Advances in Polynomial Continuation for Solving Problems in Kinematics

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Abstract

For many mechanical systems, including nearly all robotic manipulators, the set of possible configurations that the links may assume can be described by a system of polynomial equations. Thus, solving such systems is central to many problems in analyzing the motion of a mechanism or in designing a mechanism to achieve a desired motion. This paper describes techniques, based on polynomial continuation, for numerically solving such systems. Whereas in the past, these techniques were focused on finding isolated roots, we now address the treatment of systems having higher-dimensional solution sets. Special attention is given to cases of exceptional mechanisms, which have an higher degree of freedom of motion than predicted by their mobility. In fact, such mechanisms often have several disjoint assembly modes, and the degree of freedom of motion is not necessarily the same in each mode. Our algorithms identify all such assembly modes, determine their dimension and degree, and give sample points on each.

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1 Introduction

For many mechanical systems, the set of possible configurations that the links may assume can be described by a system of polynomial equations. In particular, this is true for any mechanism consisting of rigid bodies joined by any of the lower-order pairs, excepting general screw joints. Thus, prismatic, rotary, cylindrical, spherical and planar joints are all allowed. Moreover, many higher-order contact joints are also described by polynomial equations, for example, point-on-plane and line-on-plane contact. We consider both the problem of determining the possible motions of such mechanisms and the problem of finding the parameters of the mechanism such that it meets prescribed precision points. In the latter case, our methods are applicable whenever the governing equations are polynomial in the unknown parameters.

In the last decade, polynomial continuation developed into a convenient, reliable tool for solving such problems. It is a numerical process which finds all isolated roots of a polynomial system. ("Isolated" means there are no other roots in the vicinity, that is, the solution does not lie on a curve, surface, or other higher-dimensional solution set.) Starting at the roots of a suitable start system, whose roots are known, the method tracks the solution paths as the start system is continuously transformed into the target system. When the start system and the transformation procedure, called a homotopy, are chosen suitably, the endpoints of these solution paths are guaranteed to include all isolated solutions of the target system. The numerical approach of continuation seems to have been first applied to kinematics in [13, 14], as a heuristic, and later by [21] in a modern form with solid mathematical underpinnings. The modern approach, making essential use of complex numbers to avoid singularities or other degeneracies, is presented for engineers and kinematicians in [8, 26], where one may find references on the development of the method. (See also [7].) It has proven to be a powerful approach to solving kinematical problems, as exemplified by [10, 11, 23, 27]. The early determinations that the general six-revolute, single-loop spatial mechanism has 16 solutions [21] and the general Stewart-Gough platform has 40 solutions [10] helped guide the subsequent development of algebraic elimination procedures for both problems. When considering a new kinematics problem, Raghavan and Roth [11] recommend to first apply continuation to find the expected number of solutions and use that information to decide if the problem is within range of elimination techniques. Elimination algorithms tend to be faster and to have acceptable accuracy when the number of roots is moderate, but continuation tends to be faster and more accurate when the number of roots is larger. Some publicly available software for polynomial continuation is available [22, 9].

It may be that a problem does not have isolated roots, but rather has higher-dimensional solutions. In fact, in general, the solutions of a polynomial system may consist of components of several different dimensions. Simple examples can be constructed as products of polynomials, for example, if $f(x, y) = (x + y - 1)(x^2 - 1)$ and $g(x, y) = (x + y - 1)(x - y)$, then the solutions of the system $\{f = 0, g = 0\}$ consist of the line $x + y - 1 = 0$ and the two points $(x, y) = \pm(1, 1)$. It is not necessary that the polynomials be factorizable for these phenomena to arise; in fact, examples of this from kinematics will be presented herein. One, a moveable seven-bar linkage, has both a solution curve and six isolated solutions, and another, a moveable Stewart-Gough platform, has several solution curves. In both cases, the existence of the solution curves depends on the parameters of the linkage having certain special relationships; general linkages of these types have only isolated solutions.

