REDUCIBLE HYPERPLANE SECTIONS I

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Dedicated to the memory of our friend and colleague, Michael Schneider

Abstract: In this article we begin the study of $\widehat{X}$, an n-dimensional algebraic submanifold of complex projective space $\mathbb{P}^N$, in terms of a hyperplane section $\widehat{A}$ which is not irreducible. A number of general results are given, including a Lefschetz theorem relating the cohomology of $\widehat{X}$ to the cohomology of the components of a normal crossing divisor which is ample, and a strong extension theorem for divisors which are high index Fano fibrations. As a consequence we describe $\widehat{X} \subset \mathbb{P}^N$ of dimension at least five if the intersection of $\widehat{X}$ with some hyperplane is a union of $r \geq 2$ smooth normal crossing divisors $\widehat{A}_1, \dots, \widehat{A}_r$, such that for each i, $h^1(\mathcal{O}_{\widehat{A}_i})$ equals the genus $g(\widehat{A}_i)$ of a curve section of $\widehat{A}_i$. Complete results are also given for the case of dimension four when $r = 2$.

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Introduction. Let $\widehat{X}$ be an n-dimensional algebraic submanifold of complex projective space $\mathbb{P}^N$. There are many results [5, 9] describing the structure of $\widehat{X}$ under assumptions on one of its hyperplane sections. For example, there are classifications under conditions on some basic projective invariant such as the degree or genus of a curve section, or some birational invariant such as not being of general type. Though, in these results smoothness of $\widehat{X}$ or of the hyperplane section of $\widehat{X}$ is often relaxed slightly, they all assume that the hyperplane section is irreducible. In this article we begin the study of $\widehat{X}$ in terms of a hyperplane section $\widehat{A}$ which is not irreducible. We assume that $\widehat{A}$ is a union of distinct components $\widehat{A}_1, \dots, \widehat{A}_r$ where $r \geq 2$. Since normal bundles of the $\widehat{A}_i$ do not have to be ample (some of the $\widehat{A}_i$ can have ample conormal bundles), the known theory does not apply.

Since any line bundle becomes very ample after being twisted by a high enough power of a very ample line bundle, it is clear that given any divisor $\widehat{A}_1$ on a connected projective manifold X, there is a smooth divisor $\widehat{A}_2$ on $\widehat{X}$ with good transversality properties with respect to $\widehat{A}_1$ and with $\widehat{A}_1 + \widehat{A}_2$ very ample. This process produces a divisor $\widehat{A}_2$, whose invariants are generally quite large. Looking at examples suggests strongly that if $\widehat{A}$ is a very ample divisor on a projective manifold $\widehat{X}$ of dimension at least two, and if $\widehat{A}$ decomposes into irreducible components

$$\widehat{A} = \sum_{i=1}^{r} \widehat{A}_i,$$

and if all of the $\widehat{A}_i$ are "small" measured with respect to some invariant, then $(\widehat{X}, \widehat{L})$ should be "special".

In this paper we investigate the simplest problem of this type.

> **Problem A.** Describe $\widehat{X} \subset \mathbb{P}^N$ if the intersection of $\widehat{X}$ with some hyperplane is a union of $r \geq 2$ smooth normal crossing divisors, each of which has a curve section of genus 0.

This puts numerous restraints on $\widehat{X}$, e.g., the first Betti number of $\widehat{X}$ is zero, but it is very difficult to make a complete classification of the $\widehat{X} \subset \mathbb{P}^N$ satisfying this condition when $\dim \widehat{X} = 2$. This paper grew from the realization that such a problem becomes progressively easier as the dimension of $\widehat{X}$ increases. We give a complete answer when $r = 2$ and $\dim \widehat{X} = 4$ and for arbitrary r when $\dim \widehat{X} \geq 5$. Indeed for dimensions ≥ 5 we solve the following more general problem.

> **Problem B.** Describe $\widehat{X} \subset \mathbb{P}^N$ if the intersection of $\widehat{X}$ with some hyperplane is a union of $r \geq 2$ smooth normal crossing divisors $\widehat{A}_1, \ldots, \widehat{A}_r$, such that for each i, $h^1(\mathcal{O}_{\widehat{A}_i})$ equals the genus $g(\widehat{A}_i)$ of a curve section of $\widehat{A}_i$.

In the course of solving these problems we identify some of the invariants controlling this sort of problem and develop some of the theory surrounding them.

Here is the scheme of the paper. In §1 we summarize background material and give a number of examples and preliminary results that come up in the rest of the paper.

In §2 we prove some general facts. The most striking is the very useful variant of the First Lefschetz Theorem on hyperplane sections that follows from the usual First Lefschetz Theorem combined with mixed Hodge theory.

In §3 we work out the structure of pairs $(\widehat{X}, \widehat{L})$ with $\widehat{L}$ ample and $\widehat{A}_1+\cdots+\widehat{A}_r \in |\widehat{L}|$ with $r \geq 2$, and $K_{\widehat{X}} + (n-2)\widehat{L}$ not nef and big.

In §4 we prove an extension theorem which gives strong restrictions on high index Fano fibrations as divisors on manifolds of dimension $n \geq 4$. In particular using the results of §3 we solve Problem B, and we solve Problem A when $r = 2$ and the dimension of $\widehat{X}$ equals 4.

In the three dimensional case we need more stringent hypotheses, e.g., that the hyperplane section has exactly two components. The results in this case are different than those in this paper, and require a very detailed case by case argument based on the special structure of the second reduction and detailed analysis of bundles on $\mathbb{P}^1$-bundles over curves. For these reasons we are preparing a sequel dealing only with the results special to three dimensions.

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1. Preliminary material. Certain special varieties arise naturally as the building blocks of adjunction theory. We give definitions in the smooth case. For more on these varieties and on all aspects of adjunction theory we refer to the book of Beltrametti and the third author [5].

Given an integer $k \geq 0$, by a *Fano fibration of index* k, we mean a quadruple, $(\mathcal{M}, \mathcal{L}, p, Y)$ with $\mathcal{L}$ an ample line bundle on a connected projective manifold $\mathcal{M}$, a normal projective variety Y and a surjective morphism with connected fibers $p : \mathcal{M} \to Y$ with $K_{\mathcal{M}} + k\mathcal{L} \cong p^*H$ for some ample line bundle H on Y. The general fiber F of p is a Fano manifold of index k, i.e., $-K_F \cong k\mathcal{L}_F$. With appropriate interpretation, we could let k be merely rational, but this does not give much more generality (since writing k as a quotient of relatively prime integers $k = \dfrac{u}{v}$ with $u \geq 0, v > 0$, and by changing the line bundle $\mathcal{L}$ appropriately (see [5, Lemma 1.5.6]), such a fibration becomes a u-Fano fibration with the same p and Y).

Some special cases are particularly important. By a *scroll* $(\mathcal{M}, \mathcal{L})$ over a normal variety Y we mean a pair consisting of an ample line bundle $\mathcal{L}$ on a connected projective manifold $\mathcal{M}$ and a morphism $p : \mathcal{M} \to Y$ with connected fibers such that $K_{\mathcal{M}} + (f + 1)\mathcal{L} \cong p^*H$ where H is an ample line bundle on Y and $f := \dim \mathcal{M} - \dim Y$. By a *quadric fibration* $(\mathcal{M}, \mathcal{L})$ (respectively a *Del Pezzo fibration*) over a normal variety Y we mean a pair consisting of an ample line bundle $\mathcal{L}$ on a connected projective manifold $\mathcal{M}$ and a morphism $p : \mathcal{M} \to Y$ with connected fibers such that $K_{\mathcal{M}} + f\mathcal{L} \cong p^*H$ (respectively $K_{\mathcal{M}} + (f-1)\mathcal{L} \cong p^*H$) where H is an ample line bundle on Y and $f := \dim \mathcal{M} - \dim Y$. The third author introduced these and a number of the above classes of special varieties in [16]. They are better behaved than might be expected merely from the definitions, see, [5, Chapter 12], e.g., in the case of a scroll, the general fiber of p is $(\mathbb{P}^f, \mathcal{O}_{\mathbb{P}^f}(1))$, and if $f \geq 1$ and $y := \dim Y \leq 2$, then these are $\mathbb{P}^{n-y}$-bundles over Y.

We recall basic results on nefvalue morphisms, and refer the reader to [5, (1.5)] for further discussion. Given an ample line bundle $\mathcal{L}$ on a normal irreducible and reduced projective variety V of index e with at worst canonical singularities, we define the *nefvalue* of the pair $(V, \mathcal{L})$ to be the number

$$\tau := \min\{t \in \mathbb{R} | K_V + t\mathcal{L} \text{ is nef}\}.$$

The Kawamata rationality theorem [12] and the Kawamata-Shokurov Basepoint-Free theorem [12, 14] assert the following.

Theorem 1.1 (Kawamata-Shokurov) *Let $\mathcal{L}$ be an ample line bundle on a normal irreducible and reduced projective variety V of index e with at worst canonical singularities. Assume that K_V is not nef and let τ be the nefvalue of $(V, \mathcal{L})$.*

1. *τ is rational and there is a morphism with connected fibers $\nu : V \to W$ onto a normal projective variety W such that given any positive integer such that $N\tau$ and N/e are integral, there is an ample line bundle $\mathcal{H}$ on W such that $N(K_V + \tau\mathcal{L}) = \nu^*\mathcal{H}$.*

2. Further, expressing $e\tau = u/v$ with u, v coprime positive integers,

$$u \leq e(\max_{y \in W}\{\dim \nu^{-1}(y)\} + 1).$$

We need the following result due to Beltrametti, Sommese, and Wiśniewski [5, Theorem 6.4.4] (see also [6]).

Theorem 1.2 (Beltrametti, Sommese, Wiśniewski) *Let $(\mathcal{M}, \mathcal{L}, \nu, Y)$ be a Fano fibration of index $k \geq \dfrac{n}{2} + 1$ where $n := \dim \mathcal{M}$. Assume that $n > \dim Y > 0$. Then ν is the contraction of an extremal ray and $\mathrm{Pic}(\mathcal{M}) \cong \mathrm{Pic}(Y) \oplus \mathbb{Z}\mathcal{L}$.*

The important case when $\dim Y = 0$ is due to Wiśniewski [18].

Theorem 1.3 (Wiśniewski) *Let $(\mathcal{M}, \mathcal{L}, \nu, Y)$ be a Fano manifold of index $k \geq \dfrac{n}{2} + 1$ where $n := \dim \mathcal{M}$. Either $\mathrm{Pic}(\mathcal{M}) \cong \mathbb{Z}\mathcal{L}$ or $k = \frac{n}{2} + 1$, $\mathcal{M} = \mathbb{P}^{n/2} \times \mathbb{P}^{n/2}$, and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{n/2} \times \mathbb{P}^{n/2}}(1, 1)$.*

The following lemma will be useful.

Lemma 1.4 *Let L be an ample line bundle on a connected projective manifold X. Assume that there is a divisor $A + B \in |L|$ with A and B distinct irreducible divisors. Let $\pi : X \to Y$ be a surjective morphism with connected fibers onto a projective variety Y. If $\dim X - \dim Y \geq 1$ then either $\pi(A \cap B)$ is of dimension at least dimension $\dim Y - 1$; or $\pi(A \cap B) = \pi(A)$; or $\pi(A \cap B) = \pi(B)$.*

Proof. If $\dim Y \leq 1$ there is nothing to prove. Thus we can assume without loss of generality that $\dim Y \geq 2$.

