to components. While components are subsets of the countable model M, $c\ell$ is calculated in the universal domain. However, by projectivity $c\ell$ localizes to M as follows:

($*$) if $\{a\} \cup A \subset \mathscr{C}(M)$ and $a \in c\ell(A)$ then there is an $e \in \hat{\mathscr{C}}(M) \cap acl(A \cup \{a\})$ such that $a \in acl(A \cup \{e\})$.

Lemma 5.3 *There is a k such that $DIM(C) < k$ for any indecomposable component C.*

Proof: Similar to the proof of the preceding lemma, the key to the many-model argument is the following claim. Note, however, that we are working with components that may have supports of varying size rather than fibers.

Claim. Let $A_0, \ldots, A_n$ be components with independent supports, B an indecomposable component such that $B \subset c\ell(A_0, \ldots, A_n)$ and B is $c\ell-$ dependent on $A^\sharp = A_0 \cup \cdots \cup A_{n-1}$. Then, $B \subset c\ell(A^\sharp)$.

Let Y be a basis for the support of B over $A^\sharp$. By ($*$) there is $E \subset \hat{\mathscr{C}}(M)$ such that $B \subset acl(A^\sharp \cup A_n \cup E)$. Since each component contains $\hat{\mathscr{C}}(M)$, $B \subset acl(A^\sharp \cup A_n)$. By projectivity, there is an $A' \subset acl(A_n) \cap \mathscr{C}(M)$ interalgebraic with B over $A^\sharp$. If $\dim(Y) = 0$, $A' \subset \hat{\mathscr{C}}(M)$, and we have $B \subset c\ell(A^\sharp)$, as desired. So, we assume that $\dim(Y) > 0$ and work towards a contradiction of the indecomposability of B. Let Z be the support of A'. Then Z is interalgebraic with Y over $A^\sharp$, so $\dim(Y) = \dim(Z/A^\sharp) = \dim(Z)$. Let $\bar{a}' \subset A'$ be a finite tuple such that $A' \subset acl(\bar{a}' \cup \hat{\mathscr{C}})$. Since B is a component and Y is conjugate to Z there is $\bar{b}' \subset B$ realizing the conjugate over Y of $tp(\bar{a}'/Z)$. Then $DIM(\bar{b}') = DIM(\bar{a}') = DIM(A') = DIM(B/A^\sharp)$, so $B \subset c\ell(\bar{b}' \cup A^\sharp)$. Let $B' \subset B$ be a component with support Y that contains $\bar{b}'$.

Letting B' play the role of $A^\sharp$ and $A^\sharp$ play the role of A_n, repeat the above argument to yield $W \subset$ the support of B, $W \perp Y$, $\dim(W) > 0$, and a component $B'' \subset B$ with support W such that $B \subset c\ell(B' \cup B'')$. This contradicts the indecomposability of B to prove that $\dim(Y) = 0$, hence the claim is true.

Assume that there is no bound as in the statement of the lemma. Let $\Omega_0 = \{DIM(C) : C$ an indecomposable component$\}$, an infinite set. Let $\Omega = \{k_i : i < \omega\}$ be a subset of Ω such that $k_i > \sum_{j<i} k_j$ for $i > 0$. Let $B_\Omega = \{B_i : i \in \Omega\}$ be a collection of indecomposable components chosen so that, if Y_i is the support of B_i, $\{Y_i : i \in \Omega\}$ is independent. For $\Delta \subset \Omega$ let M_Δ be a countable model almost atomic over $B_\Delta = \{B_i : i \in \Delta\}$. That $\Delta \neq \Gamma \subset \Omega \implies M_\Delta \not\cong M_\Gamma$ is proved just as in Lemma 5.2 using the above claim in place of Claim 3. The details are left to the reader. This proves the lemma. □

Lemma 5.4 *Lemma 5.3 contradicts the fact that D is not locally modular.*

Proof: Let $k_0 \geq U(c)$ for all $c \in \mathscr{C}$ and k_1 the dimension of the support of C, for C an indecomposable component of maximal $c\ell-$dimension. Since D is not pseudo-projective (Lemma 1.5) there is k large enough so that if X is a closed subset of D of dimension $\geq k$ there are $a, b \in D$ such that $b \in acl(X \cup \{a\})$ and there is no $Y \subset X$ with $b \in acl(Y \cup \{a\})$ and $\dim(Y) \leq k_0$. Let C be a component with support of dimension $> k + k_1$. By Lemma 5.3 there are components $C_1, C_2 \subset C$ such that $C \subset c\ell(C_1 \cup C_2)$, C_1 is indecomposable whose support Y has dimension $\leq k_1$ and C_2 is a component whose support X is independent from Y. Notice that $\dim(X) > k$. Let $a \in Y$ and $b \in acl(X \cup \{a\})$ such that there is no $Z \subset X$ with $b \in acl(Z \cup \{a\})$ and $\dim(Z) \leq k_0$. Since C is a component there is a $c \in C$ whose support contains b. By projectivity there are $c_1 \in acl(C_1) \cap \mathscr{C}$ and $c_2 \in acl(C_2) \cap \mathscr{C}$ such that c is interalgebraic with c_1 over c_2 and with c_2 over c_1. Since $b \in \ell_1(c)$, $b \in acl(c_1, c_2)$. Let $Z_i =$ the support of c_i, for $i = 1, 2$. Since $a \in acl(X \cup \{b\}) \subset acl(X \cup Z_1)$, $a \in acl(X \cup Z_1)$. Since a and Z_1 are in Y and $Y \perp X$, $a \in acl(Z_1)$. Since $b \in acl(Z_1 \cup \{c_2\})$, $b \in acl(Z_1 \cup Z_2)$ and $\{b\} \cup Z_1$ is independent from X over Z_2. Thus, $b \in acl(Z_2 \cup \{a\})$. However, $c_2 \in \mathscr{C}$, so $\dim(Z_2) \leq U(c_2) \leq k_0$. This contradicts the choice of a, b, X to prove the lemma. □

This final contradiction proves that there are no fictional irreducible elements. Hence every element of C is $1-$based (Theorem 2.1(5)).

6 More on dependence on $1-$based groups

The existence of the structure group will be proved in Section 7. This digression establishes detailed properties of types in $1-$ based groups that will be essential in counting the number of structure groups. The results in this section apply to any $1-$ based group in a superstable theory. While forking dependence (and ranks in the superstable context) on a $1-$ based basic group G is determined by basic subgroups of $n-$ tuples, types can be more complicated. Most obviously, G^- may be greater than $\{0\}$. Factoring G by G^- will produce a group that is closer to being a pure module. In particular, a $*-$ endomorphism on G induces an endomorphism on G/G^-. However, there may also be $H \subset G^n$, a basic subgroup, and a predicate in the language for a nonzero coset of H. This further complicates the automorphism group of G. In this section we find a Fisher abelian structure as a factor group of a $1-$ based group, whenever G^- is sufficiently rich.

Definition 6.1 Let G be a $1-$ based basic group in a stable theory. A sequence $\bar{a} \in G^n$ is called *good* if it is 0 or a generic of a basic subgroup of G^n. A subgroup H of G is a *good basis* for G if for every $a \in G$, $a+H$ contains a good element independent from H.

A group in which every element is good comes close to being a Fisher abelian structure. At first glance, though, it's possible for some $n-$ tuple of this group to fail to be good. The next proposition shows that this cannot happen. More generally, it gives a sufficient condition for finding a Fisher abelian structure as a factor group of G (Theorem 6.2).

Proposition 6.1 *Let G be a $1-$ based basic group in a stable theory. Let $H \supset G^-$ be a good basis for G. Then, for every n, H^n is a good basis for G^n.*

Proof: We emphasize that H is not assumed to be definable or basic here. It is simply a subgroup of G. Add constants to the language for the elements of H so that G^- has the properties claimed for H.

Assume, inductively, that the proposition is true for all tuples from G^n and $\bar{a} = (a_0, \ldots, a_n) \in G^{n+1}$. (The result is true for $1-$ tuples by hypothesis.) Let $\bar{a}' = (a_0, \ldots, a_{n-1})$. By induction, there are $\bar{h}' \in (G^-)^n$ and $h_n \in G^-$ such that $\bar{a}' + \bar{h}'$ is a generic of a basic group K_1 and $a_n + h_n$ is a generic of a basic group K_2. Thus, we may as well assume $\bar{a}$ was chosen so that $\bar{a}'$ is a generic of K_1 and a_n is a generic of K_2.

Since the group is $1-$ based there is a $c \in acl(\bar{a}') \cap acl(a_n)$ such that $\bar{a}'$ is independent from a_n over c. By Lemma 1.26 there is a basic group K_0 and a $*-$ homomorphism $\psi_1 : K_1 \longrightarrow K_0$ such that c is interalgebraic with $\psi_1(\bar{a}')$. Let Ψ_1 be the connected component of the graph of ψ_1. Since $\bar{a}'$ is a generic of the connected group K_1 there is a d such that $\bar{a}'d$ is a generic of Ψ_1 and d is interalgebraic with c. Let K_0 be a basic group such that d is a generic of K_0.

Turning to a_n, $d \in acl(a_n)$, so there is a $*-$ homomorphism $\psi_2 : K_2 \longrightarrow K_0$ with $\psi_2(a_n) =^* d$ (by Lemma 1.20). Let Ψ_2 be the connected component of the graph of ψ_2. By Corollary 1.21 there is an $h \in G^-$ such that $(a_n + h, d) \in \Psi_2$. Thus, without loss of generality, we can assume that $a_n d$ is a generic of Ψ_2. Now, combine the defined basic groups to form another group as follows. Let K be the set of $n+2-$ tuples $\bar{c}'c_n a$ where $\bar{c}'e \in \Psi_1$ and $c_n e \in \Psi_2$. We will show that $(\bar{a}', a_n + h, d)$ is a generic of K^o. K is $\bigwedge-$ definable over $acl(\emptyset)$. Let $\bar{b}'b_n e$ be a generic of K^o. Then $\bar{b}'e$ is a generic of Ψ_1 and $b_n e$ is a generic of Ψ_2. There is an automorphism of the universe fixing $acl(\emptyset)$ and mapping $\bar{b}'e$ to $\bar{a}'d$. Thus, we may as well assume that $\bar{b}'e = \bar{a}'d$ and that b_n is independent from $\bar{a}'a_n d$ over $\bar{a}'d$, hence over d by transitivity. Since $a_n d$ and $b_n d$ are both generics of Ψ_2, and $a_n \underset{d}{\perp} b_n$, $a_n - b_n$ is a generic of $\Psi_0 = \{ x : (x,0) \in \Psi_2 \}$. Since G^- is a good basis there is an $h \in G^-$ such that $(a_n + h) - b_n$ is a generic of the connected component of Ψ_0. By Lemma 1.19, $stp(a_n + h/d) = stp(b_n/d)$. Thus, $stp(\bar{a}', a_n + h, d) = stp(\bar{a}', b_n, d)$, proving that $(\bar{a}', a_n + h, d)$ is a generic of K^o. The projection of K^o onto the first $n+1$ coordinates is a basic group showing that $(\bar{a}', a_n + h)$ is good. This proves the proposition. □

Using this result we will show that when G^- is a good basis, G/G^-, under the induced structure is a Fisher abelian structure. This is made precise as follows.