Recently, we developed a method for solving these more general cases by polynomial continuation [16], making essential use of the method described in [15]. Improvements to the technique have been described in [17, 18]. These are all part of a program of work outlined in [20], which coined the term “Numerical Algebraic Geometry” and laid down the basic concepts therein. The algorithm whose use is described in this paper is an extension to [22], not yet publicly available due to its experimental state of development.

2 Three Example Problems

In this section, we introduce three example problems: a planar seven-bar structure, a moveable Stewart-Gough platform and a problem in spatial body guidance. The solutions to these will be discussed later, after a brief outline of our methods.

2.1 A Seven-Bar Structure

This problem tests a known result from the kinematics of planar linkages. Suppose we are given a collection of seven rigid planar pieces: one quadrilateral, two triangles, and four line segments with vertices labeled as shown at the left of Fig. 1.

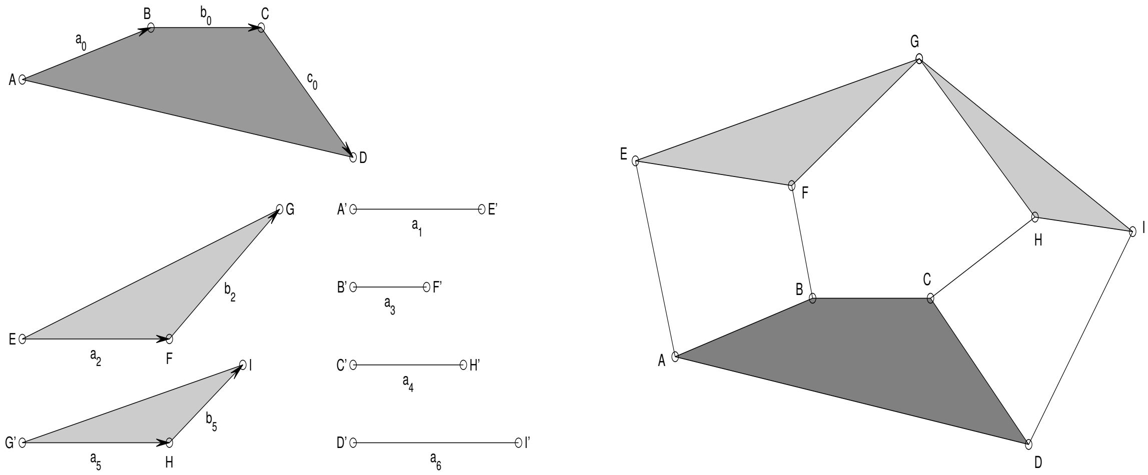

Figure 1: Find all possible assemblies of pieces at the left into a 7-bar mechanism.

We wish to assemble the pieces so as to align A with A', B with B', etc. It is not permitted to flip the pieces over, but they can be translated and rotated in any fashion within the plane. One such assembly is shown at the right of Fig. 1. The problem is to find all possible assemblies. It is simplest to hold one of the links, say the quadrilateral, in a fixed location and determine the locations of the remaining links.

Using the formulation in [25] for problems of this type, the problem can be written as a system of polynomial equations:

$$\theta_j \hat{\theta}_j = 1, \qquad j = 1, \ldots, 6 \tag{1}$$

$$\begin{aligned} -a_0 + a_1\theta_1 + a_2\theta_2 - a_3\theta_3 &= 0 \\ -b_0 + b_2\theta_2 + a_3\theta_3 - a_4\theta_4 + a_5\theta_5 &= 0 \\ -c_0 + a_4\theta_4 + b_5\theta_5 - a_6\theta_6 &= 0 \end{aligned} \tag{2}$$

$$\begin{aligned} -\bar{a}_0 + \bar{a}_1\hat{\theta}_1 + \bar{a}_2\hat{\theta}_2 - \bar{a}_3\hat{\theta}_3 &= 0 \\ -\bar{b}_0 + \bar{b}_2\hat{\theta}_2 + \bar{a}_3\hat{\theta}_3 - \bar{a}_4\hat{\theta}_4 + \bar{a}_5\hat{\theta}_5 &= 0 \\ -\bar{c}_0 + \bar{a}_4\hat{\theta}_4 + \bar{b}_5\hat{\theta}_5 - \bar{a}_6\hat{\theta}_6 &= 0 \end{aligned} \tag{3}$$