Assume that $\dim \pi(A \cap B) < \dim Y - 1$. Let H be a very ample line bundle on Y. If $\pi(A)$ and $\pi(B)$ both properly contain $\pi(A \cap B)$, we can find general $D_1, \ldots, D_k \in |H|$ such that, letting $Y_k := D_1 \cap \ldots \cap D_k$, $Y_k \cap \pi(A) \neq \emptyset$, $Y_k \cap \pi(B) \neq \emptyset$, and $Y_k \cap \pi(A \cap B) = \emptyset$. This implies that Y_k is of dimension at least one, and $X_k := \pi^{-1}(Y_k)$ is at least two dimensional. Since the D_i are general we can also assume without loss of generality that Y_k is irreducible and normal, and X_k is smooth and connected.

Let $A_k := A \cap X_k$ and $B_k := B \cap X_k$. Since both are nonempty and since $A_k + B_k$ is an ample divisor on X_k with $\dim X_k \geq 2$, we conclude that $A_k \cap B_k \neq \emptyset$. This implies that $\pi(A \cap B) \cap Y_k \neq \emptyset$. Q.E.D.

Corollary 1.5 *Let* $(\mathcal{M}, \mathcal{L}, \nu, Y)$ *be a Fano fibration of index* $k \geq \frac{n}{2} + 1$ *where* $n := \dim \mathcal{M}$*. Assume that* $n > \dim Y > 0$*. Let* $A + B \in |\mathcal{L}|$ *with* A *and* B *distinct irreducible divisors. Then* $\dim \nu(A \cap B) \geq \dim Y - 1$.

Proof. If $\dim \nu(A \cap B) < \dim Y - 1$ it follows from Lemma 1.4 that after possibly renaming, $\nu(A) = \nu(A \cap B)$ and thus that $\dim \nu(A) < \dim Y - 1$. Since ν is a contraction of an extremal ray and since A misses at least one fiber of ν we conclude that there is a divisor on Y whose inverse image is A. This contradicts the conclusion $\dim \nu(A) < \dim Y - 1$. Q.E.D.

Slightly more is true in special cases.

Corollary 1.6 *Let* $(\mathcal{M}, \mathcal{L}, \nu, Y)$ *be a Fano fibration of index* $k = n - 2$ *where* $n := \dim \mathcal{M}$*. Let* $A + B \in |\mathcal{L}|$ *with* A *and* B *distinct irreducible divisors. If* $n \geq \dim Y + 2$ *then* $\dim \nu(A \cap B) \geq \dim Y - 1$.

Proof. The result is clear if $\dim Y \leq 1$. If $\dim Y = 2$ this is a result of Beltrametti and the third author [5, Theorem 14.2.3]. If $\dim Y = 3$ this is a result of Beltrametti, the third author, and Wiśniewski, see [6, (3.2.1)] and [5, Theorem 14.1.1]. Q.E.D.

The following simple example comes up often.

Example 1.7 Let $\widehat{L}$ be an ample line bundle on a smooth n-dimensional projective manifold $\widehat{X}$. Suppose that $\mathrm{Pic}(\widehat{X}) = \mathbb{Z}$ with a generator $\mathcal{O}_{\widehat{X}}(1)$ for $\mathrm{Pic}(\widehat{X})$. If $\widehat{L} \cong \mathcal{O}_{\widehat{X}}(1)$, then every $D \in |\widehat{L}|$ is irreducible and reduced. More generally if $\widehat{L} \cong \mathcal{O}_{\widehat{X}}(s)$ and $D \in |\widehat{L}|$ decomposes $D = \sum_{i=1}^{r} D_i$ into irreducible and reduced divisors D_i, then $D_i \in |\mathcal{O}_{\widehat{X}}(s_i)|$ with $s = \sum_{i=1}^{r} s_i$.

Example 1.8 [Scrolls of arbitrary fiber and base dimensions] Let V be a rank $(n - m + 1)$ vector bundle over a smooth projective manifold Y of dimension m, and let $(X, L) = (\mathbb{P}(V), \xi_V)$. Assume that

1. there is a smooth connected divisor $Z \subset Y$ such that $V \otimes [-Z]$ has a nowhere vanishing section s;
2. V is very ample in the sense that the tautological bundle ξ_V over $\mathbb{P}(V)$ is very

ample; and

3. $K_Y + \det V$ is ample and therefore $(\mathbb{P}(V), \xi_V)$ is a scroll over Y, i.e., $K_{\mathbb{P}(V)} + (n - m + 1)\xi_V \cong \pi_V^* H$ with H ample on Y.

Letting $\widehat{s}$ denote the section of $\xi_V \otimes \pi_V^*[-Z]$ corresponding to s, $B := \widehat{s}^{-1}(0)$ is a smooth $\mathbb{P}^{n-m-1}$-bundle over Y, $A := \pi_V^{-1}(Z)$ is a smooth $\mathbb{P}^{n-m}$-bundle over Z, A meets B transversely in $\mathbb{P}(V_Z/(s_Z \otimes \mathcal{O}_Z))$, and $\xi_V = A + B$. Note that given any smooth projective manifold Y of dimension m and any integer $n \geq m+1$, we can find V and Z satisfying the above conditions. Indeed taking Z as a divisor which is very ample and such that $K_Y + (2n - 2m + 1)Z$ is ample, define $V := [Z] \oplus [2Z]^{\oplus(n-m)}$.

The following theorem shows that Example 1.8 is a common state of affairs for scrolls.

Theorem 1.9 *Let L be an ample line bundle on an n-dimensional projective manifold X. Assume that (X, L) is $(\mathbb{P}(V), \xi_V)$ for a vector bundle of rank $\geq \dim Y + 1$ over a smooth positive dimensional projective manifold Y with induced projection $\pi : X \to Y$. If there are smooth connected divisors $A_1, \ldots, A_r$ with $r \geq 2$ and with $A_1 + \cdots + A_r \in |L|$, then (after renaming if necessary):*

1. *there are smooth connected divisors $D_1, \ldots, D_{r-1}$ on Y such that $A_i = \pi^{-1}(D_i)$ for $1 \leq i \leq r-1$; and*
2. *A_r meets the generic fiber of π in a hyperplane, and if $\dim Y \leq 3$ then $A_r = \mathbb{P}(V')$ for an appropriate quotient bundle of V.*

Proof. Let F be a general fiber of π. Since L_F is a generator of $\mathrm{Pic}(F) = \mathbb{Z}$ we conclude that only one (say, after renaming, A_r) of the A_i can meet a general fiber of F of π and A_r will meet it in a hyperplane. Thus there are smooth connected divisors $D_1, \ldots, D_{r-1}$ on Y such that $A_i = \pi^{-1}(D_i)$ for $1 \leq i \leq r-1$.

Letting $f := \dim X - \dim Y$, we have $f - 1 = \dim A_r - \dim Y \geq \dim Y - 1$. $L - A_r$ is the pullback under π of a line bundle $\mathcal{P}$ on Y, i.e., $L = A_r + \pi^*\mathcal{P}$. Choose an ample line bundle $\mathcal{H}$ on Y such that $\mathcal{P} + \mathcal{H}$ and $K_Y + \det V + (f-1)(\mathcal{P} + \mathcal{H})$ are ample line bundles and $L + \pi^*(\mathcal{P} + \mathcal{H})$ is very ample. Then $(A_r, (L + \pi^*(\mathcal{P} + \mathcal{H}))_{A_r})$ is a scroll over Y. To see this note that

$$K_{A_r} + f(L + \pi^*(\mathcal{P} + \mathcal{H}))_{A_r} \quad = \quad (K_X + A_r + f(L + \pi^*(\mathcal{P} + \mathcal{H})))_{A_r}$$

$$
\begin{aligned}
&= (K_X + (f+1)L + f(\pi^*(\mathcal{P}+\mathcal{H})) - \mathcal{P})_{A_r} \\
&= \pi^*(K_Y + \det V + (f-1)(\mathcal{P}+\mathcal{H}) + \mathcal{H}).
\end{aligned}
$$

Since $\dim A_r - \dim Y \geq \dim Y - 1$ the result follows from [5, Proposition 14.1.3]. Q.E.D.

Remark 1.10 The condition on the base is merely to ensure that we had a bundle. See [5, Chapter 14.1] for what is known and conjectured about the more general case when the base is arbitrary. Indeed using the discussion following [5, Conjecture 14.1.10] it follows that the condition $\dim Y \leq 3$ can be removed.

Slightly more can be said when (X, L) is a scroll over a curve.

Theorem 1.11 *Let L be an ample line bundle on an n-dimensional projective manifold X. Assume that (X, L) is $(\mathbb{P}(V), \xi_V)$ for a vector bundle of rank ≥ 2 over a smooth curve Y with induced projection $\pi : X \to Y$. If there are $r \geq 2$ irreducible connected divisors $A_1, \ldots, A_r$ with $A_1 + \cdots + A_r \in |L|$, then (after renaming if necessary):*

1. *there are distinct points $D_1, \ldots, D_{r-1}$ on Y such that $A_i = \pi^{-1}(D_i)$ for $1 \leq i \leq r-1$; and*
2. *A_r meets the generic fiber of π in a hyperplane; if A_r has at worst isolated singularities then $A_r = \mathbb{P}(V')$ for an appropriate quotient bundle of V.*

Proof. The argument from the proof of Theorem 1.9 quickly reduces us (after renaming if necessary) to showing that if A_r is the component that meets every fiber of π, then A_r is a fiber bundle under π. This is clear if $\dim A_r = 1$, since in this case A_r meets the general fiber in a single point and is therefore a section. Since A_r has at worst isolated singularities and is a Cartier divisor of dimension ≥ 2 on a manifold, it follows that A_r is normal. Using [5, Proposition 3.2.1], it suffices to note that the fibers of π_{A_r} are of equal dimension. Q.E.D.

The results in [5, Chapter 7] contain the classification results we need from adjunction theory. Here let us summarize the main concepts from the classification.

Let $\widehat{L}$ be an ample line bundle on a connected n-dimensional projective manifold $\widehat{X}$. Assume that $n \geq 2$. Up to a short list of very special pairs $(\widehat{X}, \widehat{L})$ described in [5, Chapter 7], we have that $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef and big. If $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef and big, then there exists a pair (X, L) called the *first reduction of* $(\widehat{X}, \widehat{L})$ and a morphism $\phi : \widehat{X} \to X$ called the *first reduction map*, where:

1. X is a projective manifold and ϕ expresses $\widehat{X}$ as the blowup of X at a finite set $\mathcal{F}$;
2. $L := (\phi_* L)^{**}$ and $K_X + (n-1)L$ are ample line bundles;
3. $K_{\widehat{X}} + (n-1)\widehat{L} \cong \phi^*(K_X + (n-1)L)$ (or, equivalently, $\widehat{L} \cong \phi^* L - \phi^{-1}(\mathcal{F})$).

Furthermore if $n \geq 3$ and the first reduction of (X, L) of $(\widehat{X}, \widehat{L})$ exists, then up to a short list of very special pairs (X, L) described in [5, Chapter 7], we have that $\mathcal{K} := K_X + (n-2)L$ is nef and big. There exists a pair (X', L') called the *second reduction of* $(\widehat{X}, \widehat{L})$ and a morphism $\psi : X \to X'$ called the *second reduction map*, where:

1. X' has isolated terminal singularities and ψ expresses X as a completely explicit modification of X' with $X - \psi^{-1}(Z)$ isomorphic to $X' - Z$ for an algebraic set Z with $\dim Z \leq 1$;
2. $L' = (\phi_* L)^{**}$ is at worst 2-Cartier, $\mathcal{K}' := K_{X'} + (n-2)L'$ is an ample line bundle, and $\mathcal{K} = \psi^* \mathcal{K}' = \psi^*(K_{X'} + (n-2)L')$.

Here is an example where the fibers of the first reduction map are part of an ample divisor.