Definition 6.2 Let G be a $1-$ based basic group such that G^- is a good basis. Let $\mathscr{K}$ be the collection of basic subgroups of G^n for some n. For $K \in \mathscr{K}$, let P_K be a new predicate symbol of the same arity as K. Let L^+ be the language $\{0, +, P_K\}_{K \in \mathscr{K}}$. Let G^+ be the L^+- structure with universe G/G^-, $+$ the inherited group operation and P_K interpreted by $(K + G^-)/G^-$. G^+ is a Fisher abelian structure called the *purification* of G.

Remark 6.1 Normally, G^+ is not in G^{eq} since G^- may be undefinable. However, if α is an automorphism of G that fixes $acl(\emptyset)$ pointwise then α induces an automorphism of G^+. (Since $K \in \mathscr{K}$ is $\bigwedge-$ definable over $acl(\emptyset)$, α must fix $acl(\emptyset)$ in order to preserve K.)

Theorem 6.2 *Let G be a $1-$ based basic group in a superstable theory such that G^- is a good basis and G^+ the purification of G. Then $Th(G^+)$ is quantifier-eliminable, totally transcendental and G^+ is saturated.*

Proof: Let $M = G^+$. The following items will be established in succession, from which the theorem follows. Parts of the proof correspond closely to parallel proofs in modules. We will refer to such proofs rather than repeating the details here. After showing elimination of quantifiers the proofs are by standard methods.

(i) Every complete type over $\emptyset$ consistent with $Th(M)$ is realized in M.

(ii) $Th(M)$ is quantifier-eliminable.

(iii) For any model N of $Th(M)$ and $p \in S_1(M)$, p is a translate of the generic type of P for some $P \in L^+$.

(iv) $Th(M)$ is totally transcendental.

(v) M is saturated.

These items will follow from more specific claims about the definable subsets and types in $Th(M)$. First, some terminology is set.

For $P_K \in L^+$ we say that P_K has rank n if $U(K) = n$. We don't ascribe any properties to this rank assignment at the outset; it is simply for bookkeeping. We call $\bar{a} \in M$ a generic of $P_K(M)$ if $\bar{a} = \bar{b} + G^-$ and $\bar{b}$ is a generic of K. Equivalently, $\bar{a} \in P_K(M)$ and $\bar{a} \notin Q(M)$, for Q any element of L^+ with rank smaller than the rank of P_K. Let Γ be the group of automorphisms of M induced by elements of $Aut(G)$ that fix $acl(\emptyset)$ pointwise.

Most of the properties of the theory follow quickly from

Claim 1. Given $\bar{a} \in M^n$ let $p_{\bar{a}}(\bar{v})$ be the quantifier-free type of $\bar{a}$. Then, for any $\bar{b} \in M^n$ realizing $p_{\bar{a}}(\bar{v})$, there is an $\alpha \in \Gamma$ with $\alpha(\bar{a}) = \bar{b}$. In fact, there is a $P \in L^+$ such that $p_{\bar{a}}(\bar{v})$ is implied by $\{P(\bar{v})\} \cup \{\neg Q : Q \in L^+, M \models \neg Q(\bar{a})\}$.

For a given $\bar{a}$ let $P_K \in L^+$ have least rank among those satisfied by $\bar{a}$; i.e., $\bar{a}$ is a generic of P. Let $\bar{b}$ be another generic of P_K. By Proposition 6.1 there are $\bar{a}_0, \bar{b}_0 \in G$, each generics of K such that $\bar{a}_0 + G^- = \bar{a}$ and $\bar{b}_0 + G^- = \bar{b}$. Thus, $stp(\bar{a}_0) = stp(\bar{b}_0)$, so there is an automorphism α of G fixing $acl(\emptyset)$ and mapping $\bar{a}_0$ to $\bar{b}_0$. Since $\alpha \in \Gamma$ and $\alpha(\bar{a}) = \bar{b}$, the claim is proved.

Claim 2. If X is an $\emptyset-$ definable (in $L+$) subgroup of M^n, for some n, then $X = P(M)$ for some $P \in L^+$.

Ranging over $\bar{a} \in X$, let $P \in L^+$ have maximal rank such that $\bar{a}$ is a generic of P. By Claim 1, for any generic $\bar{b}$ of P there is an automorphism of M mapping $\bar{a}$ to $\bar{b}$, so $\bar{b} \in X$. That is, X contains all generics of P. Since any element of P is the difference of two generics, $P(M) \subset X$. Suppose there is some $\bar{b} \in X \setminus P(M)$. Let Q be such that $\bar{b}$ is a generic of $Q(M)$. As above, $Q(M) \subset X$. Since X is a group, $P(M) + Q(M) \subset X$. Using the maximality assumption on $n =$ the rank of P, the rank of any P' with $P'(M) \subset P(M) + Q(M)$ is $\leq n$. The contradiction exhibited by this fact is best calculated in G as follows. Let K be such that $P = P_K$ and H such that $Q = P_H$. Then, $K + H$ is a $\bigwedge-$ definable subgroup of G with rank $= n = U(K)$. Thus, K is the connected component of $K + H$. By standard isomorphism theorems, $K \cap H$ has bounded index in K. This contradicts that K and H are connected unless $H \supset K$ and $H = K$ or $U(H) > n$. The latter is impossible by the maximality assumption on n, so $K = H$. Thus $P(M) = Q(M)$, a contradiction that proves the claim.

Proof of (i). Let $\mathbf{X} = \{X_i : i < \omega\}$ be the sets defined by the formulas in a complete type $p(\bar{v})$ consistent with $Th(M)$. Let $P \in L^+$ have least rank such that $P(M) \in \mathbf{X}$. Let $\bar{a}$ be a generic of $P(M)$. We show that p is equivalent to $p_{\bar{a}}$ as follows. By the minimal rank assumption on P, $p_{\bar{a}}(M) \supset \bigcap \mathbf{X}$. If $p_{\bar{a}}(M) \not\subset \bigcap \mathbf{X}$ there is an $X_i \in \mathbf{X}$ and generics $\bar{b}, \bar{c} \in P(M)$ such that $\bar{b} \in X_i$ and $\bar{c} \notin X_i$. This contradicts that there is an $\alpha \in \Gamma$ such that $\alpha(\bar{b}) = \bar{c}$. Thus, $p_{\bar{a}}(M) = \bigcap \mathbf{X}$. Since $\bar{a}$ realizes $p_{\bar{a}}$ it follows, a fortiori, that p is realized in M, proving (i).

Proof of (ii). Since every complete type consistent with $Th(M)$ is equivalent to a quantifier-free type, a compactness argument shows that every formula is equivalent to quantifier-free formula, proving (ii).

Proof of (iii). This property follows from the elimination of quantifiers and the characterization of definable groups in Claim 2. It is proved like the corresponding result in modules (following the pp elimination of quantifiers).

Proof of (iv). By Claim 2, M has the d.c.c. on definable subgroups. Combining this with (iii) shows that $Th(M)$ is t.t. by counting the possible types over any model. The details for modules (due to Macintyre and Garavaglia) are found in [Pre88, Theorem 3.1].

It remains to prove that M is saturated. By Claim 1 and (i), M is $\aleph_0-$ saturated. Classical results about types in a t.t. theory imply $(*)$ M is saturated if and only if for any infinite set of indiscernibles $I \subset M$ there is an indiscernible sequence J, $I \subset J \subset M$, $|J| = |M|$. Thus, the proof of (v) reduces to

Claim 3. Let $I \subset M$ be an infinite set of indiscernibles. There is a set of indiscernibles J, $I \subset J \subset M$, with $|J| = |M|$.

By the characterization of types in (iii) there is $P_K \in L^+$ such that for $a \in I$, $I' = \{b - a : b \in I \setminus \{a\}\}$ is an independent sequence of generics in $P_K(M)$. Each element of I' is $c + G^-$ for c a generic of K, and the set of all such are independent. This set is contained in an independent set of generics of K of cardinality $|M|$. The images of these elements in M form an independent set J' of generics of $P_K(M)$ that contains I'. Then $J = \{b + a : b \in J'\}$ is an indiscernible set of cardinality $|M|$ containing I. This proves Claim 3.

By Claim 3 and $(*)$, M is saturated. This completes the proof of the theorem. □

7 Existence of the Structure Group

The ultimate goal of the section is to prove the existence of the so-called structure group of the theory T, as detailed in Subsection 7.4.

The bulk of the work is contained in the first subsection where we prove the existence of basic groups with maximally irreducible generics and the tight relationship between such groups and the rest of $\mathbf{C}$.

In the second subsection we show that a $1-$ based basic group in a superstable theory of finite rank is a sum of weight 1 basic subgroups. This result is only needed in the third subsection, where we prove that there is a bound on the number of levels for elements of $\mathbf{C}$. This result, that seems to have nothing to do with groups, is most easily proved using weight 1 elements of $\mathbf{C}$, whose existence is proved using $1-$ based basic groups. This completes the proof of Theorem 2.1(3). The bound on levels is needed for the finiteness of the list of groups H_i.

7.1 Obtaining groups with irreducible generics

Virtually all proved properties of irreducible elements are used in the existence proof of the structure group. This group is also fundamental in proving the structure theorem.

Recall the definition of maximally irreducible in Definition 4.2.

Definition 7.1 Let $\mathscr{G}_0$ be the family of groups G satisfying (1) G is connected and $\bigwedge-$definable over a finite set A with $tp(A)$ isolated, and (2) for $g \in G$ generic there is a maximally irreducible $d \in \mathbf{C}$ such that $d \downarrow A$ and d is interalgebraic with g over A.

Let $\mathscr{G}$ be the collection of groups H such that for some $G \in \mathscr{G}_0$ with domain A, H is a subgroup of G basic over A. In this case we call A the *domain* of H.

For $c \in \mathbf{C}$ and $G \in \mathscr{G}_0$ we write $G \prec c$ ($G \approx c$) if for $g \in G$ generic and d as in (2), $d \prec c$ ($d \approx c$). Slightly abusing the terminology, $G \in \mathscr{G}_0$ is called *maximally irreducible* if d as in (2) is maximally irreducible, and we say g has l levels if d has l levels. Moreover, for such g and d as in (2), if $c \in \mathbf{C}$ and $c \prec d$ we write $c \prec g$.

Remark 7.1 We defined $\mathscr{G}_0$ and $\mathscr{G}$ in two steps because it isn't immediately clear that for $H \in \mathscr{G}$ with domain A and $h \in H$ generic there is a $d \in \mathbf{C}$, $d \downarrow A$ and d interalgebraic with h over A. For our purposes it will be enough that h is the difference of two elements for which it is true; i.e., generics of $G \in \mathscr{G}_0$ containing H.

The three main ingredients in the construction of the structure group are the following propositions. Proposition 7.1 gives the existence of maximally irreducible groups. Proposition 7.2 describes how $\mathbf{C}$ is generated from such groups. Proposition 7.3 controls the number of such groups, up to $*-$ isomorphism.

Proposition 7.1 *Let c be maximally irreducible. Then there is a $G \in \mathscr{G}_0$ such that G is maximally irreducible and $G \approx c$.*

Proposition 7.2 *Let $c \in \mathbf{C}$ and B a finite set such that there is a $G \in \mathscr{G}_0$ with domain $A \subset B$, $c \prec G$ and c has the same number of levels as a generic of G. Then, if $c \notin acl(B)$, every realization of $stp(c/B)$ is interalgebraic with an element of G over $\{c\} \cup B$.*

Proposition 7.3 *Given $G, H \in \mathscr{G}_0$ such that $G \approx H$, G is $*-$ isomorphic to H.*

The first lemma simply concerns the $\approx$ relation on conjugates of $G \in \mathscr{G}_0$.