The parameters $a_0, b_0, c_0, a_1, a_2, b_2, a_3, a_4, a_5, b_5, a_6$ are complex numbers that describe the shape of the links. In Eqs.(3), $\bar{a}_i$, $\bar{b}_i$, and $\bar{c}_i$ denote the complex conjugate of a_i, b_i, and c_i. One may notice that the coefficients in Eqs.(3) are the conjugates of those in Eqs.(2). The variable θ_i represents the rotation of link i as a complex number. Solutions having $|\theta_i| = 1$ (all i) correspond to actual solutions of the geometric problem.

For generic parameters, this problem has 18 distinct solutions in complex space, see [5] for a demonstration using a different formulation. For the dimensions shown at the left of Fig. 1, eight of these are "real" solutions having $|\theta_i| = 1$.

For certain special linkages, higher-dimensional solution sets can occur. One such example can be constructed by making the two four-bar linkages $ABFEG$ and $CDIHG$ to be Roberts cognates, ([1], p. 340), so that the solution set must include a four-bar coupler curve, having degree six. Our objective will be to confirm that the known curve is discovered by the algorithm and additionally to see if any other solutions exist.

2.2 Special Stewart-Gough platforms

A generic Stewart-Gough platform consists of two rigid bodies, called the base and the endplate, joined by six legs. The legs are connected to the base and endplate by spherical joints. As a robot-manipulator, the lengths of the legs are controlled by actuators to move the endplate with six degrees of freedom, but when the leg lengths are held constant, the platform is in most cases a rigid structure. However, for certain arrangements of the joints and certain leg lengths, the structure may lose rigidity and become mobile. For a robot-manipulator, this is generally undesirable and possibly dangerous. On the other hand, the same arrangement might be useful as a mechanism, having one or more degrees of freedom.

When the six ball joints in the base and the six ball joints in the end plate are in general position, a Stewart-Gough platform has forty isolated solutions. This was first established by continuation [10] and later proven analytically [3, 24]. One formulation of the kinematic equations is as follows, where $(\mathbf{e}, \mathbf{g}) \in \mathbb{P}^7$ are Study coordinates for rigid body motion [24].

$$\begin{aligned} \mathbf{e}^T\mathbf{g} &= 0 && (4)\\ L_0\mathbf{e}^T\mathbf{e} - \mathbf{g}^T\mathbf{g} &= 0 && (5)\\ \mathbf{e}^T A_i\mathbf{e} - \mathbf{e}^T B_i\mathbf{g} &= 0, \qquad i = 1, \ldots, 5 && (6) \end{aligned}$$

Formulas for the 4×4 matrices A_i and B_i in terms of the ball joint positions and the leg lengths may be found in [24].

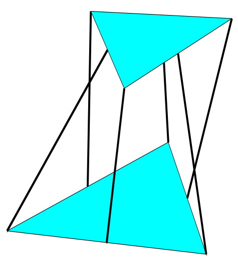

Figure 2: Griffis-Duffy Platform. Both base and endplate are equilateral triangles.

One a special kind of Stewart-Gough platform, shown in Fig.2, is called a Griffis-Duffy platform [2, 4]. In this platform, the base and endplate are triangles, with ball joints at each vertex and along each side. The legs connect a side point of one body to a vertex of the other, correspondences proceeding in order around the respective triangles. That is, labelling the joints as $P_0, \ldots, P_5$ clockwise around the base triangle and the corresponding joints of the endplate as $P'_0, \ldots, P'_5$, leg i connects P_i to P'_{i+1}, considering $P'_6 \equiv P'_0$.