Example 1.12 [Fibers of reduction maps] Let $\widehat{L}$ be a very ample line bundle on a smooth connected projective n-fold $\widehat{X}$. Assume that there is a smooth divisor $\widehat{A} \cong \mathbb{P}^{n-1}$ on $\widehat{X}$ such that $[\widehat{A}]_{\widehat{A}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and $\widehat{L}_{\widehat{A}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. This can always be achieved by taking a taking $\widehat{X}$ to be the blowup of a smooth projective manifold $\phi : \widehat{X} \to X$ at one point $x \in X$, with $\widehat{A} := \phi^{-1}(x)$, and with $\widehat{L} := \phi^* L - \widehat{A}$ for some line bundle L which is a tensor product of at least two very ample line bundles on X (or which, more generally is 2-jet ample in the sense of [4]). Then $\widehat{L} - \widehat{A}$ is spanned and choosing a general $\widehat{B} \in |\widehat{L} - \widehat{A}|$ we have $\widehat{A} + \widehat{B} \in |\widehat{L}|$ with $\widehat{A}$ and $\widehat{B}$ smooth and transverse. The usual situation is that $\widehat{B}$ is connected. This happens if either $n \geq 3$ or $n = 2$ and $(\widehat{L} - \widehat{A})^2 > 0$.

Note that $\widehat{A}$ meets $\widehat{B}$ transversely in a smooth quadric relative to $\widehat{L}$. Thus the genus of a curve section of $\widehat{A} \cap \widehat{B}$ is 0.

Notice that the example $(\widehat{X}, \widehat{L})$ arising from this construction when we choose $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ is Del Pezzo.

2. A first Lefschetz theorem for reducible hyperplane sections. In this section we collect some consequences of the first Lefschetz hyperplane section theorem and Deligne's mixed Hodge theory [8].

Throughout this section we let D be an ample divisor on a smooth connected projective manifold X and assume that $D = \sum_{i=1}^{r} D_i$ is a sum of smooth irreducible divisors on X with all subsets of the D_i's meeting transversely. For any multi-index $I = (i_0, \ldots, i_k)$ with $1 \leq i_0 < \ldots < i_k \leq r$, we let $l(I) = k$ and $D_I = D_{i_0} \cap \ldots \cap D_{i_k}$.

In this case, the mixed Hodge structure on D has been constructed in a simple and concrete way in [10, section 4]. The weight filtration has the form

$$\{0\} \subset W_0 \subset W_1 \subset \ldots \subset W_{m-1} \subset W_m = H^m(D, \mathbb{Q}),$$

and tensored with $\mathbb{C}$ is defined by using the filtration induced on the E_∞ term of the spectral sequence of the double complex $A^{p,q} = A^p(D^{[q]})$ by the filtration $\bigoplus_{r \leq k} A^{r,s}$. Here, $D^{[q]}$ is the disjoint union of all D_I with $l(I) = q$, and $A^*(D^{[q]})$ is the usual de Rham complex.

Let $d : A^{p,q} \to A^{p+1,q}$ be the usual exterior derivative and let $\delta : A^{p,q} \to A^{p,q+1}$ be given by

$$(\delta\phi)_{i_0 \ldots i_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^{\nu+p} \phi_{i_0 \ldots \widehat{i_\nu} \ldots i_{q+1}} |_{D_{i_0} \cap \cdots \cap D_{i_{q+1}}}$$

for $\phi = (\phi_{i_0 \ldots i_q})$, with $\phi_{i_0 \ldots i_q} \in A^p(D_{i_0} \cap \cdots \cap D_{i_q})$. Then the graded module associated to the weight filtration on $H^*(D)$ is the E_∞ term of the spectral sequence of the double complex $(A^{*,*}, d, \delta)$, and the usual Hodge filtration on $A^{*,*}$ induces a Hodge structure of pure weight m on E_∞^m.

Moreover, the spectral sequence degenerates at the E_2 term, i.e.,

$$E_2 = E_3 = \ldots = E_\infty \, ,$$

and it follows that $W_{m-j}/W_{m-j-1} \cong H^j(H^{m-j}(D_I, \mathbb{Q}), \delta)$, where the right-hand side is induced by the sequence

$$\cdots \to \bigoplus_{l(I)=j-1} H^{m-j}(D_I,\mathbb{Q}) \xrightarrow{\delta} \bigoplus_{l(I)=j} H^{m-j}(D_I,\mathbb{Q}) \xrightarrow{\delta} \bigoplus_{l(I)=j+1} H^{m-j}(D_I,\mathbb{Q}) \to \cdots.$$

If D is an ample divisor of a smooth projective variety X, the image of $H^m(X,\mathbb{Q}) \to H^m(D,\mathbb{Q})$ is of pure weight m, and combining that observation with the Lefschetz theorem yields the following theorem:

Theorem 2.1 *Let D be an ample divisor on a smooth, connected projective variety X, and assume that $D = \sum_{i=1}^r D_i$ is a sum of smooth irreducible divisors with all subsets of the collection of divisors meeting transversely. Then for $j + k < \dim X$*

$$0 \to H^j(X,\mathbb{Q}) \to \bigoplus_{l(I)=0} H^j(D_I,\mathbb{Q}) \to \cdots \to \bigoplus_{l(I)=k} H^j(D_I,\mathbb{Q})$$

is exact with the convention that $H^j(D_I,\mathbb{Q}) = 0$ if $l(I) \geq r$.

In particular, the case $k = 0$, together with the Lefschetz theorem, yields the following.

Corollary 2.2 *If $j < \dim X$, the restriction map $H^j(X,\mathbb{Q}) \to \bigoplus_{i=1}^r H^j(D_i,\mathbb{Q})$ is injective. Moreover, if $j < \dim D$, the restriction map $H^j(D,\mathbb{Q}) \to \bigoplus_{i=1}^r H^j(D_i,\mathbb{Q})$ is injective.*

If D has two irreducible components, say $D = A+B$, we have the following corollary of Theorem 2.1.

Corollary 2.3 *Let D be an ample divisor on a connected projective manifold X with $\dim X = 3 + j \geq 3$, and assume that $D = A + B$, where A and B are connected submanifolds that intersect transversely. Then*

$$0 \to H^i(X,\mathbb{Q}) \to H^i(A,\mathbb{Q}) \oplus H^i(B,\mathbb{Q}) \to H^i(A\cap B,\mathbb{Q}) \to 0$$

for $i \leq j$. In particular $A \cap B$ is connected if $\dim X \geq 3$.

Proof. From Theorem 2.1 with $r = 2$ we have

$$0 \to H^i(X,\mathbb{Q}) \to H^i(A,\mathbb{Q}) \oplus H^i(B,\mathbb{Q}) \to H^i(A\cap B,\mathbb{Q}) \to 0,$$

from which the conclusion follows. Q.E.D.

Let D be an ample divisor on a connected projective manifold X with $\dim X \geq 3$, and assume that $D = A + B$, where A and B are connected submanifolds that intersect transversely. We call $A \cap B$ the *hinge variety* associated to the divisor $A + B$. As a byproduct of our investigation we will give a complete classification of such divisors when the curve genus of $(A \cap B, L_{A\cap B})$ is 0 or 1, i.e., when the genus of a curve section of $(A \cap B, L_{A\cap B})$ is 0 or 1.

3. Classification up to the second reduction. In this section we let $\widehat{L}$ denote an ample line bundle on a projective manifold $\widehat{X}$ of dimension $n \geq 2$, and we assume that $\widehat{A} = \sum_{i=1}^{r} \widehat{A}_i$ is a divisor in $|\widehat{L}|$, where each $\widehat{A}_i$ is irreducible. Unless otherwise stated we do *not* assume that the $\widehat{A}_i$'s are smooth or intersect transversely. We do not assume that the line bundle is spanned because we will in the course of studying very ample line bundles need to apply these results to ample, but not necessarily spanned line bundles.

Theorem 3.1 *Let $\widehat{L}$ denote an ample line bundle on a projective manifold $\widehat{X}$ of dimension $n \geq 2$. Assume that $\widehat{A} = \sum_{i=1}^{r} \widehat{A}_i$ is a divisor in $|\widehat{L}|$ with $r \geq 2$, and with each $\widehat{A}_i$ irreducible. If $K_{\widehat{X}} + (n-1)\widehat{L}$ is not nef then either*

1. *$(\widehat{X}, \widehat{L}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $r = 2$, and each $(\widehat{A}_i, \widehat{L}_{\widehat{A}_i})$ is a line; or*
2. *$(\widehat{X}, \widehat{L}) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$, $r = 2$, and there are points $x, y \in \mathbb{P}^1$ such that $\widehat{A} = (\{x\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{y\})$; or*
3. *$(\widehat{X}, \widehat{L})$ is a scroll over a smooth curve Y, one component of $\widehat{A}$ meets the general fiber in a hyperplane, and the other components are fibers. If the $\widehat{A}_i$ have at worst isolated singularities, then all the components are smooth, and the component that meets the general fiber in a hyperplane is a scroll over a curve relative to $\widehat{L}$, and is in particular a $\mathbb{P}^{n-2}$-bundle over Y.*

Moreover, each of these cases occurs.

Proof. By adjunction theory [5, table 7.4, p.164], any polarized manifold $(\widehat{X}, \widehat{L})$ for which $K_{\widehat{X}} + (n-1)\widehat{L}$ is not nef must be one of the following:

1. $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$; or
2. $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$; or

3. $(\mathcal{Q}, \mathcal{O}_Q(1))$ where $\mathcal{Q}$ is a quadric in $\mathbb{P}^{n+1}$; or

4. a scroll over a smooth curve.

Since $r \geq 2$ and $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, we conclude from Example 1.7 that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ is not possible. The case of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ obviously occurs with $\widehat{A}_i$ a line for $i = 1, 2$. Similarly the assertion about the quadric is clear. Theorem 1.11 gives the assertion about for a scroll over a smooth curve. A scroll over a smooth curve is a special case of Example 1.8. Q.E.D.

We now assume that $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef. The next two theorems take us to the first reduction (X, L).

Theorem 3.2 *Let $\widehat{L}$ denote an ample line bundle on a projective manifold $\widehat{X}$ of dimension $n \geq 2$. Assume that $\widehat{A} = \sum_{i=1}^{r} \widehat{A}_i$ is a divisor in $|\widehat{L}|$ with $r \geq 2$, and with each $\widehat{A}_i$ irreducible. If $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef but not big, then either*

1. *$(\widehat{X}, \widehat{L})$ is a Del Pezzo manifold, i.e., $K_{\widehat{X}} = -(n-1)\widehat{L}$: these are enumerated in Theorem 3.4; or*
2. *$\widehat{X}$ is a quadric fibration over a smooth curve; moreover, if $\dim \widehat{X} \geq 3$ each fiber is irreducible and reduced; or*
3. *$\widehat{X}$ is a scroll (and a $\mathbb{P}^{n-2}$-bundle of dimension ≥ 3 over a smooth surface Y), and exactly one component of $\widehat{A}$, say $\widehat{A}_r$ meets every fiber. If $\widehat{A}_r$ is smooth and $n \geq 4$, then $\widehat{A}_r$ is a $\mathbb{P}^{n-3}$-bundle over Y.*

Proof. From adjunction theory [5, Theorem 7.3.2, p.169] we know that $\widehat{X}$ is either a Del Pezzo manifold (i.e., one with $K_{\widehat{X}} = -(n-1)\widehat{L}$), a quadric fibration over a smooth curve, or a scroll over a normal surface.

In the case of a quadric fibration, the assertion about the fibers follows easily from the fact that $\widehat{L}_F = \mathcal{O}_{\mathcal{Q}}(1)$ for any fiber F. In the case of a scroll, the assertion about the components of $\widehat{A}$ follows from the fact that $\widehat{L}^{n-1} \cdot F = 1$, as in Proposition 3.1. The fact that $\pi : \widehat{X} \to Y$ is a $\mathbb{P}^{n-3}$-bundle when $n \geq 4$ (respectively $\widehat{A}_r$ is a $\mathbb{P}^{n-2}$-bundle when $n \geq 3$) over a smooth surface follows from the discussion after Conjecture 5.3 of [1] (see also [16, Theorem 3.3]). Q.E.D.