Lemma 7.4 *(i) If $G, G' \in \mathscr{G}_0$ are conjugate over $acl(\emptyset)$, $G \approx G'$.*

(ii) Let $G \in \mathscr{G}_0$ have domain A. There is a formula $\sigma \in stp(A)$ so that if A' satisfies σ and G' is the conjugate of G over A', then $G \approx G'$.

Proof: (i) This is immediate by property (2) in the definition of $\mathscr{G}_0$ and Remark 4.4.

(ii) Let $g \in G$ be generic and $c \in \mathbf{C}$ interalgebraic with g over A, $c \downarrow A$. There is a formula $\varphi(x,y) \in tp(g,A)$ such that whenever $\models \exists x \varphi(x,B)$, $tp(B) = tp(A)$ and $\varphi(x,B)$ defines a group H such that H^o is conjugate to G. By the definability of types there is a formula $\sigma \in stp(A)$ such that $\models \sigma(B)$ and $c \downarrow B$ implies $\models \exists x \varphi(x,B)$ and there is an h satisfying $\varphi(x,B)$ interalgebraic with c over B.

Now, let A' be any set satisfying $\sigma(y)$, G' the conjugate of G over A'. Let H be a conjugate of G with domain B such that $stp(B) = stp(A')$ and $c \downarrow B$. Notice that $H \approx G'$ (by (i)). By the choice of σ there is a $B-$ definable group H_0 such that $H_0^o = H$ and a generic $h \in H_0$ interalgebraic with c over B. While h may not be in H, since it is a generic it is still routine to show that $c \approx h'$ for $h' \in H$ generic. Thus, $G \approx H$. Since $H \approx G'$, $G \approx G'$, as required.

This proves the lemma. □

The proof of Proposition 7.1 relies heavily on the following result, proved in [BH94].

Theorem 7.5 (Bouscaren-Hrushovski) *Suppose that T is stable, 1-based and c belongs to an algebraic triangle over A_0. Then, for some $A \supset A_0$, with $A \setminus A_0$ finite and c independent from A over A_0, there is a basic group G over A, which acts generically and regularly on $stp(d/A)$ for some d interalgebraic with c over A.*

Notice that the theory is assumed to be $1-$ based in this theorem, however the same proof yields the same theorem in a superstable theory of finite rank when c is $1-$ based. (Any strong type involved in the proof must be nonorthogonal to $stp(c/B)$ for some set B, hence is itself $1-$ based.) The connection between algebraic triangles and groups is used to prove Proposition 7.1 as follows.

Proof of Proposition 7.1: The existence of a basic group whose generic is interalgebraic with c over its defining parameters will follow rather quickly from the Bouscaren-Hrushovski result. Finding a group with much the same properties whose domain is isolated takes more work.

By Corollary 4.13 c belongs to an algebraic triangle. By Theorem 7.5 (and the subsequent comment) there is a finite set A, $A \downarrow c$, and a basic group G over A which acts generically and regularly on $stp(d/A)$ for some d interalgebraic with c over A. Certainly, $U(G) = U(d/A)$.

Claim 1. There is a finite set $B \supset A$, $B \downarrow c$, such that over B, c is interalgebraic with an element of G that is generic over B.

Let g be a generic of G independent from c over A. Since g is then generic over d there is a d' realizing $stp(d/A)$ interalgebraic with g over $A \cup \{d\}$. Let c' be a realization of $stp(c)$ interalgebraic with d' over A. Removing d and d' from the relationships, $c \downarrow Ag$, $cc' \downarrow A$ and c is interalgebraic with g over $A \cup \{c'\}$. Let $B = A \cup \{c'\}$.

Claim 2. Without loss of generality, $tp(A)$ is isolated.

By Corollary 4.15 every minimal type appearing in c is nonorthogonal to $\emptyset$, hence the same is true of a generic of G. Let G_0 be a group definable over B whose connected component is G (which exists by [Hru87, 2.1]). Since every element of G_0 is a sum of two generics in G, every minimal type appearing in an element of G_0 is nonorthogonal to $\emptyset$. By the definability of rank lemma, Lemma 1.2, there is a formula $\varphi(x,b)$ over B that defines G_0 and satisfies: for $n = R^\infty(G_0) = R^\infty(G)$, $\forall y[\exists x\varphi(x,y) \implies R^\infty(\varphi(x,y)) = n]$. Let $\theta(x,z)$ be a formula over B such that $\models \theta(g,c)$ and $\models \theta(x,z) \implies x$ and z are interalgebraic over B. Without loss of generality, θ is defined over b, and we write $\theta = \theta(x,z,b)$. Since $b \perp c$ the definability of types yields a formula $\sigma(z)$ over $acl(\emptyset)$ such that $\models \sigma(y)$ and $y \perp c \implies \models \exists x(\varphi(x,y) \wedge \theta(x,c,y))$. Let b_0 be independent from c such that $tp(b_0)$ is isolated and contains $\sigma(y)$.

Let $H_0 = \varphi(\mathfrak{C}, b_0)$, a group of $\infty-$ rank n and let $h \in H_0$ satisfy $\theta(x,h,b_0)$. Then h is interalgebraic with c over b_0, and since $U(c/b_0) = n$, h is a generic of H_0. Let $H = (H_0)^o$. H acts regularly on $stp(h/b_0)$ so if $k \in H$ is generic over c, it is interalgebraic over b_0c with a realization of $stp(c)$ independent from b_0c. Replacing b_0 by A proves the claim.

Claim 3. For $g \in G$ generic there is a maximally irreducible $e \in \mathbf{C}$ interalgebraic with g over A, $e \perp A$, $e \approx c$.

Arguing as in Claim 1 and choosing $g \in G$ generic over $\{c\} \cup A$, g is interalgebraic over $\{c\} \cup A$ with a realization c' of $stp(c)$. A rank calculation shows that c' is independent from $\{c\} \cup A$, hence $cc' \perp A$. By projectivity there is an $e \in \mathbf{C}$, $e \in acl(cc') \cap acl(\{g\} \cup A)$ such that $c' \in acl(e)$. These properties quickly imply that $\{c, c', e\}$ is an algebraic triangle. It follows that $e \approx c$ and $U(e) = U(c)$, hence e is maximally irreducible. Furthermore, $e \perp A$, $U(e) = U(g/A)$ and $e \in acl(\{g\} \cup A)$, so e and g are interalgebraic over A. This completes the proof of Claim 3.

Combining Claims 2 and 3 proves that G is an element of $\mathscr{G}_0$ with the desired relationship with c. This completes the proof of the theorem. $\square$

The next lemma ties an arbitrary element of $\mathbf{C}$ with an element of $\mathscr{G}$. It does the heavy lifting in the proof of Proposition 7.2.

Lemma 7.6 *Let $c \in \mathbf{C}$ and B a finite set with $c \notin acl(B)$, and let $G \in \mathscr{G}_0$ with domain $A \subset B$ such that $c \prec G$ and c has the same number of levels as a generic of G. Then, there is a definable group K with $K^o \subset G$ basic over A and $k \in K$ generic over B that is interalgebraic with c over B.*

Proof: The bulk of the proof is contained in

Claim 1. There is $h \in G$ which is a generic over B of subgroup H of G, basic over A, and a d such that $x \in Z = \{c,d,h\} \implies x \in acl(B \cup Z \setminus \{x\})$, $c \perp_B d$ and $h \perp_B d$.

Let $g \in G$ be generic over $\{c\} \cup B$. By the conditions on $\mathscr{G}_0$ there is a $b \in \mathbf{C}$ maximally irreducible element interalgebraic with g over A, $b \perp B$. By Lemma 4.12 there is an $a \in \mathbf{C}$ such that $U(a) = U(b)$ and $x \in X = \{a,b,c\} \implies x \in acl(X \setminus \{x\})$. From rank calculations it follows that both b and a are independent from $\{c\} \cup B$. Replace b by g and notice that $x \in Y = \{a,g,c\} \implies x \in acl(B \cup Y \setminus \{x\})$.

To find h satisfying the claim we must use the characterization of strong types in a $1-$based group as follows. Let $c'g'$ be a realization of $stp(cg/B \cup \{a\})$ that is independent from cg over $B \cup \{a\}$. By Lemma 1.19, $h = g - g'$ is a generic of a subgroup H, basic over A, and h is independent from $\{a,g'\} \cup B$ over A. Since $c' \in acl(\{a,g'\} \cup B)$, h is independent from $\{a,g',c'\} \cup B$ over A. Furthermore, h is interalgebraic with c over $\{a,g',c'\} \cup B$ (since the same is true of $g = h + g'$). Let $d \in acl(\{c,h\} \cup B) \cap acl(\{a,g',c'\} \cup B)$ such that ch is independent from $ag'c'$ over $\{d\} \cup B$.

We now verify that $Z = \{c,d,h\}$ satisfies the requirements of the claim. Since h is independent from $\{a,g',c'\} \cup B$ over A and $d \in acl(\{a,g',c'\} \cup B)$, h is independent from $\{d\} \cup B$ over A. By choice of $g'c'$, c is independent from $g'c'$ over $\{a\} \cup B$. Since c is also independent from a over B, c is independent from d over B. Finally, the choice of d implies that $x \in Z = \{c,d,h\} \implies x \in acl(B \cup Z \setminus \{x\})$. This proves the claim.

Claim 2. There are: (1) a finite $C \supset A$ with $tp(C/B)$ isolated, $c \underset{B}{\downarrow} C$, (2) a definable group K with $K^o \subset G$ basic over A, and (3) $k \in K$ a generic over C interalgebraic with c over C.

If $tp(d/B)$ were isolated we could set $\{d\} \cup A$ as C and h as k to prove the lemma. Since this may not hold we need to replace these elements by others with a slight change in the requirements on h.

The element h is a generic of a subgroup of G, basic over A and having $\infty-$rank some n. Notice that n is also $R^\infty(c/B)$. By the standard argument using definability of types, as in the proof of Proposition 7.1, there are: (1) d' such that $tp(d'/B)$ is isolated and $c \underset{B}{\downarrow} d'$, (2) a group K having $\infty-$rank n and definable over a finite subset of $acl(A)$, and (3) a $k \in K$ interalgebraic with c over $\{d'\} \cup B$. Notice that $R^\infty(k/\{d'\} \cup B)$ must be n, so k is a generic over $\{d'\} \cup B$. Also, $K^o \subset G$ is basic over A. Letting $C = \{d'\} \cup A$ completes the proof of the claim.

Claim 3. c and k are interalgebraic over B.

It si given that c and k are interalgebraic over $C \supset B$. We will apply Corollary 4.8 to prove the claim. Let $g_0, g_1 \in G$ be generics such that $k = g_0 - g_1$ and $g_0 g_1$ is independent from $\{c\} \cup C \cup B$ over $\{k\} \cup A$. Let $d_i \in \mathbf{C}$ be interalgebraic with g_i over A. Then, $c \in acl(\{d_0,d_1\} \cup B \cup C)$. Since $\{c,d_0,d_1\} \subset \mathbf{C}$ and $tp(C/B)$ is isolated, Corollary 4.8 implies that $c \in acl(\{d_0,d_1\} \cup B)$. Since $d_0 d_1$ are independent from $\{c\} \cup B \cup C$ over $\{k\} \cup A$, $c \in acl(\{k\} \cup B)$, proving the claim.

