AME40423 Mechanisms and Machinery
Coordinate Systems and Homogeneous Transformations

Here we will introduce the concept of homogeneous transformations and develop a geometric interpretation of them.

Homogeneous Transformations

Consider the point P in figure 1. It has coordinates in frame $\{A\}$ given by the

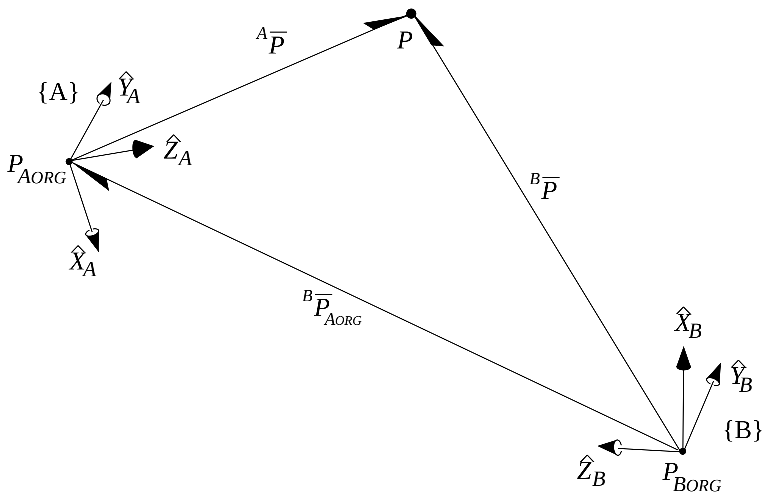

Figure 1: A Point P and its Coordinates in Frames $\{A\}$ and $\{B\}$

vector

$$ {}^A\bar{P} = \begin{bmatrix} {}^Ap_x \\ {}^Ap_y \\ {}^Ap_z \end{bmatrix} \qquad (1)$$

where the scalar components of vector ${}^A\bar{P}$ are the projections of ${}^A\bar{P}$ onto the ${}^A\hat{X}_A$, ${}^A\hat{Y}_A$ and ${}^A\hat{Z}_A$ axes respectively, i.e.

$$ {}^Ap_x = {}^A\bar{P}\cdot{}^A\hat{X}_A \ , \quad {}^Ap_y = {}^A\bar{P}\cdot{}^A\hat{Y}_A \ , \quad {}^Ap_z = {}^A\bar{P}\cdot{}^A\hat{Z}_A \quad . \qquad (2)$$

Thus ${}^A\bar{P}$ may be written as,

$$ {}^A\bar{P} = {}^Ap_x\,{}^A\hat{X}_A \ + \ {}^Ap_y\,{}^A\hat{Y}_A \ + \ {}^Ap_z\,{}^A\hat{Z}_A \ , \qquad (3)$$

and recognize that on the right hand side of eqn. (1.3), ${}^{A}p_x$, ${}^{A}p_y$ and ${}^{A}p_z$ are scalars, whereas ${}^{A}\hat{X}_A$, ${}^{A}\hat{Y}_A$ and ${}^{A}\hat{Z}_A$ are vectors (unit vectors).

Now, lets introduce a second frame {B} and suppose that the position of the origin of frame {A} relative to {B} (${}^{B}\bar{P}_{AORG}$) and orientation of frame {A} relative to {B}, (${}^{B}\hat{X}_A$, ${}^{B}\hat{Y}_A$, ${}^{B}\hat{Z}_A$) are known. What then would be the coordinates of point P in frame {B}? From figure 1 we see that ${}^{B}\bar{P}$ is the vector sum of ${}^{A}\bar{P}$ and ${}^{B}\bar{P}_{AORG}$, however, before adding these vectors, we must have the vector ${}^{A}\bar{P}$ expressed in terms of frame {B} coordinates. Therefore,

$$ {}^{B}\bar{P} = \left({}^{A}p_x \, {}^{B}\hat{X}_A \; + \; {}^{A}p_y \, {}^{B}\hat{Y}_A \; + \; {}^{A}p_z \, {}^{B}\hat{Z}_A \right) \; + \; {}^{B}\bar{P}_{AORG} \tag{4} $$

which can be written as a matrix product,

$$ {}^{B}\bar{P} = \underbrace{\begin{bmatrix} {}^{B}\hat{X}_A & {}^{B}\hat{Y}_A & {}^{B}\hat{Z}_A \end{bmatrix}}_{{}^{B}_{A}R} \underbrace{\begin{bmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \end{bmatrix}}_{{}^{A}\bar{P}} + {}^{B}\bar{P}_{AORG} \quad . $$

i.e.