For the purpose of this paper the case of dimension ≥ 3 treated in Theorem 3.4 suffices. The two dimensional case is a straightforward consequence of the classification of Del Pezzo surfaces, but has many special cases. The following simple lemma is needed.

Lemma 3.3 *Let $\widehat{L}$ be an ample line bundle on a connected projective manifold $\widehat{X}$ of dimension $n \geq 3$. Assume that $\widehat{L}^n = 2$, and that $\widehat{A} = \sum_{i=1}^{r} \widehat{A}_i \in |\widehat{L}|$ is a sum of irreducible divisors $\widehat{A}_i$. If $K_{\widehat{X}} = -(n-1)\widehat{L}$ then $r = 1$.*

Proof. Assume that $r \geq 2$. By Theorem 1.3 we can assume that $n = 3$. Since

$$2 = \widehat{L}^3 = \widehat{L}^2 \cdot \widehat{A} = \sum_{i=1}^{r} \widehat{L}^2 \cdot \widehat{A}_i,$$

we see that we can assume that $r = 2$. We know that $\widehat{L}$ is spanned by [9, page 44]. From this and Goren's theorem [9, Theorem (1.1)], we conclude that $(\widehat{A}_i, \widehat{L}_{A_i}) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ for $i = 1, 2$. From this and $K_{\widehat{X}} = -(n-1)\widehat{L}$ we conclude that the normal bundle of $\widehat{A}_i$ is $\cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ for $i = 1, 2$. This is absurd since this would allow us to contract one of the $\widehat{A}_i$, say $\widehat{A}_1$, on $\widehat{X}$, which would give rise to a holomorphic map on $\widehat{A}_2$, which has a positive dimensional image and which contracts the positive dimensional set $\widehat{A}_1 \cap \widehat{A}_2 \subset \widehat{A}_2$. Q.E.D.

Theorem 3.4 *Let $\widehat{L}$ be an ample line bundle on a connected projective manifold $\widehat{X}$ of dimension $n \geq 3$. Assume that $\widehat{A} = \sum_{i=1}^{r} \widehat{A}_i \in |\widehat{L}|$ is a sum of $r \geq 2$ irreducible divisors $\widehat{A}_i$, each having at worst isolated singularities. Assume that the $\widehat{A}_i$ meet pairwise transversely. If $K_{\widehat{X}} = -(n-1)\widehat{L}$ then either*

1. *$(\widehat{X}, \widehat{L}) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(1, 1, 1))$; or*
2. *$(\widehat{X}, \widehat{L}) = (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 1))$; or*
3. *$(\widehat{X}, \widehat{L}) = (\mathbb{P}(T_{\mathbb{P}^2}), \xi_{T_{\mathbb{P}^2}})$; or*
4. *$(\widehat{X}, \widehat{L})$ has first reduction $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ with $\widehat{X}$ the blowup of $\mathbb{P}^3$ at most one point.*

Proof. For the Del Pezzo case we use Fujita's classification [9, 8.11]. We first note that $1 \leq d \leq 8$, where $d = \widehat{L}^n$ is the degree of $\widehat{L}$. If $r > 1$ then $d \neq 1$. Indeed since $\widehat{L}$ is ample, $\widehat{L}^n = \sum_{i=1}^{r} \widehat{L}^{n-1} \cdot \widehat{A}_i \geq r$.

Moreover, $d \neq 2$ by Lemma 3.3.

The Del Pezzo manifolds with $3 \leq d \leq 8$, except for the ones listed in the proposition above, have $\mathrm{Pic}(\widehat{X}) = \mathbb{Z}$ with generator $\widehat{L}$, which contradicts the assumption that $r > 1$. Q.E.D.

Remark 3.5 It is easy to see that each of the cases listed in the above propositions can occur. Scrolls over normal surfaces are special cases of example 1.8. The blowup of $\mathbb{P}^3$ at a point is a special case of example 1.12. On $\mathbb{P}(T_{\mathbb{P}^2})$ we get a divisor by considering the section of the tautological bundle induced by a vector field on $\mathbb{P}^2$ with zero set consisting of a point and a line. To get a quadric fibration we modify example 1.8 by taking $\widehat{X}$ to be a smooth section of $2\xi_V$, where V has rank $n-m+2$, and constructing the divisor $\widehat{A}+\widehat{B}$ as in that example.

It follows from Propositions 3.1 and 3.2 that, with the exceptions listed, the first reduction (X, L) exists; i.e., $\widehat{X}$ is the blowup $\phi : \widehat{X} \to X$ of a finite set $\mathcal{F}$ on a smooth projective variety X, and $L := (\phi_* \widehat{L})^{**}$ is an ample line bundle with $K_X + (n-1)L$ ample. If $E = \phi^{-1}(x)$ is an exceptional divisor over a point $x \in X$, then $E \cong \mathbb{P}^{n-1}$ with $[E]_E = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and $\widehat{L}_E = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ (see [5, 7.3]).

It follows that if E is not a component of the ample divisor $\widehat{A}$ it can meet at most one component, say $\widehat{A}_1$, and it must meet it transversely in a smooth $\mathbb{P}^{n-2}$ lying entirely in the set of regular points of $\widehat{A}_1$ and with $\mathcal{O}_{\mathbb{P}^{n-2}}(-1)$ as the normal bundle of $E \cap \widehat{A}_1$ in $\widehat{A}_1$. In this case, blowing down E does not reduce the number of components, so that we obtain a divisor $A = \sum_{i=1}^{r} A_i$ in X, where the singularities of the A_i's are no worse than those of the $\widehat{A}_i$'s in $\widehat{X}$.

If E is a component of $\widehat{A}$, say $E = \widehat{A}_1$, then we must have $[\sum_{i=2}^{r} \widehat{A}_i]_E = \mathcal{O}_{\mathbb{P}^{n-1}}(2)$, from which it follows that either

- E meets exactly one other component $\widehat{A}_i$ with $i \neq 1$ in an $(n-2)$-dimensional smooth quadric $\mathcal{Q}$ lying entirely in the set of regular points of $\widehat{A}_i$ and with normal bundle $-L_{\mathcal{Q}}$ in $\widehat{A}_i$; or
- E meets exactly two other components $\widehat{A}_i$, $\widehat{A}_j$ with $1 < i < j$. In this case each

of $E \cap \widehat{A}_i \subset \mathrm{reg}(\widehat{A}_i)$ and $E \cap \widehat{A}_j \subset \mathrm{reg}(\widehat{A}_j)$ is a $\mathbb{P}^{n-2}$, and the normal bundles of $E \cap \widehat{A}_i$ in $\widehat{A}_i$ and of $E \cap \widehat{A}_j$ in $\widehat{A}_j$ are both isomorphic to $\mathcal{O}_{\mathbb{P}^{n-2}}(-1)$; or

- E meets exactly one other component in a singular (possibly nonreduced) quadric.

Obviously, the third alternative can not occur if the $\widehat{A}_i$ are all smooth and intersect transversely.

It is also clear that if more than one component of $\widehat{A}$ is exceptional with respect to ϕ, then no two of the exceptional components can intersect (otherwise the intersection would be a divisor in $\mathbb{P}^{n-1}$ with negative normal bundle).

We sum up these observations in the following propositions.

Theorem 3.6 *Let $\widehat{L}$ be an ample line bundle on a connected projective manifold $\widehat{X}$ of dimension $n \geq 3$. Assume that $\widehat{A} = \sum_{i=1}^{r} \widehat{A}_i$ is a divisor in $|\widehat{L}|$, where $r \geq 2$, each $\widehat{A}_i$ is irreducible with at worst isolated singularities, and $\widehat{A}_i \neq \widehat{A}_j$ for $i \neq j$. Assume that $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef and big. Let $\phi : \widehat{X} \to X$ denote the first reduction. Let $A_i = \phi(\widehat{A}_i)$. If no component of $\widehat{A}$ is a fiber of ϕ, then*

1. *ϕ is a biholomorphism in a neighborhood of any point belonging to at least two of $\widehat{A}_i$. Thus if all subsets of the $\widehat{A}_i$ have transverse intersection, then all subsets of the A_i have transverse intersection.*
2. *The map $\phi_{\widehat{A}_i} : \widehat{A}_i \to A_i$ expresses $\widehat{A}_i$ as the blowup of A_i at the finite set $\mathcal{F} \cap A_i$ of regular points of A_i with $\widehat{L}_{\widehat{A}_i} = \phi^* L_{A_i} - \phi_{\widehat{A}_i}^{-1}(\mathcal{F} \cap A_i)$. Thus the singularities of $A_i = \phi(\widehat{A}_i)$ are the same as the singularities of $\widehat{A}_i$ and the curve genus of $(\widehat{A}_i, \widehat{L}_{\widehat{A}_i})$ is the same as the curve genus of (A_i, L_{A_i}).*

Theorem 3.7 *Let $\widehat{L}$ be an ample line bundle on a connected projective manifold $\widehat{X}$ of dimension $n \geq 3$. Assume that $\widehat{A} = \sum_{i=1}^{r} \widehat{A}_i$ is a divisor in $|\widehat{L}|$, with $r \geq 2$, with each $\widehat{A}_i$ having at worst isolated singularities, and with the $\widehat{A}_i$ irreducible and meeting pairwise transversely. Assume that $\mathrm{Sing}(\widehat{A}_i) \cap \mathrm{Sing}(\widehat{A}_j) = \emptyset$ for $i \neq j$. Assume that $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef and big. Let $\phi : \widehat{X} \to X$ denote the first reduction. Assume (after renaming if necessary) that $\dim \phi(\widehat{A}_i) < \dim \widehat{A}_i$ if and only if $i = s+1, \ldots, r$. Then for $i > s$, $\phi(\widehat{A}_i)$ is a point x_i, $\widehat{A}_i$ meets at least one other $\widehat{A}_j$ with $j \leq s$ and at most* *two other such $\widehat{A}_j$. If it meets just one other $\widehat{A}_j$, then x_i is a nondegenerate quadratic singularity of $\phi(\widehat{A}_j)$. Moreover, $s \geq 1$ and ϕ maps $\widehat{A}$ onto a divisor $A = \sum_{i=1}^{s} A_i$.*

We now pass to the first reduction; that is we consider an n-dimensional projective manifold X with an ample divisor $A = \sum_{i=1}^{s} A_i$ and assume that the line bundles $L = [A]$ and $K_X + (n-1)L$ are both ample. *We do not assume that the A_i's are smooth or that their intersections are transversal.* Adjunction theory gives us the following proposition.

Theorem 3.8 *Let L be an ample line bundle on a connected n-dimensional projective manifold X. Assume that $K_X + (n-1)L$ is ample. Assume that there is an $A \in |L|$ with $A = \sum_{i=1}^{s} A_i$ for distinct irreducible divisors A_i. Then $K_X + (n-2)L$ is nef, except in the following cases:*

1. *$(X, L) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.*
2. *$(X, L) = (\mathcal{Q}_3, \mathcal{O}_{\mathcal{Q}_3}(2))$, where $(\mathcal{Q}_3, \mathcal{O}_{\mathcal{Q}_3}(1))$ is a smooth quadric 3-fold in $\mathbb{P}^4$.*
3. *$(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.*
4. *X is a $\mathbb{P}^2$-bundle over a smooth curve Y, $\nu : X \to Y$ with $2K_X + 3L \cong \nu^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Y. Thus $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for a general fiber F. There are two possibilities:*
   - *(a) Except for one of the A_i which meets a general fiber in a smooth conic, the A_i are fibers; or*
   - *(b) Two of the A_i are scrolls over Y, each meets a fiber of ν in a line, and the remaining A_i are fibers of ν.*

Moreover, all of these cases can occur with $s > 1$.