This completes the proof of the lemma. $\square$

Proof of Proposition 7.2: Let c, B and G be as hypothesized. By Lemma 7.6 c is interalgebraic over B with a generic $k \in K$, for K a definable group with $K^o \subset G$ basic over A. The set of realizations of $stp(k/B)$ is contained in $k + K$. Since every realization of $p = stp(c/B)$ is interalgebraic over B with a realization of $stp(k/B)$, $p(\mathfrak{C}) \subset acl(\{c\} \cup K^o \cup B)$. This proves the theorem. $\square$

Proof of Proposition 7.3: Let G have domain A and H domain B. Let $g \in G$ be generic over $A \cup B$ and $h \in H$ generic over $\{g\} \cup A \cup B$. There are maximally irreducible $c, d \in \mathbf{C}$ such that $c \downarrow A \cup B$, $d \downarrow \{c\} \cup A \cup B$, c is interalgebraic with g over A and d is interalgebraic with h over B. Since c and d are maximally irreducible, $c \approx d$ and $c \downarrow d$, Corollary 4.13 yields a $b \in \mathbf{C}$ such that $\{b, c, d\}$ is an algebraic triangle. So, g is generic over $C = \{b\} \cup A \cup B$, h is generic over C and g is interalgebraic with h over C. By Corollary 1.22, G and H are $*-$ isomorphic. $\square$

Since the elements of $\mathscr{G}_0$ may not be definable over $acl(\emptyset)$, $G \in \mathscr{G}_0$ may not be contained in $\mathbf{C}$. The following discussion of the irreducibility of the elements of G clarifies the picture and is used in subsequent proofs.

Lemma 7.7 *Let $G \in \mathscr{G}_0$ have domain A. Then a generic of G is irreducible in $T(A)$.*

Proof: Let $g \in G$ be generic and $c \in \mathbf{C}$ interalgebraic with g over A and $c \downarrow A$. It suffices to show that c is irreducible over A. Suppose to the contrary that there are $b_0, b_1 \in acl(\{c\} \cup A)$ such that $U(b_i/A) < U(c/A) = U(c)$ ($i = 0,1$) and $tp(c/\{b_0, b_1\} \cup A)$ is isolated. Since c is $1-$based there are $d_i \in acl(c) \cap acl(\{b_i\} \cup A)$ such that b_i is interalgebraic with d_i over A ($i = 0,1$). Since c is independent from $\{b_0, b_1\} \cup A$ over $d_0 d_1$, $tp(c/d_0 d_1)$ is isolated. Moreover, $U(d_i) = U(d_i/A) = U(b_i/A) < U(c)$ ($i = 0,1$), contradicting the irreducibility of c to prove the lemma. $\square$

Further analysis of the elements of $G \in \mathscr{G}_0$ will pass through the following notion. Remember that the decomposition of elements into levels is valid in any superstable theory of finite rank.

Definition 7.2 Let T_0 be a superstable theory of finite rank, G a $1-$based basic group and c a generic of G. By Lemma 1.26 there is a basic group H and a $*-$ homomorphism ψ of G onto H such that $\psi(c)$ is interalgebraic with $\underline{c}$. Then H is called the *top filter group* of G and ψ the *top filter map* of G. H may be denoted $\underline{G}$.

Remark 7.2 For a generic $c \in G$, $\underline{c}$ is essentially unique so any two top filter groups of G have interalgebraic generics, hence are $*-$ isomorphic.

Lemma 7.8 *Let $G \in \mathscr{G}_0$ have domain A. Then every element of G is irreducible in $T(A)$.*

Proof: We work in $T(A)$, where G is $\bigwedge-$ definable over $\emptyset$. Let $\psi : G \longrightarrow \underline{G}$ be a top filter map of G. Notice that for any $b \in G$, $U(b/\psi(b)) \leq 1$. [For $a \in G$ generic, $U(a/\psi(a)) = 1$, so the kernel of ψ has rank 1.]

Claim. Whenever $b \in G$ and $c \in acl(b)$ depends on b over $\psi(b)$, b is algebraic in c.

The irreducibility of the generics of G guarantees this property when b is a generic. Suppose, towards a contradiction, that b is an arbitrary element of $G\setminus G^-$, $c\in acl(b)$ depends on b over $\psi(b)$ and $b\notin acl(c)$. Let $a_0\in G$ be a generic, $a_0 \perp b$ and $a_1=a_0+b$. Since b and c are interalgebraic over $\psi(b)$ and $b\notin acl(a_0,\psi(b))=acl(a_0,\psi(a_1))$, $c\notin acl(a_0,\psi(a_1))$. Since c is algebraic in a_0a_1, a_1 depends on ca_0 over $\psi(a_1)$. Thus, $a_1\in acl(ca_0)$, by the irreducibility of a_1. Since $b=a_1-a_0$ this implies that $b\in acl(c,a_0)$. However, $c\in acl(b)$ and $b \perp a_0$ then imply that $b\in G^-$, a contradiction that proves the claim.

Claim. $b\in G\setminus G^- \implies tp(b/\psi(b))$ is nonisolated.

If $b\notin acl(\psi(b))$, then $stp(b/\psi(b))$ is a translate of the connected component of the kernel of ψ. This minimal set is properly weakly minimal. Since every element of G is a bone, $tp(b/\psi(b))$ is minimal-by-finite, hence nonisolated. Thus, it suffices to reach a contradiction from the assumption that $b\in acl(\psi(b))$ and $b\notin G^-$. Let a_0 be a generic of G, $a_0 \perp b$, and $a_1=a_0+b$. The assumption on b implies that $a_1\in acl(a_0,\psi(a_1))$, hence $a_1\in acl(a_0)$, by the irreducibility of a_1. This contradicts that $b\notin G^-$, to prove the claim.

Now suppose that $b\in G\setminus G^-$ is reducible, $c_0,c_1\in acl(b)$, $U(c_i)<U(b)$ ($i=0,1$) and $tp(b/c_0,c_1)$ is isolated. Since $tp(b/\psi(b))$ is nonisolated (second claim) there is a some c_i, say c_0 that is not in $acl(\psi(b))$. Since c_0 depends on b over $\psi(b)$, $b\in acl(c_0)$ by the first claim. This contradicts that $U(c_0)<U(b)$, finally proving the lemma. □

Lemma 7.9 *Let $G\in\mathcal{G}_0$ have domain A. Then for any $g\in G\setminus G^-$ there is a $d\in\mathbf{C}$, $d \perp A$, such that d is interalgebraic with g over A.*

Proof: Pick $g\in G\setminus G^-$ and notice that the lemma is true if g is a generic simply from the definition of $\mathcal{G}_0$. Let g_0 be a generic of G $A-$independent from g and $g_1=g_0+g$. Let $c_i\in\mathbf{C}$, $c_i \perp A$, and c_i interalgebraic with g_i over A ($i=0,1$). Since all elements of $\mathbf{C}$ are $1-$based there is a $c\in acl(c_0,c_1)\cap acl(\{g\}\cup A)$ such that c_0c_1 is independent from $\{g\}\cup A$ over c. While g is irreducible over A (by Lemma 7.8) it isn't entirely clear that c is irreducible, so we continue the proof as follows.

Notice that $c \perp c_0$ and c_1 is interalgebraic with c over c_0. If c_1 depends on $\underline{c}$ over $c_0\underline{c}_1$, then $c_1\in acl(c_0,\underline{c})$ (by the irreducibility of c_1) contradicting that $c\in acl(c_0,c_1)$ and $c \perp c_0$. Thus, c_1 depends on c over $c_0\underline{c}_1\underline{c}$, implying that $stp(c/\underline{c})$ is nonorthogonal to a properly weakly minimal type. Since c is a bone (simply because it is in $acl(c_0,c_1)$) it follows that $tp(c/\underline{c})$ is nonisolated. Thus, by Lemma 3.10, there is a $d\in acl(c)\cap\mathbf{C}$ that depends on c over $\underline{c}$. A fortiori, d has the same number k of levels as c, which is also the number of levels of g in $T(A)$. Since $d \perp A$, d also has k levels over A (Corollary 3.4). Then, $d\in acl(\{g\}\cup A) \implies g \not\perp_{\underline{g}} d$ in $T(A) \implies g\in acl(\{d\}\cup A)$ (since g is irreducible in $T(A)$). This proves the lemma. □

Remark 7.3 For M a model, Proposition 7.2 will allow us to translate the realizations of $stp(c)$, $c \in \mathbf{C} \cap M$, into a definable group with connected component in $\mathscr{G}$. For the structure group we need the same property to hold using only finitely many elements of $\mathscr{G}$ over a finite $A \subset M$. The missing ingredient is the bound on the number of levels of elements of $\mathbf{C}$. Once this is proved, Proposition 7.3 implies that there are finitely many elements of $\mathscr{G}_0$ up to $*-$ isomorphism. This will be pulled together in Subsection 7.4.

7.2 More properties of $1-$ based groups

To execute the proof of that there is a bound on the number of levels of elements of $\mathbf{C}$ requires that we can reduce attention to weight 1 elements. This is done by working inside groups and using Proposition 7.1. Notice that this is true for any $1-$ based group in a supserstable theory of finite rank.

Lemma 7.10 *Let G be a $1-$ based basic group in a superstable theory of finite rank. Then G is a sum of weight 1 basic subgroups.*

Proof: Recall from Corollary 3.5 that for c any element, $wt(c) = wt(\ell_1(c)$. This is proved by induction on the rank of G. If a generic of G has only one level, then it is easy to use Lemma 1.26 to show that G is the sum of its rank 1 (hence weight 1) subgroups. Thus, we can assume that G has more than one level. Let $\underline{G}$ be a top filter group of G with ψ the corresponding top filter map. $\underline{G}$ itself is a $1-$ based basic group and $U(\underline{G}) < U(G)$. If $\underline{G}$ has weight 1 then for $c \in G$ generic, $wt(G) = wt(c) = wt(\ell_1(c)) = wt(\ell_1(\underline{c})) = wt(\underline{G}) = 1$, and we are done. So, we assume that $\underline{G}$ does not have weight 1, but, by induction, it is the sum of weight 1 basic subgroups. Let $K_0, \ldots, K_m \subset \underline{G}$ be weight 1 basic groups such that $\underline{G} = \sum K_i$. Let G_i be a basic subgroup of G such that for a a generic of G_i there is a generic b of K_i with $\psi(a) =^* b$. Since $wt(\underline{G}) > 1$, $U(K_i) < U(\underline{G})$ and $U(G_i) < U(G)$. Thus, each G_i is the sum of weight 1 basic subgroups.

We claim that $\sum G_i = G$. To see this let b be a generic of $\underline{G}$ and $b_i \in K_i$, $i \leq m$, such that $b = b_0 + \cdots + b_m$. Let $a_i \in G_i$ be such that $\psi(a_i) =^* b_i$ and $a = a_0 + \cdots + a_m$. Then, $\psi(a) =^* b$, implying that a is a generic of G. (The rank of $tp(a)$ is the rank of $tp(b)$ plus the rank of the kernel of ψ, which is also the formula for computing the rank of G.) Thus, $G = \sum G_i$.

Since each G_i is the sum of weight 1 basic subgroups and $\sum G_i = G$, the lemma is proved. □

7.3 Finitely many levels

This subsection has the one goal of showing there is a bound on the number of levels of elements of $\mathbf{C}$, completing the proof of Theorem 2.1(3). This is needed in the proof of the Structure Group Theorem. Predictably, assuming that there is no bound on the number of levels of irreducibles we will contradict the few countable models assumption.