We consider two special Griffis-Duffy platforms, first identified in [4] . In the first specialization, the base and endplate triangles are equilateral and the ball joints on the sides are at the midpoints. Figure 2 is of this type, with the endplate is scaled to 70% of the base. The second case, a further specialization of the first one, is to make the base and endplates congruent and to make all six leg lengths equal. For convenience, let us call these two special cases the Griffis-Duffy I and Griffis-Duffy II platforms.

Griffis-Duffy I platforms are a members of a class of Stewart-Gough platforms that are called *architecturally singular*. For general leg lengths, these platforms have no solutions: they cannot be assembled. However, if one specifies a general position of the endplate with respect to the base and sets the leg lengths to match, then the platform has one-degree-of-freedom of motion. In [4], the motion of Griffis-Duffy I and II platforms are analyzed. We treated both these cases numerically, expecting only to confirm the results in [4]. However, we found instead several surprises that we explain in the results section below.

2.3 A Spatial Body-Guidance Problem

Our final example problem concerns the synthesis of a mechanism to guide a rigid body through six spatial positions. The spatial problem may be regarded as a generalization of the classical planar Burmester problem: given five placements of a moving body in the plane, find the points of the moving body that lie on a common circle fixed in the plane. These points are called "circle points," and the centers of the fixed circles are called "center points." There are in general six centerpoint/circle-point pairs. If we specify only four placements instead of five, we get a center-point curve and a circle-point curve, each of which are cubic [1]. These curves are useful for designing a four-bar linkage to carry the body through the specified locations: a so-called body-guidance design problem.

A related problem, due to Shönflies, considers a body moving in space rather than in the plane, and asks for points of the body that lie on a common fixed sphere for several given placements of the body in space. Seven general positions determine 20 center-point/sphere-point pairs, a result proven by

Shönflies in 1886 [1]. Solutions computed by continuation were reported in [26], and a reduction to a degree-20 polynomial in one variable given in [6]. These solutions can be useful for designing a seven-bar spatial mechanism to guide a body through the specified precision points.

We consider a variation of this problem in which only six placements are given. For general positions, these will determine a center-point curve and a corresponding sphere-point curve. (Properly speaking, letting $y \in \mathbb{R}^3$ be the center point and $x \in \mathbb{R}^3$ the corresponding sphere point, there is a single center-point/sphere-point curve in (x, y). Projecting this curve onto y only gives the center-point curve in the fixed space, and projecting onto x gives the sphere-point curve in the moving body.) It is natural to wonder if the center-point/sphere-point curve is a single irreducible piece and to determine its degree.

For $i = 1, \ldots, 6$, let $p_i \in \mathbb{R}^3$ denote the position of the body and let $R_i \in \mathbb{R}^{3\times 3}$ be the rotation matrix for its orientation. Equations for the system can be written as, for $i = 1, \ldots, 5$, see [26],

$$y^T(R_i - R_0)x + y^T(p_i - p_0) - (p_i^T R_i - p_0^T R_0)x - (p_i^T p_i - p_0^T p_0)/2 = 0. \quad (7)$$

It may be seen from the bilinear structure of this system that slicing the solution curve by adding a single linear equation in (x, y) will yield at most 20 solutions, while slicing it with a linear equation in just x or y alone will yield at most 10 solutions.[1] These represent limits on the degree and bi-degree of the curve, but do not tell us the exact degrees or whether the curve is irreducible. Our methods will determine these questions.

3 Outline of the Method

Given a system of m polynomial equations in n variables, $f(x) : \mathbb{C}^n \to \mathbb{C}^m$, how can we determine the dimensions of its solution components and how can we represent those solutions numerically? First, we must define what we mean by "components." Notice that, even though in kinematics we are generally interested only in real solutions, at this point we are considering the equations over the complexes. This simplifies the problem considerably; we will worry about real components later.