Proof. The proposition is simply a restatement of [5, Theorem 7.3.4, p.171], except for the claim that each example occurs. Cases 1 through 3 are evident. An example of case 4(a) is obtained by taking $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ over $\mathbb{P}^1$ with $0 < a \leq b$ and letting $L := 2\xi + F$, where ξ is the tautological bundle and F is a fiber. An example of case 4(b) is obtained with the same X, by letting $L := (\xi + F) + (\xi + F)$. Q.E.D.

Theorem 3.9 *Let L be an ample line bundle on a connected n-dimensional projective* *manifold X. Assume that $K_X + (n-1)L$ is ample. Assume that there is an $A \in |L|$ with $A = \sum_{i=1}^{s} A_i$ for distinct irreducible divisors A_i. Assume that $K_X + (n-2)L$ is nef but not big. Let $\nu : X \to Y$ be the nefvalue morphism associated to (X, L), i.e., ν is a surjective morphism with connected fibers onto a normal projective variety Y, and $K_X + (n-2)L \cong \nu^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Y. Then one of the following must hold.*

1. *(X, L) is a Mukai variety; i.e., $K_X = -(n-2)L$. If $s > 1$ then either $\dim X \leq 5$ or $(X, L) = (\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1,1))$.*
2. *(X, L) is a Del Pezzo fibration over a smooth curve under ν.*
3. *(X, L) is a quadric fibration over a normal surface under ν.*
4. *(X, L) is a scroll over a normal 3-fold under ν.*

Proof. This proposition is a restatement of [5, Theorem 7.5.3, p.176], except for the second statement in case 1, which follows immediately from Theorem 1.3. Q.E.D.

As an application of the above results we have the following results we will need below.

Theorem 3.10 *Let $\hat{L}$ be an ample and spanned line bundle on an n-dimensional projective manifold $\widehat{X}$. Assume that $n \geq 3$ and that there are two smooth transverse divisors $\hat{A}, \hat{B}$ on $\widehat{X}$ with $\hat{A} + \hat{B} \in |\hat{L}|$. Assume that the genus $g(h)$ of a curve section of $h := \hat{A} \cap \hat{B}$ is 0. Then either:*

1. *$(\widehat{X}, \hat{L})$ is a scroll over a smooth curve; or*
2. *$(\widehat{X}, \hat{L})$ is Del Pezzo (see Theorem 3.4); or*
3. *$(\widehat{X}, \hat{L})$ is a quadric fibration over a smooth curve (there are two possibilities: the general possibility plus the special three dimensional case over a rational curve with $\hat{A}$ and $\hat{B}$ both $\mathbb{P}^1$-bundles); or*
4. *$\widehat{X}$ is a scroll over a smooth surface, and only one component, say $\hat{A}$ meets every fiber, and is a $\mathbb{P}^{n-3}$-bundle over the base surface; $\hat{B}$ is the inverse image under the scroll projection of a rational curve on the surface;*
5. *$K_{\widehat{X}} + (n-1)\hat{L}$ is nef and big and either $\hat{A}$ or $\hat{B}$ is a fiber of the first reduction*

mapping associated to $(\widehat{X}, \widehat{L})$; or

6. *$K_{\widehat{X}} + (n-1)\widehat{L}$ is nef and big with first reduction (X, L) and neither $\widehat{A}$ nor $\widehat{B}$ is a fiber of the first reduction mapping $\phi : \widehat{X} \to X$ associated to $(\widehat{X}, \widehat{L})$. Let $A := \phi(\widehat{A})$ and $B := \widehat{B}$. The following cases occur:*
   (a) *$(X, L) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$; or*
   (b) *$(X, L) = (\mathcal{Q}_3, \mathcal{O}_{\mathcal{Q}_3}(2))$, where $(\mathcal{Q}_3, \mathcal{O}_{\mathcal{Q}_3}(1))$ is a smooth quadric 3-fold in $\mathbb{P}^4$; or*
   (c) *$(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$; or*
   (d) *X is a $\mathbb{P}^2$-bundle over a smooth curve Y, $\nu : X \to Y$ with $2K_X + 3L \cong \nu^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Y. Thus $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for a general fiber F. There are two possibilities:*
      i. *After renaming if necessary B is a fiber and A meets a general fiber in a smooth conic; or*
      ii. *A and B are scrolls over $Y \cong \mathbb{P}^1$, each meets a fiber of ν in a line.*

Proof. By theorem 3.1, if $K_{\widehat{X}} + (n-1)\widehat{L}$ is not nef then $(\widehat{X}, \widehat{L})$ is a scroll over a smooth curve. We may therefore assume that $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef. Then by theorem 3.2, $K_{\widehat{X}} + (n-1)\widehat{L}$ is big except when $(\widehat{X}, \widehat{L})$ is Del Pezzo; or a quadric fibration over a smooth curve; or a scroll over a smooth surface.

Now assume that $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef and big and denote by (X, L) its first reduction. Assume that neither $\widehat{A}$ nor $\widehat{B}$ is a fiber of the first reduction map $\phi : \widehat{X} \to X$ and call $A = \phi(\widehat{A})$, $B = \phi(\widehat{B})$.

By Theorem 3.8 $K_X + (n-2)L$ is nef unless $(X, L) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$; or $(X, L) = (\mathcal{Q}_3, \mathcal{O}_{\mathcal{Q}_3}(2))$, where $(\mathcal{Q}_3, \mathcal{O}_{\mathcal{Q}_3}(1))$ is a smooth quadric 3-fold in $\mathbb{P}^4$; or $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$; or X is a $\mathbb{P}^2$-bundle over a smooth curve Y, $\nu : X \to Y$ with $2K_X + 3L \cong \nu^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Y, and $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for a general fiber F. In the last case (after possibly renaming), A meets a fiber in a smooth conic and B is a fiber; or both A and B are scrolls over $Y \cong \mathbb{P}^1$.

These cases account for the cases in part 6 of the theorem.

Hence we may now assume that $K_X + (n-2)L$ is nef. But then $\phi(h)$ is isomorphic to h, and $2g(h) - 2 = (K_X + (n-2)L) \cdot A \cdot B \cdot L^{n-3} \geq 0$, contrary to $g(h) = 0$. Q.E.D.

Theorem 3.11 *Let $\widehat{L}$ be an ample and spanned line bundle on an n-dimensional projective manifold $\widehat{X}$. Assume that $n \geq 3$ and that there are two smooth transverse divisors $\widehat{A}, \widehat{B}$ on $\widehat{X}$ with $\widehat{A} + \widehat{B} \in |\widehat{L}|$. Assume that the genus $g(h)$ of a curve section of $h := \widehat{A} \cap \widehat{B}$ is 1. Then either:*

1. *$n = 3$ and $(\widehat{X}, \widehat{L})$ is a quadric fibration over an elliptic curve with $\widehat{A}$ and $\widehat{B}$ $\mathbb{P}^1$-bundles over the elliptic curve; or*
2. *$\widehat{X}$ is a scroll over a smooth surface, and only one component, say $\widehat{A}$ meets every fiber, and when $n \geq 4$, $\widehat{A}$ is a $\mathbb{P}^{n-3}$-bundle over the base surface; $\widehat{B}$ is the inverse image under the scroll projection of an elliptic curve on the surface; or*
3. *$\widehat{X}$ is a $\mathbb{P}^2$-bundle over an elliptic curve Y, $\nu : \widehat{X} \to Y$ with $2K_{\widehat{X}} + 3\widehat{L} \cong \nu^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Y. Thus $(F, \widehat{L}_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for a general fiber F; $\widehat{A}$ and $\widehat{B}$ are $\mathbb{P}^1$-bundles; and each meets a fiber of ν in a line.*
4. *The first reduction $\phi : (\widehat{X}, \widehat{L}) \to (X, L)$ exists with L, $K_X + (n-1)L$ ample, and with $K_X + (n-2)L$ nef. Neither $\widehat{A}$ nor $\widehat{B}$ are exceptional divisors for ϕ and ϕ is a biholomorphism in a neighborhood of h. Letting $A := \phi(\widehat{A})$ and $B := \phi(\widehat{B})$ we have*
   - *(a) (X, L) is Mukai with $\dim X \leq 5$ or $(X, L) = (\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1,1))$; or*
   - *(b) (X, L) is a Del Pezzo fibration over a smooth curve under ν: after renaming, $\widehat{A}$ is a fiber and $\widehat{B}$ meets the general fiber in a Del Pezzo manifold;*
   - *(c) X is three-dimensional and (X, L) is a quadric fibration over a smooth surface under ν;*
   - *(d) X is four-dimensional and (X, L) is a scroll over a normal threefold under ν.*
   - *(e) $K_X + (n-2)L$ is nef and big and either A or B is a divisor which gets mapped to a point under the second reduction mapping $\psi : X \to X'$ associated to (X, L).*

Proof. We let $g(h) = 1$ denote the genus of a curve section of $(h, \widehat{L}_h)$. Note that by Theorem 3.1 $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef. By Theorem 3.2 either $K_{\widehat{X}} + (n-1)\widehat{L}$ is big or one of the following happens.

1. $(\widehat{X}, \widehat{L})$ is a Del Pezzo manifold, i.e., $K_{\widehat{X}} = -(n-1)\widehat{L}$;
2. $\widehat{X}$ is a quadric fibration over a smooth curve and since $\dim \widehat{X} \geq 3$ then each fiber is irreducible and reduced;
3. $\widehat{X}$ is a scroll over a smooth surface, and only one component, say $\widehat{A}$ meets every fiber; moreover, if $\dim \widehat{X} \geq 4$, then $\widehat{A}$ is a $\mathbb{P}^{n-3}$-bundle over the base surface.

Note the first cannot happen since it would yield the absurdity

$$0 = 2g(h) - 2 = (K_{\widehat{X}} + (n-2)\widehat{L}) \cdot \widehat{L}^{n-3} \cdot \widehat{A} \cdot \widehat{B} = -\widehat{L} \cdot h < 0.$$

In the second case we have either

1. $\widehat{B}$ is a fiber and $\widehat{A}$ is a divisor meeting a general fiber in a quadric. In this case the genus of a curve section of $\widehat{A} \cap \widehat{B}$ is zero; or
2. $\dim \widehat{X} = 3$ and $\widehat{A}$, $\widehat{B}$ are $\mathbb{P}^1$ bundles meeting in a section. This is possible if the base curve is elliptic, giving the first case in the theorem.

In the third case we can get an elliptic curve as a curve section of the intersection. This gives the second case of the theorem.

Thus we can assume without loss of generality that $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef and big. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Note that neither $\widehat{A}$ nor $\widehat{B}$ is an exceptional fiber of the first reduction map since a curve section of h would then be a smooth rational curve. Thus $\widehat{A}$ and $\widehat{B}$ are mapped to smooth divisors A, B in X and any exceptional fibers do not meet $\widehat{A} \cap \widehat{B}$, thus, a neighborhood of $\widehat{A} \cap \widehat{B}$ in $\widehat{X}$ is isomorphic to a neighborhood of $A \cap B$ in X.

By Theorem 3.8 $K_X + (n-2)L$ is nef unless:

1. $(X, L) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$
2. $(X, L) = (\mathcal{Q}_3, \mathcal{O}_{\mathcal{Q}_3}(2))$, where $(\mathcal{Q}_3, \mathcal{O}_{\mathcal{Q}_3}(1))$ is a smooth quadric 3-fold in $\mathbb{P}^4$
3. $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$
4. X is a $\mathbb{P}^2$-bundle over a smooth curve Y, $\nu : X \to Y$ with $2K_X + 3L \cong \nu^*\mathcal{H}$ for

an ample line bundle $\mathcal{H}$ on Y. Thus $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for a general fiber F. There are two possibilities:

(a) After renaming if necessary, A meets a fiber in a smooth conic and B is a fiber; or

(b) Both A and B are $\mathbb{P}^1$-bundles over Y.