Lemma 7.11 *Assume that there is no bound on the number of levels of elements of* $\mathbf{C}$. *Then there is a set of elements* $\mathbf{P}$ *with the following properties.*

(1) *Each* $c \in \mathbf{P}$ *is in* $\mathbf{C}$ *and* $wt(c) = 1$.
(2) *There is no bound on the number of levels of elements of* $\mathbf{P}$.

Proof: Notice that if each element of $\mathscr{G}_0$ were basic over $\emptyset$ there would be nothing left to prove, given Lemma 7.10. Here we need to transfer properties from an element of $\mathscr{G}_0$ to $\mathbf{C}$. Let $G \in \mathscr{G}_0$ have domain A and k levels. By Lemma 7.10 G is a sum of weight 1 subgroups. An element with k levels cannot be a sum of elements with fewer than k levels, so there is an element g of G with k levels and $wt(g/A) = 1$. By Lemma 7.9 there is a $d \in \mathbf{C}$, $d \perp A$, and interalgebraic wih g over A. Since $d \perp A$, $wt(d) = 1$.

Letting $\mathbf{P}$ be the collection of all elements d obtained in this manner proves the lemma. $\square$

Proposition 7.12 *There is an* l *such that every element of* $\mathbf{C}$ *has* $\leq l$ *levels.*

Proof: Suppose to the contrary that there is no such bound and let $\mathbf{P}$ be as guaranteed in the preceding lemma. Let $\Omega = \{ k : c \in \mathbf{P} \text{ has } k \text{ levels} \}$, by assumption an infinite set. For $n \in \Omega$ let $a^n \in \mathbf{P}$ be an element with k levels, chosen so that $A = \{ a^n : n \in \Omega \}$ is independent. Let $P_n = \{ c : tp(c) = tp(a^n) \}$, for $n \in \Omega$. For $n \in \Omega$, $P_{>n} = \bigcup_{m>n} P_m$, and similarly for other subscripts using $<$, $\leq$, etc.

For $X \subset \Omega$ let $A_X = \{ a^n : n \in X \}$. Let $M_X \supset A_X$ be a countable model almost atomic over A_X. Since $A_X \subset \mathbf{C}$, $\mathbf{C}(M_X) \subset acl(A_X)$, as usual. A key technical property on dependence is the following.

Claim 1. If $b \in P_i$ and $b \in acl(C_0 \cup C_1)$, where $C_0 \subset P_{<i}$ and $C_1 \subset P_{\geq i}$, then $b \not\perp C_1$.

Suppose to the contrary that $b \perp C_1$. Then b has i levels over C_1 (Corollary 3.4). Since C_0 has $< i$ levels b is independent from C_0 over $\underline{b} \cup C_1$. This contradicts that $b \in acl(C_0 \cup C_1)$.

To prove that $X \neq Y \implies M_X \ncong M_Y$ we need to establish

Claim 2. $i \in X$ if and only if there is a $b \in P_i(M_X)$ which is independent from all independent finite sequences from $P_{>i}(M_X)$.

This claim will require several steps and make use of the fact that $c \in \mathbf{P} \implies wt(c) = 1$. Let $J_0 = \{ a^n : n \in X \text{ and } n \leq i \}$ and $J_1 = \{ a^n : n \in X \text{ and } n > i \}$.

Subclaim. If $i \notin X$ and $b \in P_i(M_X)$, then for some finite $C \subset J_1$, $b \not\perp C$.

For such a b, $b \in acl(J_0 \cup C)$ for some finite $C \subset J_1$. By Claim 2, $b \not\perp C$, proving the subclaim.

Now consider the case $i \in X$. Then, $a^i \in J_0$. Let $\bar{b} = \{b_0, \ldots, b_n\}$ be an independent sequence of elements of $P_{>i}(M_X)$. The claim will be proved when we show that $a^i \perp \bar{b}$, which will take several steps.

Since $\bar{b} \subset acl(J_0 \cup J_1)$ and all elements of $\mathbf{C}$ are $1-$ based, there are $a_j \in acl(J_0)$ and $c_j \in acl(J_1)$ such that $x \in X = \{a_j, b_j, c_j\} \implies x \in acl(X \setminus \{x\})$. Notice that $(*)$ $a_j \perp c_j$ (since $J_0 \perp J_1$), $b_j \not\perp c_j$ (by Claim 1), and $a_j \perp b_j$ (using that $wt(b_j) = 1$). These conditions also imply that $wt(c_j) = 1$. The main step in showing that $\bar{b}$ is independent from J_0 (hence a^i) is the proof of

$$\bar{c} = \{c_0, \ldots, c_n\} \text{ is independent.} \tag{5}$$

This is established using weight, $(*)$ and the independence of $\bar{b}$. Assume, inductively, that $C_m = \{c_0, \ldots, c_{m-1}\}$ is independent. Thus, $wt(C_m) = m$. Since $\{b_0, \ldots, b_m\}$ is independent and $b_j \not\perp C_m$, for $j < m$ (by $(*)$), $b_m \perp C_m$. Since $b_j \not\perp c_j$ (by $(*)$) and $wt(c_j) = 1$, $c_m \perp C_m$, establishing (5).

Since $\bar{c} \subset acl(J_1)$, $\bar{c} \perp J_0$. By $(*)$, $U(c_j) = U(b_j)$, so by (5) and the independence of $\bar{b}$, $U(\bar{b}) = U(\bar{c})$. Adding that $\bar{c}$ and $\bar{b}$ are interalgebraic over J_0, $U(\bar{b}/J_0) = U(\bar{c}/J_0) = U(\bar{c}) = U(\bar{b})$. That is, $\bar{b}$ is independent from J_0. Since $a^i \in J_0$ we have proved Claim 2.

The property described in Claim 2 is invariant under isomorphisms, so $X \neq Y \implies M_X \not\cong M_Y$. This contradicts that T has few countable models to prove the proposition. □

Remark 7.4 This completes the proof of Theorem 2.1(3).

Corollary 7.13 *Among the elements of* $\mathbf{C}$ *there are finitely many* $\approx$ *classes.*

Proof: By Corollary 4.16 there are finitely many top strong types up to nonorthogonality. Thus, for any l there are finitely many $\approx$ classes among the elements of $\mathbf{C}$ with l levels. Since there is a bound on the number of levels of elements of $\mathbf{C}$ the corollary is proved. □

Corollary 7.14 *There are finitely many elements of* $\mathscr{G}_0$ *up to* $*-$ *isomorphism.*

Proof: There are finitely many $\approx$ classes among elements of $\mathbf{C}$ by Corollary 7.13, thus there are finitely many $\approx$ classes among elements of $\mathscr{G}_0$. By Proposition 7.3, there are finitely many elements of $\mathscr{G}_0$, up to $*-$ isomorphism. □

7.4 Proof of the Structure Group Theorem

Here the Structure Group Theorem, stated below, is proved using the tools developed in earlier subsections. This is a critical intermediary result on the way to the Structure Theorem. We also set the notation on the structure group used subsequently.

Definition 7.3 A *structure group for* T is a sorted group $\hat{H}$ having the following properties: (1) There is a finite A such that $tp(A)$ is isolated and each sort of $\hat{H}$ is connected and $\bigwedge-$definable over A. (2) Each sort of $\hat{H}$ is in $\mathscr{G}_0$. (3) Each element of $\mathscr{G}_0$ is $*-$isomorphic to a sort of $\hat{H}$.

The set A is called the *domain* of $\hat{H}$.

Lemma 7.15 *There is a structure group for* T.

Proof: Using Lemma 7.4(ii) and that there are finitely many $\approx$ classes among elements of $\mathscr{G}_0$, there are: a finite A such that $tp(A)$ is isolated, $H_0,\ldots,H_n \in \mathscr{G}_0$ with domain A such that each $G \in \mathscr{G}_0$ is $\approx$ to H_i, for some $i \le n$. By Corollary 7.14 each $G \in \mathscr{G}_0$ is $*-$isomorphic to H_i, for some $i \le n$. Thus, the sorted group $\hat{H}$ constructed with the H_i's is a structure group for T. $\square$

Results about $\mathscr{G}_0$ in Subsection 7.1 lead to the following characterization of a structure group as a basis for $\mathbf{C}$.

Lemma 7.16 *Let* $\hat{H}$ *be a structure group with domain* A. *Then for each* $c \in \mathbf{C}$, *every realization of* $stp(c/A)$ *is interalgebraic with an element of* $\hat{H}$ *over* $\{c\} \cup A$.

Proof: By condition (3) in the definition of a structure group there is a sort H_i of $\hat{H}$ such that H_i has the same number of levels as c and $c \approx H_i$. By Proposition 7.2, every realization of $stp(c/A)$ is interalgebraic with an element of H_i over $\{c\} \cup A$, to prove the lemma. $\square$

Theorem 7.17 (Structure Group Theorem) *Let* M *be a model of* T *and* $\hat{H}$ *a structure group with domain* $A \subset M$. *Then there is a finite set* B, $A \subset B \subset M$, *such that* $\mathbf{C}(M) \subset acl(\hat{H}(M) \cup B)$.

Proof: By Proposition 4.18 (using that there is a bound on the $U-$rank of elements of $\mathbf{C}$) there is a finite set $A' \subset \mathbf{C}(M)$ such that every element of $\{stp(c/A) : c \in \mathbf{C}(M)\}$ is realized in $acl(A' \cup A)$. Let $B = A' \cup A$. Then, by Lemma 7.16, $\mathbf{C} \subset acl(\hat{H}(M) \cup B)$. $\square$

8 Counting structure groups

The ultimate goal of this section is a proof of the stucture theorem. The route is through proving that a particular class of Fisher abelian structures (the purification of the structure group) has finite representation type. First, we set some notation. The assumption that $\hat{H}^-$ is a good basis may not hold in the original theory, but by Lemma 8.1 it holds in after fixing some finite set. (Indeed we must fix some finite domain to find a structure group with domain $\emptyset$.) The effect of setting such parameters on the proof of the structure theorem for an arbitrary model will be discussed in due time.

Notation.

- $\hat{H}$ denotes a structure group over $\emptyset$ and $\hat{H}^-$ is a good basis.
- $\hat{H}^+$ denotes the purification of $\hat{H}$ in the language L^+ as described in Definition 6.2.
- A subgroup A of $\hat{H}$ is called an $\hat{H}-$*space* if $A = acl(A) \cap \hat{H}$. For $A \subset \hat{H}-$, A^+ denotes $(A + \hat{H}^-)/\hat{H}^-$ as a subset of $\hat{H}^+$ in the language L^+.
- Let $\mathscr{C} = \{A^+ : A \text{ is an } \hat{H}-\text{space}\}$.
- Let $\mathscr{C}_0 = \{\hat{H}(M)^+ : M \models T\}$.
- Let $\mathscr{A} = Ab(L^+)$.

Ultimately, it will be shown that $\mathscr{C}$ has finite representation type.

Remark 8.1 $\mathscr{C}_0 \subset \mathscr{C}$ (clear) and $\mathscr{C}_0$ and $\mathscr{C}$ have the same countable elements (by Corollary 4.6).

Remark 8.2 (i) A basic $K \subset \hat{H}^n$ is $\bigwedge-$definable over some finite $e \in acl(\emptyset)$. [Just by being basic we know it is $\bigwedge-$definable over $acl(\emptyset)$. Since every $\bar{a} \subset acl(\mathbf{C})$ is a bone, for any $\bar{a} \in \hat{H}^n$ there is a finite $e \in acl(\emptyset)$ such that $tp(\bar{a}/e) \vdash stp(\bar{a})$.]