[1]These follow from the two-homogeneous Bezout numbers for the equations. The coefficient of $\alpha^3\beta^3$ in $(\alpha+\beta)^6$ is 20, and the coefficient of $\alpha^3\beta^3$ in $\alpha(\alpha+\beta)^5$ is 10. See [1, p.138] for an alternative deduction of these degrees.

It is well-known from algebraic geometry, that the solution set can be broken up into *irreducible components*. These are the natural pieces into which one would divide up the set: individual points, curves, surfaces, etc. To be more precise, an irreducible algebraic set is one that cannot be expressed as the sum of a finite number of proper algebraic subsets. (An algebraic set is the solution set of some system of polynomial equations.) For example, a line is irreducible: it can be considered the union of an infinite number of points or cut up into a finite number of pieces by inequalities, but neither of these violates its irreducibility. On the other hand, the union of two lines or the union of a line and a point distinct from the line are both reducible. The decomposition of a solution set into its irreducible components is analogous to the factorization of a polynomial in one variable: there are a finite number of solution points, although some may appear with multiplicity.

One should note that irreducible components are determined in complex space. The real part of a complex curve may have several disjoint pieces. A familiar example is that a four-bar coupler curve may have two disjoint circuits, but both are part of the same complex coupler curve; that is, both circuits are given by the same sixth-degree coupler curve equation. The exceptions are when the coupler curve equation factors. For example, the coupler curve of a parallelogram linkage factors into a circle and a fourth-degree curve, and these two pieces are its irreducible components.

A crucial fact used in our algorithms is that an algebraic set S of dimension k in $\mathbb{C}^n$ meets every almost every linear subspace of dimension $n-k$ in a nonzero, finite number of points, where the number of points is equal to the degree d of the set. This is where the use of complex space is very useful, because a real algebraic curve in real 3-space might not intersect a given plane, but when extended to their complex counterparts, the two will meet, albeit possibly at complex-valued points. Let L_{n-k} denote the set of all linear subspaces of dimension $n-k$ in $\mathbb{C}^n$. Then, the members of this set that do not meet S in d isolated points are an algebraic subset of L_{n-k}. On the other hand, almost every line in L_{n-k} completely misses any given algebraic set of dimension less than k. Again, the exceptions form an algebraic subset.

In outline, our algorithm [16] computes the irreducible decomposition, using the facts just stated, as follows. Assuming that the given equations are not all identically zero, we start by looking for components of dimension $n-1$. We do so by intersecting the solution set with a randomly chosen line. With probability one, this line meets the components of dimension $n-1$

in a finite number of isolated points, which we compute using continuation. We call these points *witness points*. The line misses all lower-dimensional components entirely. We may now move the line around and follow the solution points to collect samples on the solution set. By fitting a polynomial to the sample set [16] or by connecting points by a continuation path [18], we identify which witness points belong to the same irreducible components, and in so doing discover the degree of each. This completes the discovery and decomposition of solution sets of dimension $n-1$.

We may now proceed to dimension $n-2$, this time cutting with a random plane (a member of L_2). With probability one, this will hit the $(n-2)$-dimensional components in a finite number of isolated points and miss the lower-dimensional components. It does, however, intersect the higher-dimensional components. We use a second continuation to get the isolated points, but we may also get some points on the higher-dimensional components. But this is not a problem, since we can use the information already gleaned in the previous round to detect these and cast them out. We then, as before, collect the witness points into irreducible components to complete the work at dimension $n-2$.

The algorithm moves down the dimensions sequentially, until finally, at dimension zero, we find the isolated solutions to the system.

The final output is a list of all irreducible components found at each dimension. For each of these, we have witness points. The number of witness points is equal to the degree of the irreducible component, and they all lie on a common $(n-k)$-dimensional linear subspace. Starting at a witness point, we can, by continuation, sample as many points as we like from any component.