Only the last case with Y an elliptic curve is possible, which gives the third case of the theorem.

If $K_X + (n-2)L$ is nef but not big then by Theorem 3.9 one of the following occurs. Let $\nu : X \to Y$ be the nefvalue morphism associated to (X, L), i.e., ν is a surjective morphism with connected fibers onto a normal projective variety Y, and $K_X + (n-2)L \cong \nu^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Y. Then one of the following must hold.

1. (X, L) is a Mukai variety; i.e., $K_X = -(n-2)L$. Either $\dim X \leq 5$ or $(X, L) = (\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^3}(1,1))$.
2. (X, L) is a Del Pezzo fibration over a smooth curve under ν; or
3. (X, L) is a quadric fibration over a normal surface under ν; or
4. (X, L) is a scroll over a normal 3-fold under ν.

Note that $0 = 2g(h) - 2 = (K_X + (n-2)L) \cdot L^{n-3} \cdot A \cdot B = \nu^*\mathcal{H} \cdot A \cdot B \cdot L^{n-3}$. Thus h is contained in a fiber of the map ν. Thus the first case that (X, L) is a Mukai variety can occur. The second can also with A a fiber and B meeting the general fiber in a Del Pezzo manifold. In the remaining cases we note that since $A \cap B$ must go to a point under ν we must have that either A or B also goes to a point, by Lemma 1.4. Using Corollary 1.6 we see that in the last two cases we must have that $\dim X - \dim \nu(X) = 1$. Note the base surface of the quadric fibration is smooth by a theorem of Besana [7].

Thus we can assume without loss of generality that $K_X + (n-2)L$ is nef and big.

Let $\psi : X \to X'$ be the second reduction map. By [2, Theorem 0.2.1] ψ is an isomorphism outside of a union of irreducible divisors of X, and for each such divisor D, either $\psi(D)$ is a point, or (D, L_D) is a scroll over a curve.

Observe that at most one of A and B is one of these exceptional divisors.

Now we shall show that exactly one of A and B is exceptional. To see this, assume that neither A nor B is exceptional. Then since $K_X + (n-2)L \cong \psi^*\mathcal{K}'$ for an ample line bundle $\mathcal{K}'$ on X' and

$$0 = 2g(h) - 2 = (K_X + (n-3)L + A + B) \cdot L^{n-3} \cdot A \cdot B = \psi^*\mathcal{K}' \cdot h \cdot L^{n-3},$$

h must go to a point under ψ. Choosing general $D_1, \ldots, D_{n-3} \in |L|$ we can slice A, B and consider the situation in dimension $n = 3$. Since neither A nor B is an exceptional divisor we conclude that $L = A + B$ meets each exceptional divisor in a curve containing two copies of h. This is impossible by an inspection of the list of exceptional divisors given in [2, Theorem 0.2.1].

Thus, after renaming if necessary, we may assume that B is exceptional. Since $K_X + (n-2)L \cong \psi^*\mathcal{K}'$ for some ample line bundle $\mathcal{K}'$ on X', we have

$$\begin{aligned} 2g(h) - 2 &= (K_X + (n-3)L + A + B) \cdot L^{n-3} \cdot A \cdot B \\ &= (K_X + (n-2)L) \cdot L^{n-3} \cdot A \cdot B \\ &= \psi^*\mathcal{K}' \cdot L^{n-3} \cdot A \cdot B = \psi_B^*\mathcal{K}' \cdot L_B^{n-3} \cdot A_B. \end{aligned}$$

If B goes to a point under ψ we have that $\psi_B^*\mathcal{K}'$ is trivial and thus $\psi_B^*\mathcal{K}' \cdot L_B^{n-3} \cdot A_B = 0$.

Otherwise, (B, L_B) is a scroll over a curve C under ψ. We shall show that this situation cannot occur. Choosing general $D_1, \ldots, D_{n-3} \in |L|$ we can slice A, B and reduce to the case of dimension $n = 3$. In this case we have $2g(h) - 2 = \psi_B^*\mathcal{K}' \cdot A_B$. Since $\mathcal{K}'_C$ is ample we have that $\psi_B^*\mathcal{K}'$ is numerically equivalent to a positive multiple tf of some fiber f of $\psi_B : B \to C$. Since $L \cdot f = 1$ and $B \cdot f = -1$ we have that $A \cdot f = 2$ and thus $2g(h) - 2 = \psi_B^*\mathcal{K}' \cdot A_B = tf \cdot A = 2t > 0$. Q.E.D.

4. High index Fano fibrations as components of divisors. In this section we investigate smooth, connected projective varieties that contain an ample divisor with one or more components that are k-Fano fibrations.

Our main tool in studying the structure of high dimensional projective manifolds with an ample divisor equal to a union of scrolls is the following proposition and its corollary.

Theorem 4.1 *Suppose that $\mathcal{L}$ is an ample line bundle on a normal irreducible projective variety V with at worst canonical singularities. Assume that A is an irreducible and reduced Cartier divisor on V with at worst canonical singularities. Assume that K_A is not nef, let τ denote the nefvalue of $(A, \mathcal{L}_A)$, and let $\phi : A \to Y$ be the nefvalue morphism of $(A, \mathcal{L}_A)$. Assume that $K_V + \tau\mathcal{L}$ is nef. There is a morphism $\overline{\phi} : V \to \mathcal{Y}$ with connected fibers from V onto a normal projective variety $\mathcal{Y}$ and an embedding $i : Y \to \mathcal{Y}$ such that the diagram*

$$\begin{array}{ccc} A & \subset & V \\ \phi \downarrow & & \downarrow \overline{\phi} \\ Y & \xrightarrow{i} & \mathcal{Y} \end{array}$$

commutes. If F is an irreducible positive dimensional fiber of $\overline{\phi}$ and $F \not\subset A$, then $F \cap A = \emptyset$ and $N(K_V + \tau\mathcal{L})_F \cong \mathcal{O}_F$ for positive integers N such that $N\tau$ is integral and NK_V is Cartier.

Proof. We first observe that $K_V + \tau\mathcal{L} + A$ is nef. For if C is an irreducible effective curve with $C \not\subset A$, then $(K_V + \tau\mathcal{L} + A) \cdot C = (K_V + \tau\mathcal{L}) \cdot C + A \cdot C \geq 0$, since A is effective and $K_V + \tau\mathcal{L}$ is nef. On the other hand, if $C \subset A$, then $(K_V + \tau\mathcal{L} + A) \cdot C = (K_A + \tau\mathcal{L}_A) \cdot C \geq 0$, since $(K_A + \tau\mathcal{L}_A)$ is nef.

Therefore by Kleiman's criterion [13] $t(\tau\mathcal{L} + (T-1)(K_V + \tau\mathcal{L} + A))$ is an ample line bundle for all positive integers T and all positive integers t such that $t(\tau\mathcal{L} + (T-1)(K_V + \tau\mathcal{L} + A))$ is a line bundle.

Let e denote the index of V, i.e., assume that e is the smallest positive integer such that eK_V is Cartier. By the Kawamata-Shokurov Basepoint-Free theorem we can choose a positive integer N' such that N'/e and $N'\tau$ are integral and $N'(K_V + \tau\mathcal{L})$ is spanned by global sections. Let e' denote the index of A. Choose an integer $N > 0$ which is a positive multiple of N' and such that N/e' is integral and $N(K_A + \tau\mathcal{L}_A) = \phi^*\mathcal{H}$ for a very ample line bundle $\mathcal{H}$ on Y. Note that $N(K_V + \tau\mathcal{L}) = K_V + \tau\mathcal{L} + (N-1)(K_V + \tau\mathcal{L})$. Using the sequence

$$0 \to K_V + \tau\mathcal{L} + (N-1)(K_V + \tau\mathcal{L} + A) \to N(K_V + \tau\mathcal{L} + A) \to N(K_A + \tau\mathcal{L}_A) \to 0$$

we see that $\Gamma(N(K_V + \tau\mathcal{L} + A)) \to \Gamma(N(K_A + \tau\mathcal{L}_A))$ is surjective, and thus that $N(K_V + \tau\mathcal{L} + A)$ is spanned in a neighborhood of A by global sections. Furthermore, since $N(K_V + \tau\mathcal{L})$ is spanned, the bundle $N(K_V + \tau\mathcal{L} + A) = N(K_V + \tau\mathcal{L}) + NA$ is spanned at all points of $V - A$. Thus $N(K_V + \tau\mathcal{L} + A)$ is spanned.

Replace N by a positive multiple $\widehat{N}$ such that the morphism associated to $|\widehat{N}(K_V + \tau\mathcal{L} + A)|$ has connected fibers and a normal image $\mathcal{Y}$. The map $\overline{\phi} : V \to \mathcal{Y}$ associated to $\widehat{N}(K_V + \tau\mathcal{L} + A)$ extends ϕ as asserted.

To see the last assertion about F, note that $|\widehat{N}(K_V + \tau\mathcal{L} + A)|$ maps F to a point. Thus since $\widehat{N}(K_V + \tau\mathcal{L})$ is spanned, and $A \cap F \neq F$, $\widehat{N}(K_V + \tau\mathcal{L})$ is trivial on F. Q.E.D.

Corollary 4.2 *Let $\mathcal{L}, V, A, Y, \phi, \tau$ be as in Theorem 4.1. Let B be any irreducible subvariety of V with* $\mathrm{Pic}(B) = \mathbb{Z}$. *If V is smooth and* $\dim A - \dim Y + \dim B > \dim V$ *then either $B \cap A = \emptyset$ or there exists a $y \in Y$ with $B \subset \phi^{-1}(y)$.*

Proof. Let $n := \dim V$. Suppose that $B \cap A \neq \emptyset$ and $B \not\subset A$. Then A_B is effective, and therefore, since $\mathrm{Pic}(B) = \mathbb{Z}$, it is ample. In addition, $(K_V + \tau\mathcal{L})_B$ is nef; so from Kleiman's criterion we see that $(K_V + \tau\mathcal{L} + A)_B$ is ample. Therefore the map $\overline{\phi}$ is finite to one on B. Let F' be an irreducible component of a fiber of ϕ that meets $A \cap B$. Observing that $\dim F' \geq \dim A - \dim Y$ we get

$$\dim F \cap B \geq \dim F' + \dim B - n \geq \dim A - \dim Y + \dim B - n > 0,$$

which contradicts the finite-to-oneness of $\overline{\phi}_B$. Thus $B \subset A$. The same argument shows that in fact $B \subset \phi^{-1}(y)$ for some $y \in Y$. Q.E.D.

We remark that Corollary 4.2 includes the case when $B = V$.

We now apply Corollary 4.2 to study the structure of the morphism $\psi : V \to W$ associated to $N(K_V + \tau\mathcal{L})$. First, we make the following general observation.

Lemma 4.3 *Let $\mathcal{L}$ be an ample line bundle on a projective variety V with at worst canonical singularities and index e. Suppose the bundle $N(K_V + \tau\mathcal{L})$ is spanned and non-trivial for some rational $\tau > 0$ and some positive integer N with $N\tau$ and N/e integral. Let η be the induced morphism, and assume that the general fiber of η has* *positive dimension. Then* $\dim V - \dim \eta(V) \geq \tau - 1$.

Proof. Let F be a general fiber of η. Our assumptions about the singularities of V assure that F has canonical singularities of index e and that $N(K_F + \tau\mathcal{L}_F) \cong \mathcal{O}_F$. Thus by Theorem 1.1 we have the assertion. Q.E.D.

We now come to the main result of this section.