(ii) That a basic subgroup K of $\hat{H}^n$ used in defining L^+ is $\bigwedge-$definable over some finite $e \in acl(\emptyset)$, but perhaps not over $\emptyset$, effects how we apply the few countable models hypothesis. The operational hypothesis is that there are $< 2^{\aleph_0}$ countable groups $\hat{H}(M)$ up to isomorphism over $\emptyset$, rather than $acl(\emptyset)$. This will be factored into the proof at the appropriate place.

First, we show that by fixing a finite set we can assume that $\hat{H}^-$ is a good basis.

Lemma 8.1 *For $\hat{H}_0$ a structure group there is a good basis for $\hat{H}_0$ that is contained in the algebraic closure of a finite set.*

Proof: Being a good basis represents a richness with respect to realizing strong types. (If, for example, $a \in \hat{H}_0$ and B is an $\hat{H}_0-$space, then $a + B$ contains a good element when B contains a realization of $stp(a)$.) Just using Proposition 4.18 we can find an $\hat{H}_0-$space which is algebraic in a finite set and realizes every strong type realized in $\hat{H}$. The fact that we need a good element independent from B complicates the construction.

For simplicity, add constants to the language for the domain of $\hat{H}_0$. Let $X_0, \ldots, X_m$ list minimal sets, each over a finite subset of $acl(\emptyset)$, such that any minimal set appearing in some element of $\mathscr{C}$ is nonorthogonal to one of the X_i's. (A finite list suffices since there are finitely many $\approx$ classes by Corollary 7.13.) Let $\mathbf{X} = X_0 \cup \cdots \cup X_m$ and notice that $\ell_1(a) \subset \mathbf{X}$, for any $a \in \hat{H}_0$. Let k be $> U(\ell_1(a))$ for any $a \in \hat{H}_0$.

Choose $A \subset \mathbf{X}$ such that $A = acl(A) \cap \mathbf{X}$ and for each $i \leq m$, $\dim(A \cap X_i) \geq k$, and A is algebraic in a finite set. Let $B' \subset \mathbf{C}$ be a maximal set such that $acl(B' \cup A) \cap \mathbf{X} = A$. We will eventually show that $B = \hat{H}_0 \cap acl(B' \cup A)$ is a good basis, algebraic in a finite set. One part is contained in the following claim.

Claim 1. B' is algebraic in a finite set.

Suppose $\{a_i : i < \omega\} \subset B'$ and let $A_i = \{a_0, \ldots, a_{i-1}\}$. We will show by induction that for all $i \leq l$ (where l is the maximum number of levels in $\mathbf{C}$) there is an m such that $acl(\{a_0, \ldots, a_m\} \cup A) \supset \ell_i(a_j)$ for all j. Since $\ell_1(B') \subset A$ is finite dimensional this is true when $i = 1$. Suppose that $\ell_i(a_j) \subset acl(\{a_0, \ldots, a_m\} \cup A)$ for all j and let $P = \{stp(a_j/A_{m+1} \cup A) : j < \omega\}$. Since there is a bound on the ranks of the elements of $\mathscr{C}$, Proposition 4.18 applies to yield an $n \geq m$ such that each $p \in P$ is realized in $acl(A_{n+1} \cup A)$. In particular, $stp(\ell_{i+1}(a_j)/A_{m+1} \cup A)$ is realized in $acl(A_{n+1} \cup A)$ for each j. Suppose j is such that $\ell_{i+1}(a_j) \not\subset acl(A_{n+1} \cup A)$. This yields a $c \in \ell_{i+1}(a_j)$ with $stp(c/\ell_i(a_j))$ minimal such that $c \notin acl(A_{n+1} \cup A)$. By choice of n there is a $c' \in acl(A_{n+1} \cup A)$ realizing $stp(c/\ell_i(a_j))$. Let l be such that $stp(c/\ell_i(a_j))$ is nonorthogonal to X_l. Since X_l is modular over A there is a $d \in X_l$ interalgebraic with c over $\{c'\} \cup \ell_i(a_j) \cup A)$. Then $d \in acl(B' \cup A) \cap \mathbf{X}$ and $d \notin A$. This contradicts the choice of B', to prove the claim.

Let $Q = \{stp(a/A_0) : a \in \hat{H}_0 \text{ and } A_0 = \ell_1(a) \cap A\}$. Most of the work in showing that $B = \hat{H}_0 \cap acl(B' \cup A)$ is a good basis is contained in the following.

Claim 2. Every element of Q is realized in B.

Let e be a realization of $p \in Q$ with $U(e/B)$ minimal. If $e \notin B$ the maximality of B' forces there to be an $a \in \mathbf{X} \setminus A$ algebraic in $B' \cup A \cup \{e\}$. Let $d \in acl(e)$ be such that a and d are interalgebraic over $B' \cup A$. Let $E_0 = acl(B' \cup A) \cap acl(e)$ and notice that $E_0 \cap A = \ell_1(e) \cap A$ has $U-$rank $\leq k$. Then $stp(d/B' \cup A)$ is minimal and modular (since $stp(a/A)$ is minimal and modular), $U(d/E_0) = 1$ and there is $a' \in A$ with $stp(a'/E_0)$ nonorthogonal to $stp(d/E_0)$ (since A was chosen with $\dim(X_i(A)) > k$, for all i). It follows that there is a $d' \in acl(B' \cup A)$ realizing $stp(d/E_0)$. Let e' be the image of e under an automorphism mapping d to d' over E_0. Then, e' realizes p and $U(e'/B' \cup A) < U(e/B' \cup A)$. This contradiction proves the claim.

Claim 3. B is a good basis.

Assume to the contrary that $a \in \hat{H}_0$ has $U(a)$ minimal such that $a + B$ does not contain a good element independent from B. Let $A_0 = \ell_1(a) \cap \ell_1(B)$. By Claim 2 there is a $b \in B$ realizing $stp(a/A_0)$. First assume that $A_0 \not\subset acl(\emptyset)$. Then by the characterization of strong types in a $1-$based group, $c = a - b$ realizes a type of $U-$rank $< U(a)$. By the minimal rank assumption there is a good element of $c + B = a + B$ independent from B; a contradiction. Otherwise, $A_0 \subset acl(\emptyset)$; i.e., $\ell_1(a)$ is independent from $\ell_1(B)$. Thus, a is independent from B, implying that $a - b$ is a good element independent from B.

Since B is a good basis, algebraic in a finite set, the lemma is proved. □

For $\hat{H}_0$ an arbitrary structure group with domain A_0 there is a good basis $B \subset \hat{H}_0$ that is algebraic in a finite set. Thus, adding constants to the language produces a structure group $\hat{H}$ satisfying the conditions specified at the beginning of the section.

Since the domain of a structure group realizes an isolated type over $\emptyset$, an arbitrary countable model M contains the domain of a structure group. It is not clear, though, that

M contains a good basis. We'll get around this deficiency in the final proof of the structure theorem. For now, though, we operate under the assumptions stated above.

Much of the work in counting structure groups is reflected in the following, which is proved with a series of lemmas.

Theorem 8.2 *$\mathscr{C}$ has finite representation type.*

Recalling the definition $\mathscr{C}$, the following characterization of algebraic closure on $\hat{H}$ helps describe the elements of $\mathscr{C}$.

Lemma 8.3 *Let $\mathscr{K}$ be the set of basic subgroups of $\hat{H}$ that are the graphs of $*-$ homomorphisms. For any basic subgroup K of $\hat{H}$ let $P_K \in L^+$ be the predicate representing K. Then the subgroup A^+ of $\hat{H}$ is in $\mathscr{C}$ if and only if for $\Psi \in \mathscr{K}$, if Ψ is the graph of a $*-$ homomorphism from K to H and $\bar{a} \in A^+$ satsifies P_K there is a $b \in A^+$ such that $\bar{a}b$ satisfies P_Ψ.*

Proof: This follows from the characterization of strong types in a $1-$ based group and the fact that every tuple in $\hat{H}^+$ is a generic of some group $P(\hat{H}^+)$, for $P \in L^+$. A review of the proof of Proposition 6.1 and Definition 6.2 should be sufficient to fill in the details. $\square$

Remark 8.3 $\Psi \in \mathscr{K}$ defines a $*-$ homomorphism ψ from a basic K to a basic H. If $\psi(\bar{a}) =^* b$ and $\bar{a}^+, b^+$ the images of $\bar{a}, b$ in $\hat{H}^+$. Then b^+ is the unique $x \in \hat{H}^+$ such that $P_\Psi(\bar{a}^+, x)$.

Remark 8.4 Let $f \in Aut(\mathfrak{C})$. The basic groups used as sorts in $\hat{H}$ are all $\bigwedge-$ definable over $\emptyset$, so f preserves $\hat{H}$. Of course, f permutes $\hat{H}^-$, so f induces a group homomorphism of $\hat{H}^+$ respecting the sorts. If K is a basic subgroup of $\hat{H}^n$ (for some n), $\bigwedge-$ definable over $e \in acl(\emptyset)$, then fK is a basic subgroup, $\bigwedge-$ definable over $f(e)$. Thus, f permutes the basic groups used to define $\hat{H}^+$ from $\hat{H}$, so it may not produce an L^+- isomorphism. However, it follows easily from the characterization of $\mathscr{C}$ in Lemma 8.3 that if $A \in \mathscr{C}$ then $fA \in \mathscr{C}$. This will be an important consideration in applying the few models assumption to count the countable elements of $\mathscr{C}$.

Closure of $\mathscr{C}$ under direct sums and the linkage between direct sums and forking dependence is specified in the following lemma.

Lemma 8.4 *$\mathscr{C}$ is closed under direct sums. Moreover, if $A^+, B^+ \in \mathscr{C}$, where $A, B \subset \hat{H}$, then $A^+ + B^+$ is a direct sum if and only if $A \downarrow B$.*

Proof: Closure under direct sums follows from generalizing the second clause to arbitrary sums, a detail left to the reader. First suppose that $A^+, B^+ \in \mathscr{C}$, where $A, B \subset \hat{H}$, and $A \downarrow B$. Let $C = A + B$. It follows directly from Lemma 8.3 that $C^+ \in \mathscr{C}$. [If $\Psi \in \mathscr{K}$ as in Remark 8.3, $\bar{c} \subset C$ and $d \in \hat{H}^+$ is such that $P_\Psi(\bar{c}, d)$ then there are $\bar{a}d_0 \subset A$ and $\bar{b}d_1 \subset B$

such that $\bar{c} = \bar{a} + \bar{b}$, $P_\Psi(\bar{a}, d_0)$ and $P_\Psi(\bar{b}, d_1)$. Then, $d = d_0 + d_1 \in C$.] To see that C is a direct sum let $P_K \in L^+$, $\bar{a}^+ \subset A^+$, $\bar{b}^+ \subset B^+$ such that $P_K(\bar{a}^+, \bar{b}^+)$, where $\bar{a}\bar{b}$ is a generic of K. Notice that $\bar{b}$ is a generic of a basic group H. Since $\bar{a} \downarrow \bar{b}$ there is a generic $\bar{b}'$ of H independent from $\bar{a}\bar{b}$ such that $\bar{a}\bar{b}'$ is a generic of K. Thus, $\bar{0}\bar{d} \in K$, where $\bar{d} = \bar{b}' - \bar{b}$. Since $\bar{d}$ and $\bar{b}$ are both generics of H, $P_K(\bar{0}, \bar{b}^+)$. Similarly, $P_K(\bar{a}^+, \bar{0})$, showing that $C^+ = A^+ \oplus B^+$.