In the foregoing description of the method, we have skipped over the issue of sets having multiplicity greater than one. These are higher-dimensional analogues of multiple roots of a polynomial in one variable. The interested reader is referred to [16] for details. Also, we should note that an efficient approach to computing witness points at every dimension is given in [15].

4 Solutions of the Examples

We now return to the sample problems described in §2 and discuss the results found by our algorithms.

4.1 A Mobile Seven-Bar Linkage

As described above, we may construct a mobile seven-bar linkage using Roberts cognates. A particular example is as follows. First, choose

$$\begin{aligned} &b_0 = 0,\ b_2 = -0.11 + 0.49i,\ a_2 = 0.46,\ a_5 = 0.41, \\ &c_0 = 1.2,\ \alpha = 0.6 + 0.8i,\ \beta = e^{1.8i}. \end{aligned} \tag{8}$$

Then, derive the remaining parameters as

$$\begin{aligned} &a_3 = a_5,\ \gamma = b_2/a_2,\ b_5 = a_5\gamma,\ a_0 = c_0/\gamma,\ a_4 = |b_2|, \\ &a_1 = |a_0 + a_3\alpha - a_4\beta/\gamma|,\ a_6 = |a_4\beta - b_5\alpha - c_0|. \end{aligned}$$

The computations for this example begin with a test for a solution curve. To obtain witness points on any motion curves that might exist, we intersect the solution set with a random plane. This means we add random linear equation to the system and use the homotopy in [15] to find all solution points. The extra equation is incorporated using a slack variable. As indicated in the first line of Table 1, this homotopy requires tracking 48 solution paths. After 8.7s, we find six witness points and 42 "non-solutions," the latter being marked by having a nonzero slack variable. The six witness points represent the sixth-degree coupler curve, shown in Fig. 3(a). The coefficients of the coupler curve equation are found by sampling and fitting, confirming in the process that no equation of lower than degree six fits the data. In this way, the algorithm confirms that the coupler curve is one irreducible component of degree six. In the second stage of the algorithm, summarized in line two of Table 1, the 42 non-solutions from the first stage are used as start points for a homotopy that will lead to the isolated roots. After 3.7s, we find that there are six potential witness points, with the other 36 solutions diverging to roots at infinity. Using the coupler curve equation from the first stage, we confirm that none of the six finite points is on the coupler curve, hence they are all true witness points. Each of these is a rigid assembly of the links, one of which is shown in Fig. 3(b).

The greatest cost in execution time on this problem was the 42s used to construct an interpolating polynomial for the coupler curve. To only confirm that the six witness points are on one irreducible component, double precision floating-point arithmetic is sufficient. However, to test the solutions at the next stage by the method of [16], we need an accurate interpolating polynomial. For this purpose, we used multi-precision arithmetic routines to

level	WitnessGenerate					WitnessClassify		
	$\#\mathbf{x}(t)$	#ns	$\#\widehat{W}$	$\#\infty$	cpu time	1	0	cpu time
1	48	42	6	0	8s 670ms	6	0	42s 180ms
0	42	—	6	36	3s 700ms	0	6	290ms
total	90	42	12	36	12s 370ms	6	6	42s 470ms

Table 1: Execution summary for 7-bar mechanism (see text for explanation).

compute sample points to 40 decimal places, which is computationally expensive. A more efficient approach [18] is used on the next example problem, but we report here the earlier approach.

The isolated solutions have a simple physical interpretation. Regard the linkage as two four-bars, $ABFEG$ and $CDIHG$, joined at a common coupler point G. If we disconnect these linkages at the coupler point, both sweep out the same coupler curve. The isolated solutions come where the coupler curve self-intersects, that is, at it's double points. Observe that in Fig. 3(b), four-bar $ABFEG$ is positioned to move horizontally along the coupler curve whereas four-bar $CDIHG$ is ready to move along the near vertical portion of the curve. Since these motions are incompatible, the assembled structure is immobile in this configuration. We may reverse the roles to get a second isolated solution at this same double point of the coupler curve. A general four-bar coupler curve has three double points, and since this class of moveable seven-bar structures will have two isolated solutions associated to each double point, there will be in general six isolated solutions. In the example worked here, only one double point is real, hence two of the six isolated solutions are real.