Theorem 4.4 *Let $\mathcal{L}$ be an ample line bundle on an n-dimensional connected projective manifold, V. Let A be an irreducible divisor on V with at worst canonical singularities and with index e'. Assume that K_A is not nef and that τ is the nefvalue of $(A, \mathcal{L}_A)$ and $\phi : A \to Y$ is the nefvalue morphism of $(A, \mathcal{L}_A)$. Further assume that $K_V + \tau'\mathcal{L}$ is nef for some rational τ' satisfying $0 \leq \tau' \leq \tau$. Let $\psi : V \to W$ be the morphism (which exists by the Kawamata-Shokurov Basepoint-Free Theorem) from V onto a normal projective variety W such that $\psi^*\mathcal{H} \cong N(K_V + \tau'\mathcal{L})$ where $\mathcal{H}$ is ample and N is a positive integer such that $N\tau'$ is integral. Assume that $\tau' \geq (n+3)/2$. Then we can conclude the following.*

1. *If $\dim W < \dim V$, then A is the pullback under ψ of a divisor in W; moreover, ψ is of maximal rank in a neighborhood of a general fiber lying in A.*
2. *Assume that $\dim W = \dim V$. If the image under ϕ of the singularities of A is not all of Y, then for a general fiber f of $\phi : A \to Y$, it follows that $\mathrm{Pic}(f) = \mathbb{Z}$, $A_f < 0$, and $K_V + \tau'\mathcal{L}$ is ample on f.*

Proof. We first consider the case in which $\dim W < \dim V$. Applying Lemma 4.3 gives $\dim V - \dim W \geq \tau' - 1$. Applying the same lemma to $\phi : A \to Y$, we get $\dim A - \dim Y \geq \tau - 1 \geq \tau' - 1$.

Thus, if F is a general fiber of ψ, we have

$$\dim A - \dim Y + \dim F \geq 2(\tau' - 1) \geq n + 1.$$

Moreover, since $\tau' \geq (n+3)/2 > (\dim F + 2)/2$, Theorem 1.3 implies that $\mathrm{Pic}(F) = \mathbb{Z}$. Therefore, from Corollary 4.2 we get that either $F \cap A = \emptyset$ or $F \subset \phi^{-1}(y)$ for a $y \in Y$. The latter alternative is clearly impossible if F is a general fiber of ψ and A is a divisor of V. Therefore, A_F is trivial; in other words, A is in the kernel of the restriction map

$\mathrm{Pic}(V) \to \mathrm{Pic}(F)$.

Since $\dim W < \dim V$, the bundle $N(K_V + \tau'\mathcal{L})$ is spanned but not ample, and thus τ' is the nefvalue of $(V, \mathcal{L})$. In that case it is known from Theorem 1.2 that ψ is a Mori contraction, and, in particular, the sequence $\mathrm{Pic}(W) \to \mathrm{Pic}(V) \to \mathrm{Pic}(F)$ is exact. Therefore, the divisor A is in the image of $\mathrm{Pic}(W)$; i.e., it is the pullback under ψ of a divisor on W.

To see that ψ is of maximal rank in a neighborhood of a general fiber F lying in A, note that F is smooth since F is a general fiber of the morphism ψ_A. Since $K_F = -\tau'\mathcal{L}_F < 0$, we have $H^1(F, \mathcal{O}_F) = 0$. Letting $\mathcal{N}_{F\backslash A}$ (respectively, $\mathcal{N}_{F\backslash V}$) be the normal bundle of F in A (respectively, in V), we have the short exact sequence:

$$0 \to \mathcal{N}_{F\backslash A} \to \mathcal{N}_{F\backslash V} \to [A]_F \to 0.$$

Using the facts that $[A]_F \cong \mathcal{O}_F \cong \mathcal{N}_{F\backslash A}$, and that $H^1(F, \mathcal{O}_F) = 0$, we conclude that $\mathcal{N}_{F\backslash V} = \mathcal{N}_{F\backslash A} \oplus [A]_F = \mathcal{N}_{F\backslash A} \oplus \mathcal{O}_F$, and therefore that $\mathcal{N}_{F\backslash V}$ is also trivial. Using $H^1(F, \mathcal{O}_F) = 0$, we have that $H^1(F, \mathcal{N}_{F\backslash V}) = 0$. It follows from deformation theory that around any point $p \in F$ we have a neighborhood of the form $U \times \mathcal{V}$, where

1. $p \in U \subset F$ and $\mathcal{V}$ is transverse to F with ψ giving a biholomorphism of $\mathcal{V}$ with an open neighborhood $\psi(\mathcal{V})$ of $\psi(p)$; and
2. under the biholomorphism $\mathcal{V} \to \psi(\mathcal{V})$, $\psi_{U\times\mathcal{V}}$ is identified to the product projection $U \times \mathcal{V} \to \mathcal{V}$.

Therefore, $\psi_{U\times\mathcal{V}}$ maps $\mathcal{V}$ one-to-one onto W, and hence has maximal rank at p.

We next consider the case when $\dim W = \dim V$. Let f be a general fiber of $\phi : A \to Y$. Note that if the image under ϕ of the singularities of A is not all of Y, then a general fiber f is smooth. Applying theorem 1.3 gives $\mathrm{Pic}(f) = \mathbb{Z}$.

Since the map ϕ is induced by $N(K_A + \tau\mathcal{L}_A)$, we have

$$(K_A + \tau\mathcal{L}_A)_f = (K_V + \tau\mathcal{L})_f + A_f = 0.$$

Moreover, since $(K_V + \tau\mathcal{L})_f$ is nef, we have $A_f \leq 0$.

We claim that, in fact, $A_f < 0$; for, if $A_f = 0$, then $(K_V + \tau L)_f = 0$ as well. Therefore $\tau = \tau'$ and the map $\psi : V \to W$ contracts f. Furthermore, the same argument we gave in the previous case shows that the normal bundle $\mathcal{N}_{f\backslash A}$ of f in A is trivial, and, therefore the normal bundle $\mathcal{N}_{f\backslash V}$ in V must also be trivial because $[A]_f$ is trivial. Also, $K_f < 0$, so $H^1(f, \mathcal{N}_{f\backslash V}) = 0$. But then the fibers of ψ fill out an open set around f, and therefore $\dim W < \dim V$, contrary to hypothesis.

From $A_f < 0$ we get $(K_V + \tau\mathcal{L})_f > 0$, so that $(K_V + \tau\mathcal{L})_f$ is ample as asserted. Q.E.D.

Many nonexistence results follow from the above results. We restrict ourselves to a few illustrative results, the first of which we need below.

Theorem 4.5 *Let $\mathcal{L}$ be an ample line bundle on an n-dimensional connected projective manifold V. Assume that $K_V + (n-1)\mathcal{L}$ is nef and big and that $n \geq 4$. Let $A_1, \ldots, A_r$ be $r \geq 2$ distinct irreducible divisors whose union is connected, e.g., whose union is the reduction of an ample divisor. Assume that each $(A_i, \mathcal{L}_{A_i})$ is either a scroll over a curve, a quadric, or $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$. Then $n = 4$ and all of the $(A_i, \mathcal{L}_{A_i})$ are scrolls.*

Proof. First we assume that $n \geq 4$ and that not all of the $(A_i, \mathcal{L}_{A_i})$ are scrolls over a curve. After renaming if necessary we can assume that $(A_2, \mathcal{L}_{A_2})$ is not a scroll. Let A_1 (after possibly renaming) be an A_i distinct from A_2 such that $A_1 \cap A_2 \neq \emptyset$. Let $\phi_1 : A_1 \to Y$ be the nefvalue morphism of $(A_1, \mathcal{L}_{A_1})$. By hypothesis the image of ϕ_1 is either a point or a curve. Let $\overline{\phi_1} : V \to \mathcal{Y}$ be the extension of ϕ_1 that exists by Theorem 4.1. Since $\mathrm{Pic}(A_2) = \mathbb{Z}$, Corollary 4.2 implies that $A_2 \subset \overline{\phi_1}^{-1}(\phi_1(x))$ for some $x \in A_1 \cap A_2$. This is absurd since A_1 and A_2 are distinct irreducible divisors.

Thus we have reduced to the situation when $n \geq 5$ and all of the $(A_i, \mathcal{L}_{A_i})$ are scrolls. Let $\phi_i : A_i \to Y_i$ be the nefvalue morphism of $(A_i, \mathcal{L}_{A_i})$. After renaming if necessary we can assume that A_1 and A_2 meet. Let $x \in A_1 \cap A_2$. Let B denote the fiber $\phi_2^{-1}(\phi_2(x))$. Note that $B \cong \mathbb{P}^{n-2}$ and $\mathrm{Pic}(B) = \mathbb{Z}$. Let $\overline{\phi_1} : V \to \mathcal{Y}_1$ be the extension of ϕ_1 that exists by Theorem 4.1. Note that $B \subset \overline{\phi_1}^{-1}(\phi_1(x))$ by Corollary 4.2. From rigidity of proper maps we conclude that given any fiber F of the map ϕ_2 we have that $\overline{\phi_1}(F)$ is a point. From Theorem 4.1 we see that $K_V + (n-1)\mathcal{L}$ must be trivial when restricted to any fiber of ϕ_2 not contained in A_1. But this implies that a Zariski open set of A_2 is contained in a component of the exceptional set of the first adjunction map. This gives the absurdity that $A_2 \cong \mathbb{P}^{n-1}$. Q.E.D.

Here is the solution to Problem B of the Introduction for dimensions at least five.

Corollary 4.6 *Let $\widehat{L}$ be an ample and spanned line bundle on an n-dimensional connected projective manifold $\widehat{X}$ with $n \geq 5$. Assume that there is a normal crossing divisor $\widehat{A} = \widehat{A}_1 + \cdots + \widehat{A}_r \in |\widehat{L}|$. Assume that $r \geq 2$. If for each i the genus of a curve section of $(\widehat{A}_i, \widehat{L}_{\widehat{A}_i})$ equals $h^1(\mathcal{O}_{\widehat{A}_i})$, then $(\widehat{X}, \widehat{L})$ is a scroll over a smooth curve, one component of $\widehat{A}$ meets each fiber in a hyperplane, and the other components are fibers.*

Proof. By [9, Theorem 11.7], Theorem 3.1, and Theorem 4.5, each $(\widehat{A}_i, \widehat{L}_{\widehat{A}_i})$ is a scroll over a curve. Then since $n > 4$, Theorem 4.5 rules out the possibility that $K_{\widehat{X}} + (n-1)\widehat{L}$ could be nef and big.

Now suppose that $K_{\widehat{X}} + (n-1)\widehat{L}$ is nef but not big. By theorem 3.2, $(\widehat{X}, \widehat{L})$ is either a Del Pezzo manifold as in Theorem 3.4, a quadric fibration over a smooth curve, or $\widehat{X}$ is a scroll over a normal surface. However, the Del Pezzo manifolds exhibited in Theorem 3.4 all have $\dim \widehat{X} \leq 4$, so the first case does not occur. Likewise, the latter two cases are inconsistent with the $(\widehat{A}_i, \widehat{L}_{A_i})$'s being scrolls over curves.

Hence we conclude that $(\widehat{X}, \widehat{L})$ is not nef. So by theorem 3.1, $(\widehat{X}, \widehat{L})$ is a scroll over a smooth curve, all the components of $\widehat{A}$ but one are fibers, and the remaining one meets each fiber in a hyperplane. Q.E.D.

We need the following useful lemma.

Lemma 4.7 *Let $\widehat{L}$ be a very ample line bundle on a four-dimensional connected projective manifold $\widehat{X}$. Let A and B be two smooth divisors on $\widehat{X}$ meeting transversely in a nonempty manifold h. Assume that $(A, \widehat{L}_A)$ and $(B, \widehat{L}_B)$ are scrolls over smooth curves $\mathcal{A}$ and $\mathcal{B}$ respectively with scroll projections $a : A \to \mathcal{A}$ and $b : B \to \mathcal{B}$ respectively. If $K_{\widehat{X}} + 3\widehat{L}$ is nef and big, then (a, b) maps h isomorphically onto $\mathcal{A} \times \mathcal{B}$.*

Proof. Using the argument of Theorem 4.5 we see that if $K_{\widehat{X}} + 3\widehat{L}$ is nef and big then no fiber of a meets a fiber of b in a positive dimensional set. In particular $a(h) = \mathcal{A}$ and $b(h) = \mathcal{B}$.