Conversely, suppose that $C^+ = A^+ \oplus B^+$. Suppose that $\bar{a} \subset A$ and $\bar{b} \subset B$ and K is such that $\bar{a}\bar{b}$ is a generic of K. Since $P_K(\bar{a}^+, \bar{0})$ it is impossible for $\bar{a}$ to depend on $\bar{b}$. This proves the lemma. $\square$

Lemma 8.5 *If $B \in \mathscr{A}$ is an elementary extension of $A \in \mathscr{C}$ (and $|B| < |\mathfrak{C}|$) then B is isomorphic over A to an element of $\mathscr{C}$.*

Proof: Let Δ be the atomic diagram of B. Since B is an elementary extension of A and A is a submodel of $\hat{H}^+$, Δ is consistent with the theory of $(\hat{H}^+, a)_{a \in A}$. By Theorem 6.2, $\hat{H}^+$ is saturated. Thus, B is isomorphic over A to a submodel B' of $\hat{H}^+$. That $B' \in \mathscr{C}$ follows from Lemma 8.3 and the fact that B' is elementarily equivalent to $A \in \mathscr{C}$. This proves the lemma. $\square$

The hard step in the proof of Theorem 8.2 is the following result. Note that $A^+ \in \mathscr{C}$ is called *finitely generated* if A is algebraic in a finite subset; equivalently, A is the closure of a finite set under $*-$ homomorphisms.

Lemma 8.6 *Every finitely generated element of $\mathscr{C}$ is pure-injective.*

Proof: Suppose, to the contrary, that $A \in \mathscr{C}$ is generated by $\bar{a}$ and does not have the descending chain condition on pp-definable subgroups. Let φ_i, $i < \omega$, be $pp-$ formulas such that $\varphi_0(A) \supset \varphi_1(A) \supset \dots$ is a properly descending chain of subgroups. We would like to proceed as in [Pre88, 2.10] to find $2^{\aleph_0}$ many $pp-$ types over $\bar{a}$ which are consistent with $Th(A)$, contradicting the few models assumption. However, since the basic subgroups of $\hat{H}$ are only definable over $acl(\emptyset)$ the constructed types may be conjugate via $\Delta = Aut(\mathfrak{C})$ (see Remark 8.4). We need to refine Prest's argument in the following way. Let $p(x)$ be the $pp-$ type generated by $\{\varphi_i : i < \omega\}$ (i.e., $\psi \in p$ if and only if ψ is a pp-formula and there is a finite $X \subset \omega$ with $Th(A) \vdash \bigwedge_{i \in X} \varphi_i \longrightarrow \psi$.) There is an elementary extension B of A containing an element b with $p = pp^B(b) =$ the $pp-$ type of b in B. By Lemma 8.5 we may as well assume that $B \in \mathscr{C}$. Let $\bar{a}' = \bar{a}b$, Δ' the elements of Δ which fix $\bar{a}'$. Notice that we may assume φ_i to be $\Delta'-$ invariant. [Let $\varphi^0, \dots, \varphi^m$ be the $\Delta'-$ conjugates of φ_i. Since b realizes p, each of these formulas is in p. Hence, $\psi = \bigwedge_{j \le m} \varphi^j$ is a $\Delta'-$ invariant formula in p which implies φ_i. Replacing each φ_i by such a ψ produces an invariant set of formulas that generates p.]

Let $\Phi = \{\varphi : i < \omega\}$. An element $a \in A$ has finitely many images under the elements of Δ'; i.e., a finite $\Delta'-$ orbit. Let $\Psi = \{\psi : \psi$ is a formula (over $\bar{a}'$) that defines the $\Delta'-$ orbit of some $a \in A\}$. Having set this notation we define, recursively, the $2^{\aleph_0}$ many types which are not $\Delta'-$ conjugate as follows.

At stage k in the recusion we have, for each $s \in 2^k$, an element a_s and $\varphi^k \in \Phi$ (perhaps not φ_k) such that

$$\text{for all } s,t \in 2^k \text{ and } f \in \Delta',\ fa_s - a_t \notin \varphi^k(A). \tag{6}$$

Assume that we have completed the construction at stage k and $s \in 2^k$. We find $a_{s\frown 0}$ and $a_{s\frown 1}$ as follows. Let $a_{s\frown 0}$ be any element of $a_s + \varphi^k(A)$, $\psi \in \Psi$ the formula that defines the $\Delta'-$ orbit of $a_{s\frown 0}$. Since the formulas in Φ define a properly descending chain there is a $\varphi \in \Phi$ such that $\varphi(A) \subset \varphi^k(A)$ and $[\varphi^k(A) : \varphi(A)] > |\psi(A)|$. Thus, there is an element $a_{s\frown 1} \in a_s + \varphi^k(A)$ such that there is no $c \in \psi(A)$ with $a_{s\frown 1} - c \in \varphi(A)$. This formula φ depends on s, but by choosing a possibly "smaller" formula in Φ we can assume that the same formula φ^{k+1} works for each $s \in 2^k$. To verify that this satisfies the desired property in (6) suppose that $f \in \Delta'$, $s,t \in 2^{k+1}$ and $fa_s - a_t \in \varphi^{k+1}(A)$. Since (6) holds for k, and $\varphi^{k+1} \longrightarrow \varphi^k$, s and t must have the same restriction to k; i.e., $s = u\frown 0,\ t = u\frown 1$ for some $u \in 2^k$. This contradicts that for all c in the $\Delta'-$ orbit of $a_{s\frown 0}$, $c - a_{s\frown 1} \notin \varphi^{k+1}(A)$. This completes the construction.

For $s \in 2^\omega$ let $p_s = \{\varphi^k(x - a_{s\restriction k}) : k < \omega\}$. The construction guarantees that for $s \neq t$ in 2^ω, p_s and p_t are not $\Delta'-$ conjugate. By Lemma 8.5, there is a countable model M_s with p_s realized in $\hat{H}(M_s)^+$. The M_s's are pairwise nonisomorphic over an appropriate finite set, contradicting the few models assumption. □

Corollary 8.7 *The endomorphism ring of a finitely generated indecomposable element of $\mathscr{C}$ is local.*

Proof: This follows from Lemma 8.6 by a result of Zimmermann quoted as [Pre88, 4.27] and [Zie84, 4.3]. While this result of Zimmermann's is about modules rather than the more general abelian structures we have here, the same proof works. □

The following is used in the final many-model argument.

Lemma 8.8 *If $A \in \mathscr{C}$ is finitely generated then $\{fA : f \in Aut(\mathfrak{C})\}$ contains finitely many structures, up to isomorphism in L^+. Moreover, if A is indecomposable then fA, for $f \in Aut(\mathfrak{C})$, is indecomposable.*

Proof: Let A be generated by $\bar{a}^+$, a generic of P_K, where $\bar{a}$ is a generic of K. Notice that A is simply the closure of $\bar{a}^+$ under homomorphisms in L^+ with domain P_K. There are finitely many basic groups $K_0, \ldots, K_l$ that are conjugates of K under $Aut(\mathfrak{C})$. If $\bar{b}$ and $\bar{c}$ are generics of K_i and $B, C \in \mathscr{C}$ are generated by $\bar{b}^+$ and $\bar{c}^+$, respectively, then B and C isomorphic as L^+- structures. Thus, there are finitely many L^+- structures in $\{fA : f \in Aut(\mathfrak{C})\}$.

Checking the definition of a direct sum, it is clear that fA is indecomposable whenever A is indecomposable. □

Lemma 8.9 *There are, up to L^+- isomorphism, finitely many finitely generated indecomposable elements of $\mathscr{C}$.*

Proof: Suppose, to the contrary, that there are infinitely many such Fisher abelian structures. Let Δ be the permutations of $\hat{H}^+$ induced by the elements of $Aut(\mathfrak{C})$. By Lemma 8.8 if $A \in \mathscr{C}$ is indecomposable and $f \in \Delta$, then fA is also indecomposable. By Corollary 8.7 there is a family $\{A_i : i < \omega\}$ of finitely generated indecomposables such that for $i \neq j$ there is no $f \in \Delta$ such that fA_i is isomorphic to A_j as L^+- structures. For $X \subset \omega$ let $A_X = \bigoplus_{i\in X} A_i$ and let M_X be a countable model with $\hat{H}(M_X)^+ = A_X$ (see Remark 8.1). Now suppose that X and Y are subsets of ω, $i \in X \setminus Y$ and f is an isomorphism from M_X onto M_Y. Then f (considered as an element of Δ) induces a bijection of A_X onto A_Y, so $A_Y = \bigoplus_{j\in X} fA_j$. By Corollary 8.7(i) the endomorphism ring of each of the A_j's (and fA_j's) is local. Thus, by Azumaya's Lemma (Lemma 1.29) fA_i is isomorphic to some $A_j,\ j \in Y$. This contradicts our choice of the A_j's, proving the lemma. □

With this set of results in place we can complete the proof of the section's principle theorem.

Proof of Theorem 8.2. The following sequence of results verifies the hypotheses of Lemma 1.30. It is clear that $\mathscr{C}$ is closed under pure subgroups and every finite subset of $\hat{H}^+$ is contained in a finitely generated element. $\mathscr{C}$ is closed under direct sums by Lemma 8.4. Every finitely generated element of $\mathscr{C}$ is pure-injective by Lemma 8.6. There are finitely many finitely generated indecomposable elements of $\mathscr{C}$ by Lemma 8.9. By Lemma 1.30, $\mathscr{C}$ has finite representation type. □

Except for quoting previously proved results the Structure Theorem is implied by the following. At this point we remove the assumption that $\hat{H}$ is a structure group with domain $\emptyset$. Theorem 8.2 will be applied by finding a structure group over a finite subset of a fixed model and working over this set.

Theorem 8.10 *For M a model of the theory there is a finite set $C \subset M$ and a set J of tuples from $\mathbf{C}(M)$ which is $C-$ independent such that there are finitely many strong types over C realized in J and $\mathbf{C}(M) \subset acl(C \cup J)$.*

Proof: First let $C_0 \subset M$ be a finite set such that there is a structure group $\hat{H}$ with domain C_0. By the Structure Group Theorem (Theorem 7.17) there is a finite set C, $C_0 \subset C \subset M$, such that $\mathbf{C}(M) \subset acl(\hat{H}(M) \cup C)$. Add constants to the language for the elements of C.

Let $A = \hat{H}(M)$. It is not clear that A contains a good basis, so we augment A as follows. Let B be a good basis for $\hat{H}$, $B \perp A$, and $A^* = acl(A+B) \cap \hat{H}$. Applying preceding results of the section to $\hat{H}^+ = \hat{H}/B$, by Theorem 8.2, there is a direct sum decomposition $A^*/B = \bigoplus_{i\in X} A_i^+$, where the A_i^+'s are finitely generated and there are finitely many of them, up to isomorphism (in the appropriate class $Ab(L^+)$ of Fisher abelian structures). As the notation suggests we assume that A_i is an $\hat{H}-$ space containing B. Notice that there is a k such that $U(A_i/B) \leq k$, since there are finitely many A_i^+'s, up to L^+- isomorphism.

For $i \in X$ let $\bar{a}_i \subset A \cap A_i$ be such that $\bar{a}_i^+ = (\bar{a}_i + B)/B$ generates A_i^+. By Lemma 8.4, $A_X = \{\bar{a}_i : i \in X\}$ is $B-$ independent. Since $A \downarrow B$,

$$A_X \text{ is independent.} \tag{7}$$

Ideally, we'd like to take A_X as J since $A \subset acl(A_X)$. If the algebraic closure of the original set C contained a good basis we could require that the $\bar{a}_i$ are generics of basic groups, and that there are finitely many such groups since there are finitely many A_i^+'s (up to isomorphism). In this more general setting additional work is needed to find a set of generators with the properties required for J. The next claim identifies these elements.