4.2 Griffis-Duffy Platforms

We begin the analysis of the Griffis-Duffy platforms by searching for solution curves. Starting with 128 paths (the total degree of Eqs.4), the intersection of the curves with a random plane gives forty witness points. Recall that a general Stewart-Gough platform has forty isolated solutions. We may say that because the Griffis-Duffy I and II platforms are architecturally singular, all forty solutions have moved up from isolated points to solution curves. Computation of the witness points takes about one minute.

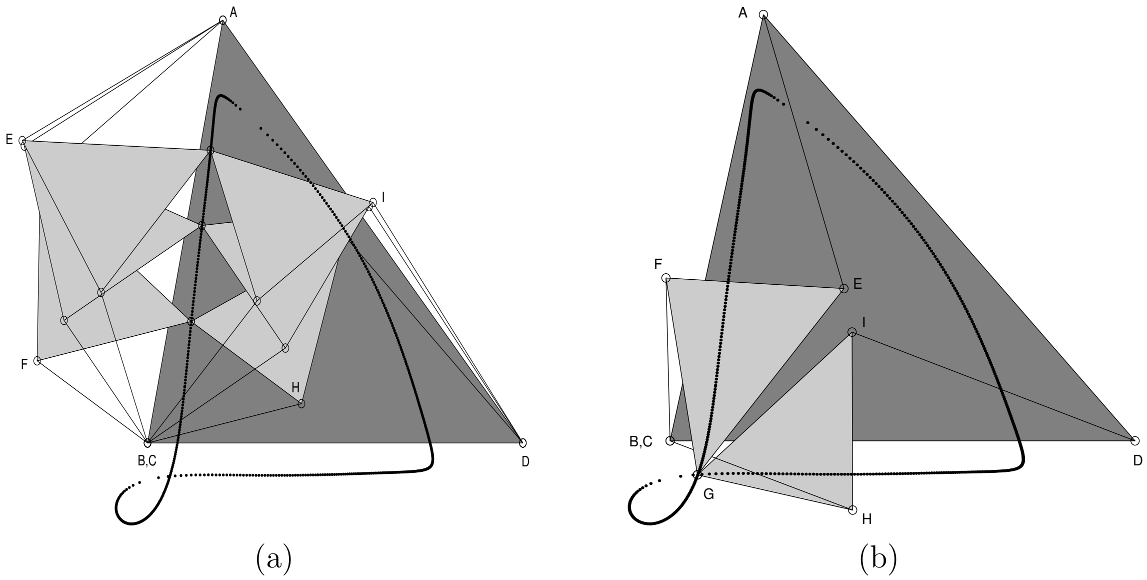

Figure 3: Assemblies of a seven-bar linkage: (a) a solution curve of degree six and (b) one of six isolated solutions.

In this example, it is highly desirable to avoid the interpolation step that was so expensive in the seven-bar example. This is because the number of monomials to compute grows exponentially as the dimension and the degree increase. Also, high degree polynomials are numerically difficult, so expensive multi-precision arithmetic is needed. Instead of interpolation, we can apply the monodromy algorithm of [18] to detect irreducible components, using linear traces to validate the groupings [19].

4.3 Griffis-Duffy I

In this case, the monodromy algorithm of [18] predicts that the 40 witness points break into 12 lines and an irreducible curve of degree 28. Monodromy predicts this in 33.4s and linear traces validate the groupings in 4.8s. (For comparison, we also ran the interpolation algorithm and found that it requires 1h 19m to compute a interpolant of degree 28.) The twelve lines all satisfy the equation $\mathbf{e}^T\mathbf{e} = 0$, which means that they do not give physically meaningful configurations. This is because $\mathbf{e}$ is a quaternion representing the rotation of the endplate, and a solution $(\mathbf{e}, \mathbf{g})$ is meaningful only if it can be rescaled to $\mathbf{e}^T\mathbf{e} = 1$. Therefore, we find that the Griffis-Duffy I platform has a single irreducible motion curve of degree 28.