$$ {}^{B}\bar{P} = \; {}^{B}_{A}R \; {}^{A}\bar{P} \; + \; {}^{B}\bar{P}_{AORG} \quad . \tag{5} $$

The (3×3) matrix, ${}^{B}_{A}R$, in eqn. (1.5) is the direction cosine matrix defining the orientation of {A} relative to {B}. It is an orthogonal matrix and its inverse is its transpose. The community at large refers to this as the "rotation matrix of {A} relative to {B}".

By using homogeneous coordinates, the matrix multiplication and vector addition in eqn. (1.5) can be brought into a single matrix multiplication.

$$ \begin{bmatrix} {}^{B}\bar{P} \\ \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{B}_{A}R & & {}^{B}\bar{P}_{AORG} \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}\bar{P} \\ \\ 1 \end{bmatrix} \tag{6} $$

In this eqn., the (4×1) column vectors are the *homogeneous coordinates* of P in frames {B} and {A} and the (4×4) matrix is the *homogeneous transformation of*

{A} *relative to* {B}, denoted as ${}^B_AT$. Recognize that the homogeneous transformation defines the position and orientation of frame {A} relative to frame {B}. Eqn. (1.6) may now be written in the compact form,

$$ {}^B\bar{P} = {}^B_AT \ {}^A\bar{P} \tag{7} $$

where it is understood that ${}^B\bar{P}$ and ${}^A\bar{P}$ are homogeneous coordinates of P, that is, they are (4×1) column vectors which consist of the cartesian coordinates of P as the first three elements and the number 1 as the fourth element. We make no notational distinction between the vector of a point's cartesian coordinates and the vector of its homogeneous coordinates. It is understood that a homogeneous transformation must multiply a vector of homogeneous coordinate.

Another frame {C} is introduced and the position and orientation of {B} relative to {C} is known. Then, in an identical fashion, the coordinates of P in {C} may be written from the coordinates of P in {B} as,

$$ {}^C\bar{P} = {}^C_BR \ {}^B\bar{P} \ + \ {}^C\bar{P}_{BORG} \tag{8} $$

and now by substitution of (1.5),

$$ {}^C\bar{P} = {}^C_BR \ {}^B_AR \ {}^A\bar{P} \ + \ {}^C_BR \ {}^B\bar{P}_{AORG} \ + \ {}^C\bar{P}_{BORG} \quad . \tag{9} $$

and one can see that when several of these coordinate transformations are concatenated, this expression grows rapidly. In terms of homogeneous transformations and homogeneous coordinates, eqn. (1.9) is compactly written as,

$$ {}^C\bar{P} \ = \ {}^C_BT \ {}^B_AT \ {}^A\bar{P} = {}^C_AT \ {}^A\bar{P} \quad , \tag{10} $$

where,

$$ {}^C_AT \ = \ {}^C_BT \ {}^B_AT \tag{11} $$

which demonstrates how homogeneous transformations are concatenated. Notice how the B subscript in the first term on the right hand side seems to have cancelled with the B superscript in the second term. Additional concatenations are possible, for example, as we will soon see,

$$ {}^{0}_{6}T = {}^{0}_{1}T \; {}^{1}_{2}T \; {}^{2}_{3}T \; {}^{3}_{4}T \; {}^{4}_{5}T \; {}^{5}_{6}T \tag{12} $$

and imagine doing that symbolically without the use of homogeneous transformations! The student should be aware that although the homogeneous transformation makes it simpler to write out concatenated transformations, in the end, it is in fact more computationally expensive, since the bottom row of the homogeneous transformation (the "dummy row") must be carried along through all multiplications and may be dropped only at the final multiplication. So, the advantage is only in notational compactness, but this advantage is significant and in the opinion of most, worth the computational cost.