Claim 1. There is a finite set $Y \subset X$ and for $j \in X \setminus Y$ a tuple $\bar{b}_j$ such that

(1) $\bar{b}_j \subset A$ and $\bar{b}_j$ is interalgebraic with $\bar{a}_j$ over $A_Y = \sum_{i \in Y} A_i$;
(2) for $B_j =$ the $\hat{H}-$ space generated by $\bar{b}_j \cup B$, $A^*/B = A_Y^+ \oplus \bigoplus_{j \in X \setminus Y} B_j^+$;
(3) $\bar{b}_j$ is a generic of a basic group K_j.

Since $U(\bar{a}_j) \leq k$ and $\bar{a}_j \subset \mathbf{C}$, $P = \{stp(\bar{a}_i) : i \in X\}$ satisfies the hypotheses of Proposition 4.18. Thus, by this proposition there is a finite set $Y \subset X$ such that every element of P is realized in $acl(A_Y)$. By the characterization of strong types in a $1-$ based group, for $j \in X \setminus Y$ there is a $\bar{b}_j$ generic of a basic group K_j, $\bar{b}_j \downarrow A_Y$, and $\bar{b}_j$ is interalgebraic with $\bar{a}_j$ over A_Y. Let B_j be the $\hat{H}-$ space generated by $\bar{b}_j \cup B$. Moreover, since $\{\bar{a}_i : i \in Y\} \cup \{b_j : j \in X \setminus Y\}$ is independent, the sum of the generated subgroups of A^*/B is direct: $A^*/B = A_Y^+ \oplus \bigoplus_{j \in X \setminus Y} B_j^+$. This proves the claim.

Claim 2. There is a finite $Z \subset X \setminus Y$ such that every K_j is $*-$ isomorphic to K_l for some $l \in Z$.

While we haven't shown that the B_j^+'s are indecomposable, since $U(B_j/B) \leq k$, and $\mathscr{C}$ has finite representation type, there is a finite $Z \subset X \setminus Y$ such that any B_j^+ is L^+- isomorphic to B_l^+, for some $l \in Z$. If B_j^+ is $*-$ isomorphic to B_l^+ there is $\bar{c} \subset K_j$, generic over B and interalgebraic with $\bar{b}_l$ over B. It follows that K_j is $*-$ isomorphic to K_l, proving the claim.

Thus, possibly replacing $\bar{b}_j$ by a tuple interalgebraic with it, we can write $\{\bar{b}_j : j \in X \setminus Y\}$ as the independent $\bigcup_{j \in Z} J_j$, where J_j is a set of generics of K_j. Then, $J = \{\bar{a}_i : i \in Y\} \cup \bigcup_{l \in Z} J_l$ is an independent subset of A, $A \subset acl(J)$ and $\{stp(\bar{c}) : \bar{c} \in J\}$ is finite. This proves the theorem. □

9 Proof of the Structure Theorem

We conclude with a proof of the Structure Theorem, restated here.

Theorem 9.1 (Structure Theorem) *Let T be a countable superstable theory of finite rank with $< 2^{\aleph_0}$ many countable models. Then for M a countable model of T there is a* *finite $A \subset M$ and a $J \subset M$ such that M is prime over $A \cup J$, J is $A-$ independent and $\{stp(a/A) : a \in J\}$ is finite.*

Proof: By Theorem 8.10 there is a finite $A_0 \subset M$ and an A_0- independent set J_0 such that $\bar{a} \in J_0 \implies \bar{a} \subset \mathbf{C}(M)$, $\mathbf{C}(M) \subset acl(A_0 \cup J_0)$ and $\{stp(a/A_0) : a \in J_0\}$ is finite. By Corollary 1.11 there is a finite $A_1 \supset A_0$ and A_1- independent set $J_1 \subset \mathbf{J}(M)$ such that $\mathbf{J}(M)$ is almsot atomic over $A_1 \cup J_1$ and $\{stp(a/A_1) : a \in J_1\}$ is finite. Since any type realized in $\mathbf{C}$ is orthogonal to any type realized in $\mathbf{J}$, $J_1 \underset{A_1}{\downarrow} J_0$. If necessary, absorb a finite subset of J_0 into A_0, forming A such that J_0 is $A-$ independent.

Even though M is almost atomic over $X = \mathbf{J}(M) \cup \mathbf{C}(M)$ (Theorem 2.1(6)) to use Lemma 1.18(ii) to show M is prime over an $A-$ independent set like $J_0 \cup J_1$ requires an additional condition. Namely, that for any $a \in M$ there is a finite $B \subset A \cup J_0 \cup J_1$, $tp(a/B)$ has no forking extension over X. Obtaining this property requires augmenting $J_0 \cup J_1$, slightly.

There are finitely many $\omega-$ categorical strongly minimal sets appearing in elements of $\mathbf{C}$. Since each is nonorthogonal to $\emptyset$ the nonorthogonality classes are represented by some $D_0, \ldots, D_m$ such that each D_i is strongly minimal over some $e_i \in acl(\emptyset)$. Absorb the e_i into A. Let J_2 be $A-$ independent from $J_0 \cup J_1$ such that J_2 is a basis for $D_0(M) \cup \cdots \cup D_m(M)$ over $J_0 \cup J_1 \cup A$. Finally, let $J = J_0 \cup J_1 \cup J_2$, an $A-$ independent set such that $\{stp(a/A) : a \in J\}$ is finite and M is almost atomic over $J \cup A$.

Claim. For any $a \in M$ there is a finite $B \subset J$ such that $tp(a/A \cup B)$ has no forking extension over $J \cup A$.

Take as $B \subset J$ a finite set such that $a \underset{B \cup A}{\downarrow} J \cup A$. We can choose B large enough so that $J' = J \setminus B$ is $A-$ independent from B. If there is a forking extension of $tp(a/B)$ over $J \cup A$ there is an a' realizing $tp(a/B)$ and a finite $X \subset \ell_1(J' \cup A)$ such that $a' \underset{B \cup A}{\not\downarrow} X$ (using that J' is $A-$ independent from B). Since X is a subset of some minimal sets it can be written as $X = X_0 \cup \{x\}$ where $x \in acl(\{a'\} \cup X_0 \cup B \cup A)$. Since J contains bases for all minimal sets appearing in $\ell_1(J')$, this would contradict that a is independent from J' over $B \cup A$. This proves the claim.

Thus, M is almost atomic over $A \cup J$ and for each $a \in M$ there is a finite $J' \subset J$ such that $tp(a/A \cup J')$ has no forking extension over $A \cup J$. By Lemma 1.18(ii), M is prime over $A \cup J$. This proves the theorem. □

References

[BH94] E. Bouscaren and E. Hrushovski. On one-based theories. *J. Symbolic Logic*, 59(2):579–595, 1994. preprint.

[BM82] John T. Baldwin and Ralph McKenzie. Counting models in universal Horn classes. *Algebra Universalis*, 15:359–384, 1982.

[Bou83] E. Bouscaren. Countable models of multidimensional $\aleph_0-$ stable theories. *J. of Symbolic Logic*, 48(2):377–383, 1983.

[Bue85] Steven Buechler. The geometry of weakly minimal types. *J. of Symbolic Logic*, 50(4):1044–1053, 1985.

[Bue86] Steven Buechler. Locally modular theories of finite rank. *Ann. of Pure and Appl. Logic*, 30:83–94, 1986.

[Bue87a] Steven Buechler. Classification of small weakly minimal sets I. In J.T. Baldwin, editor, *Classification Theory*, Lecture Notes in Mathematics 1292, pages 32–71. Springer-Verlag, Berlin/ Heidelberg/ New York, 1987.

[Bue87b] Steven Buechler. Isolated types in a weakly minimal set. *J. of Symbolic Logic*, 52(2), 1987.

[Bue87c] Steven Buechler. Nontrivial types of $U-$rank 1. *J. of Symbolic Logic*, 52(2):548–551, 1987.

[Bue88] Steven Buechler. The classification of small weakly minimal sets II. *J. of Symbolic Logic*, 53(2):625–635, 1988.

[Bue91] Steven Buechler. Pseudoprojective strongly minimal sets are locally projective. *J. of Symbolic Logic*, 56(4):1184–1194, 1991.

[Bue96] Steven Buechler. *Essential Stability Theory*. Springer-Verlag, Berlin/Heidelberg/New York, 1996.

[CHL85] G. Cherlin, L. Harrington, and A.H. Lachlan. $\aleph_0-$ categorical, $\aleph_0-$ stable structures. *Annals of Pure and Applied Logic*, 29:103–135, 1985.

[Fis77] E. Fisher. Abelian structures I. In *Abelian Group Theory, Lecture notes in mathematics, vol. 616*, pages 270–322. Springer-Verlag, 1977.

[HP87] E. Hrushovski and A. Pillay. Weakly normal groups. In *Logic Colloquium '85*, pages 233–244. North-Holland, Amsterdam and New York, 1987.

[Hru87] E. Hrushovski. Locally modular regular types. In J.T. Baldwin, editor, *Classification Theory*, Lecture Notes in Mathematics 1292, pages 132–164. Springer-Verlag, Berlin/ Heidelberg/ New York, 1987.

[Las88] M. C. Laskowski. Uncountable theories that are categorical in a higher power. *J. of Symbolic Logic*, 53(2):512–530, 1988.

[Mak84] M. Makkai. A survey of basic stability theory. *Israel J. of Math.*, 49(1-3):181–238, 1984.

[New90] L. Newelski. A proof of Saffe's conjecture. *Fundamenta Mathematica*, 134:143–155, 1990.

[New94] Ludomir Newelski. On U-rank 2 types. *Trans. Amer. Math. Soc.*, 344(2):553–581, 1994.

[Pil96] Anand Pillay. *Geometric Stability Theory*. Oxford Logic Guides 32. Oxford University Press, Oxford, 1996.

[Pre84] M. Prest. Rings of finite representation type and modules of finite Morley rank. *J. of Algebra*, 88:502–533, 1984.

[Pre88] M. Prest. *Model Theory and Modules*. London Math. Soc. Lecture Note Series 130. Cambridge University Press, 1988.

[PS85] A. Pillay and C. Steinhorn. A note on nonmultidimensional theories. *J. of Symbolic Logic*, 50(4):1020–1024, 1985.

[Saf82] J. Saffe. On Vaught's conjecture for superstable theories. preprint, 1982.

[She90] S. Shelah. *Classification Theory*. North-Holland, Amsterdam and New York, 1990. Revised Edition.

[Ste80] C. Steinhorn. *On logics that express "there exist uncountably many"*. PhD thesis, Univ. of Wisconsin-Madison, 1980.

[Vau61] R. Vaught. Denumerable models of complete theories. In *Infinitistic Methods*, pages 303–321. Pregamon, London, 1961.

[Zie84] M. Ziegler. Model theory of modules. *Ann. of Pure and Appl. Logic*, 26:149–213, 1984.

[Zil81] B. Zil'ber. Some problems concerning categorical theories. *Bulletin de Academie Polonaise des Sciences, Série des Sciences Mathématiques*, 29:47–49, 1981.

[Zil84] B. Zil'ber. Strongly minimal countably categorical theories,II,III. *Siberian Mathematics Journal*, 25, 1984.