On the face of it, this result seems to be at odds with the result of Husty and Karger [4], who give a degree 20 polynomial that vanishes on the motion curve. However, on closer inspection, there is no contradiction: Husty and Karger first eliminate the positional variables $\mathbf{g}$ and work only with $\mathbf{e}$. We find that the degree 28 curve in the full coordinates $(\mathbf{e}, \mathbf{g})$ drops to only degree 20 when projected onto the rotational component $\mathbf{e}$.

4.4 Griffis-Duffy II

This special case of the Griffis-Duffy I platform also has 12 lines corresponding to degenerate assemblies, but now the curve of degree 28 breaks up into lower-degree irreducible components. It takes 27.6s for the monodromy algorithm of [18] to group the 28 witness points into five sets: four of the five have cardinality six, and one set has four points. Validation of these groups by linear traces takes an additional 4.3s. (Again, purely for purposes of comparison, we also compute interpolating polynomials. This time it is much cheaper, only 2m 34s, because the degree of the components is much lower.)

In this case, the comparison of our results with those in [4] are more striking: we find five components of degree $\{6, 6, 6, 6, 4\}$ whereas Husty and Karger report four components of degree $\{6, 4, 4, 4\}$. Two of the differences are resolved similarly as in the Griffis-Duffy I case. That is, we find that two of the quartics reported in [4] are indeed fourth-degree in $\mathbf{e}$, but they are degree six in $(\mathbf{e}, \mathbf{g})$. However, Husty and Karger, working with symbolic computation guided by hand, missed one of the sextic components. This shows the value of a general algorithm and demonstrates the effectiveness of our numerical approach.

4.5 Center-point/Sphere-Point Curves

Briefly stated, slicing with a random plane, our approach finds 20 witness points on the center-point/sphere-point curve. Testing these with the monodromy algorithm, we find that they all connect, so we may conclude that the center-point/sphere-point curve is an irreducible curve of degree 20. Further testing shows that the center-point curve and the sphere-point curve are each degree 10. These results confirm that the upper bounds on these degrees predicted from the bi-linearity of Eqs.7 are exact.

5 Conclusions

From the experience of the last decade, polynomial continuation has been known to be a reliable and convenient way to find solutions to problems in kinematics. However, until recently, these methods were limited to finding isolated roots, which limited the kinds of problems that could be addressed. Of particular difficulty are overconstrained mechanisms, which have more degrees of freedom of movement than expected from the usual mobility calculation. These may have a mixture of isolated solutions (rigid assemblies) and motion curves of various dimensions. Also, it can happen that the motion at one dimension is composed of more than one irreducible piece.

We have developed software, well documented in the applied math literature, for solving these more difficult problems. This paper reports on the application of the new methods to several problems in kinematics: a overconstrained planar mechanism, Griffis-Duffy examples of moveable Stewart-Gough platform structures, and center-point/sphere-point curves for six spatial positions. In each case, our numerical methods give a complete irreducible decomposition of the solution set. For the planar seven-bar, the results are consistent with known theory: two Roberts cognates four-bars share a degree-six coupler curve and the three double points of the coupler curve each give two rigid assemblies of the seven-bar. Calculations for the center-point/sphere-point curve also agree with prior theory. However, for the Griffis-Duffy platforms, we get a bit of a surprise: we find some differences from the results published by Husty and Karger. Of these, differences in the degrees of components were not true contradictions, but are due to the use by Husty and Karger of a projection onto rotational coordinates whereas we work the problem in both position and rotation coordinates. However, for the Griffis-Duffy II platform, our algorithm finds a sixth-degree component that they missed. This shows the usefulness of the numerical approach to find new results or check results found by other means.

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