First note that (a, b) is one-to-one on h. If not there are two points that are identified by a and by b. Since $\widehat{L}$ is very ample and the fibers of a and b are linear, we conclude that the line ℓ between these two points is in the fibers of a and b. Thus $\ell \subset h$. This contradicts the fact that no fiber of a can meet a fiber of b in a positive dimensional set. The same argument shows that no tangent vector of h can be mapped to zero by the differential of (a, b). Q.E.D.

Lemma 4.7 is very strong. It lets us use Theorems 3.10 and 3.11 to solve Problem A of the Introduction under the assumptions that $r = 2$ and $n = 4$.

Corollary 4.8 *Let $\widehat{L}$ be a very ample line bundle on a four-dimensional connected projective manifold $\widehat{X}$. Assume that there is a normal crossing divisor $\widehat{A} + \widehat{B} \in |\widehat{L}|$. If the genus of a curve section of $(\widehat{A}, \widehat{L}_{\widehat{A}})$ and the genus of a curve section of $(\widehat{B}, \widehat{L}_{\widehat{B}})$ both equal zero, then*

1. *$(\widehat{X}, \widehat{L})$ is a scroll over $\mathbb{P}^1$; or*
2. *$(\widehat{X}, \widehat{L}) = (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1))$; or*
3. *$(\widehat{X}, \widehat{L}) \cong (\mathbb{P}^(V), \xi_V)$ where $V \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ and (after possibly renaming) $\widehat{B}$ is equal to the inverse image of a conic in $\mathbb{P}^2$ under the scroll projection, $(\widehat{A}, \widehat{L}_A) \cong (\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1,1))$, and $\widehat{A}$ is the unique divisor $\in |\widehat{L} - \widehat{B}|$); or*
4. *$(\widehat{X}, \widehat{L})$ is Mukai, i.e., $-K_{\widehat{X}} = 2\widehat{L}$.*

Proof. By Theorem 3.1 either $K_{\widehat{X}} + 3\widehat{L}$ is nef or else $(\widehat{X}, \widehat{L})$ is a scroll over a curve, which gives the first case of the theorem.

Thus we may assume that $K_{\widehat{X}} + 3\widehat{L}$ is nef. Now using Theorem 3.2, we see that $K_{\widehat{X}} + 3\widehat{L}$ is big except when $(\widehat{X}, \widehat{L})$ is a Del Pezzo manifold, a quadric fibration over a smooth curve, or a scroll over a smooth surface. In the Del Pezzo case, we have $(\widehat{X}, \widehat{L}) = (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1))$ by Theorem 3.4.

The case of a quadric fibration over a curve is not possible. To see this let $\pi : \widehat{X} \to C$ be the quadric fibration morphism onto the smooth curve C. There is an ample line bundle H on C with $K_{\widehat{X}} + 3\widehat{L} \cong \pi^* H$. Since the general fiber of π is irreducible, we see that after renaming $\widehat{A}$ is a fiber and $\widehat{B}$ meets the general fiber in a smooth quadric. Also since C is a curve, all of the fibers of $\pi_{\widehat{B}}$ have equal dimension. Since the curve genus of $(\widehat{B}, L_{\widehat{B}})$ is 0 we have

$$-2 = (K_{\widehat{X}} + \widehat{B} + 2\widehat{L}_{\widehat{B}}) \cdot \widehat{B} \cdot \widehat{L}^2 = (\pi^* H - \widehat{A}) \cdot \widehat{B} \cdot \widehat{L}^2.$$

But $\pi^* H$ is numerically equivalent to $t\widehat{A}$ for some $t \geq 1$ and $(\widehat{A} \cap \widehat{B}) \cdot \widehat{L}^2 = 2$ so we have the contradiction that $-2 = 2(t-1) \geq 0$.

We now work out the case of a scroll $\pi : \widehat{X} \to Y$ over a surface.

First note that both $\widehat{A}$ and $\widehat{B}$ are scrolls over curves. If not then after possibly renaming, $\widehat{A}$ would be $\mathbb{P}^3$ or a three-dimensional quadric. In these cases $\mathrm{Pic}(\widehat{A}) \cong \mathbb{Z}$ so $\pi(\widehat{A})$ can only be three dimensional or a point, according to Corollary 4.2. Since $\dim Y = 2$ we conclude that $\widehat{A}$ is contained in a fiber of π. This is impossible since π is a $\mathbb{P}^2$-bundle ([16, Theorem 3.3], [5, Chapter 12]), while $\dim \widehat{A} = 3$.

Thus $\widehat{A}$ and $\widehat{B}$ are scrolls over curves. Since the genera of curve sections of $\widehat{A}$ and $\widehat{B}$ are both zero, $\widehat{A}$ and $\widehat{B}$ are both scrolls over $\mathbb{P}^1$.

After renaming if necessary we conclude by Theorem 1.9 that $\widehat{B} = \pi^{-1}(C)$ for a smooth rational curve C, and $\widehat{A}$ meets a general fiber in a linear $\mathbb{P}^1$. Since $\widehat{L}$ is very ample and since the fibers of both π and the scroll projection $q : \widehat{A} \to \mathbb{P}^1$ are linear $\mathbb{P}^2$'s, we see that a fiber of q is a section of π. Thus $Y \cong \mathbb{P}^2$, and therefore

$$[\widehat{B}] \cong \pi^* \mathcal{O}_{\mathbb{P}^2}(b) \quad \text{with} \quad b = 1 \quad \text{or} \quad b = 2.$$

Since $(\widehat{X}, \widehat{L})$ is a scroll over $\mathbb{P}^2$ we have by definition that $K_{\widehat{X}} + 3\widehat{L} \cong \pi^* \mathcal{O}_{\mathbb{P}^2}(c)$ with $c \geq 1$. We claim that $c = 1$ and $b = 2$. To see this observe that a curve section of $\widehat{A}$ is rational so we have

$$-2 = (K_{\widehat{A}} + 2\widehat{L}_{\widehat{A}}) \cdot \widehat{L}^2_{\widehat{A}} = (K_{\widehat{X}} + 3\widehat{L} - \widehat{B}) \cdot \widehat{A} \cdot \widehat{L}^2 = \pi^* \mathcal{O}_{\mathbb{P}^2}(c - b) \cdot \widehat{A} \cdot \widehat{L}^2.$$

From this we conclude that $c - b < 0$. Since $c > 0$ and $b = 1$ or 2 we see that $c = 1$ and $b = 2$.

Let $V := \pi_* \widehat{L}$. Two distinct fibers of $q : \widehat{A} \to \mathbb{P}^1$ give two disjoint sections of π therefore we obtain an exact sequence with $x, y, z \in \mathbb{Z}$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(x) \to V \to \mathcal{O}_{\mathbb{P}^2}(y) \oplus \mathcal{O}_{\mathbb{P}^2}(z) \to 0.$$

This sequence splits since the first cohomology of any line bundle on $\mathbb{P}^2$ vanishes. Hence V is a direct sum of line bundles. Since $K_{\mathbb{P}^2} + \det V \cong \mathcal{O}_{\mathbb{P}^2}(b) \cong \mathcal{O}_{\mathbb{P}^2}(1)$, we conclude that $\det V \cong \mathcal{O}_{\mathbb{P}^2}(4)$. Thus $V \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ with B equal to the inverse image of a conic in $\mathbb{P}^2$ under π and with $\widehat{A} \in |\widehat{L} - \widehat{B}|$. Since

$$\pi_*(\widehat{L} - \widehat{B}) \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2},$$

we see that $(\widehat{A}, \widehat{L}_{\widehat{A}}) \cong (\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1,1))$. From this explicit description existence follows.

Now assume that $K_{\widehat{X}} + 3\widehat{L}$ is nef and big. Then by Theorem 4.5, $\widehat{A}$ and $\widehat{B}$ are both scrolls over $\mathbb{P}^1$. Also, $h = \widehat{A} \cap \widehat{B}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ by Lemma 4.7.

A fiber of the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ induced by the inclusion of h in $\widehat{A}$ is a smooth $\mathbb{P}^1$ inside a fiber $\mathbb{P}^2$ of $\widehat{A} \to \mathbb{P}^1$, hence is of degree 1 or 2. Likewise for the map from $\widehat{B}$. Then $\widehat{L}_h$ is either $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$, $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1)$, $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)$, or $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2)$ and we have $g(h) \leq 1$.

If the genus of a curve section of h is zero then, by Theorem 3.10, either $\widehat{A}$ or $\widehat{B}$ is a fiber of the first reduction map or $(\widehat{X}, \widehat{L}) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$. But the first case would contradict $\widehat{A}$ and $\widehat{B}$'s both being scrolls. The second case $(\widehat{X}, \widehat{L}) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ is also not possible. Indeed we must have $\widehat{A}, \widehat{B} \in |\mathcal{O}_{\mathbb{P}^4}(1)|$. In this case $\widehat{A} \cong \mathbb{P}^3$, $\widehat{L}_{\widehat{A}} \cong \mathcal{O}_{\mathbb{P}^3}(2)$, and the genus of a curve section of $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ is one and not zero.

Otherwise the sectional genus is one. According to Theorem 3.11, this implies that the first reduction (X, L) of $(\widehat{X}, \widehat{L})$ exists with $K_X + 2L$ nef and the first reduction map a biholomorphism in a neighborhood of $\widehat{A} \cap \widehat{B}$. Note that $\widehat{X} = X$. To see this suppose there were an exceptional fiber E for the first reduction map. Then E meets $\widehat{A} + \widehat{B}$, so we may assume without loss of generality that it meets $\widehat{A}$. Since $\widehat{A}$ is a scroll $\pi : \widehat{A} \to C$ over a curve we see that E would have to contain at least one curve of a fiber of π. This would force $\widehat{A}$ to equal E, in contradiction to $\widehat{A}$'s being a scroll. Thus $K_{\widehat{X}} + 2\widehat{L}$ is nef.

Let $\nu : \widehat{X} \to Y$ denote the morphism with connected fibers onto a normal projective variety Y such that $K_{\widehat{X}} + 2\widehat{L} \cong \nu^* H$ for an ample line bundle H on Y. Since $0 = (K_{\widehat{X}} + 2\widehat{L}) \cdot \widehat{A} \cdot \widehat{B} \cdot \widehat{L}$, we conclude that ν maps $\widehat{A} \cap \widehat{B}$ to a point. Since $\widehat{A} \cap \widehat{B}$ contains curves in fibers of the scroll maps of $\widehat{A}$ and $\widehat{B}$ we conclude that ν maps $\widehat{A} + \widehat{B}$ to a point. Thus ν maps $\widehat{X}$ to a point. Thus $-K_{\widehat{X}} = 2\widehat{L}$. Q.E.D.

The solution of the following general problem would complement for reducible divisors the results we know for irreducible ample divisors [5, Chapter 6] and [3].

Problem C. Let $\widehat{L}$ be a very ample line bundle on a connected n-dimensional projective manifold $\widehat{X}$. Classify $(\widehat{X}, \widehat{L})$ when there exists a divisor $\widehat{A} = \widehat{A}_1 + \cdots + \widehat{A}_r \in |\widehat{L}|$ with all $\widehat{A}_i$ degenerate, e.g., with $\kappa(K_{\widehat{A}_i} + (\dim \widehat{A}_i - k)\widehat{L}_{\widehat{A}_i}) < \dim \widehat{A}_i$ if $n \geq k+1$, where k is either 1, 2, or 3.

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