Sharp Matsusaka-type theorems on surfaces*

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Abstract

Let L be an ample line bundle on a smooth surface S. We give sharp lower bounds for tL to be k-jet ample based on the nefvalue of the pair (S, L). We also give sharp lower bounds for tL to be k-spanned.

Introduction

Let L be an ample line bundle on a smooth projective surface S.

In §2 of these notes we restate Fernández Del Busto's Matsusaka-type theorem from [8] using the notion of nefvalue of the pair (S, L). As a consequence we can use standard adjunction theory results to give sharp and stronger forms of the results in [8].

In §3 we prove a Matsusaka-type theorem for line bundles on surfaces (see Theorem (3.3)). We extend the arguments of Fernández Del Busto [8] to cover the case of k-spanned line bundles, and make some improvements that give better estimates even in the k-jet ample case studied in [8].

In the case of k-spannedness we get a numerical bound for tL to be k-spanned, t positive integer, which is linear in k, while the corresponding bound in [8] for tL to be k-jet ample is quadratic in k (see Theorem (3.3)).

The notions of k-spannedness, k-very ampleness and k-jet ampleness are k-th order embedding notions, which have been extensively studied in the last few years (see (1.2) and also §§8.5, 10.7 of the book [6] for more references on this topics).

In §1 we recall some background material we need.

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1 Background material

1.1 Notation. We work over the complex field $\mathbb{C}$. By *variety* we mean an irreducible and reduced projective scheme, V. We denote the structure sheaf by $\mathcal{O}_V$.

Basically we use the standard notation from algebraic geometry. Let us only fix the following.

$h^i(\mathcal{F})$, the complex dimension of $H^i(V, \mathcal{F})$, for a coherent sheaf $\mathcal{F}$ on V, $i \geq 0$;

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$\Gamma(L)$, the space of the global sections of a line bundle, L, on a variety V; we say that L is *spanned* if it is spanned at all points of V by $\Gamma(L)$;

$|L|$, the complete linear system associated to a line bundle L;

$\approx$ (respectively $\sim$), the linear (respectively numerical) equivalence of divisors;

$V^{[r]}$ = the Hilbert scheme of all 0-dimensional schemes $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of V with $\mathrm{length}(\mathcal{O}_{\mathcal{Z}}) = r$. Since we are working in characteristic zero, $\mathrm{length}(\mathcal{O}_{\mathcal{Z}}) = h^0(\mathcal{O}_{\mathcal{Z}})$.

For a divisor $D \in \mathrm{Pic}(V) \otimes \mathbb{Q}$ we denote

$$\{D\}, \text{ the integral part and } \lceil D \rceil = -\{-D\}, \text{ the rounding up.}$$

Line bundles and divisors are used with little (or no) distinction. We almost always use the additive notation.

1.2 k-th order embeddings. (See [3], [5], [4]) Fix a nonnegative integer k. Let L be a line bundle on a smooth projective variety X of dimension n. We say that L is *k-very ample* if the restriction map $\Gamma(L) \to \Gamma(\mathcal{O}_{\mathcal{Z}}(L))$ is onto for any 0-cycle $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \in X^{[k+1]}$.

We say that L is *k-spanned* if $\Gamma(L) \to \Gamma(\mathcal{O}_{\mathcal{Z}}(L))$ is onto for any *curvilinear* 0-cycle $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \in X^{[k+1]}$, i.e., $\mathrm{length}(\mathcal{O}_{\mathcal{Z}}) = k+1$ and $\mathcal{Z}$ is locally contained in a smooth curve. More explicitly, if $\mathrm{Supp}(\mathcal{Z}) = \{x_1, \dots, x_r\}$, $x_1, \dots, x_r$ distinct points, then $\mathcal{Z}$ is defined by the ideal sheaf $\mathcal{J}_{\mathcal{Z}}$ such that $\mathcal{J}_{\mathcal{Z}}\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{O}_{X,x}$ if $x \notin \{x_1, \dots, x_r\}$ and $\mathcal{J}_{\mathcal{Z}}\mathcal{O}_{X,x_i}$ is generated by $(u_{i1}, \dots, u_{in-1}, u_{in}^{k_i})$ at x_i, where $(u_{i1}, \dots, u_{in})$ are local coordinates at x_i on X, with $\sum_{i=1}^{r} k_i = k+1$. We also say in this case that k_i is the *local length* of $\mathcal{Z}$ at x_i, $i = 1, \dots, r$.

Let $x_1, \dots, x_r$ be r distinct points on X. Let $\mathfrak{m}_i$ be the maximal ideal sheaves of the points x_i, $i = 1, \dots, r$. Consider the 0-cycle $\mathcal{Z} = k_1 x_1 + \dots + k_r x_r$, $\sum_{i=1}^{r} k_i = k+1$, defined by the ideal sheaf $\mathcal{J}_{\mathcal{Z}}$ such that $\mathcal{J}_{\mathcal{Z}}\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{O}_{X,x}$ if $x \notin \{x_1, \dots, x_r\}$ and $\mathcal{J}_{\mathcal{Z}}\mathcal{O}_{X,x_i}$ is isomorphic to the k_i-th power of the maximal ideal $\mathfrak{m}_i\mathcal{O}_{X,x_i}$ of the local ring $\mathcal{O}_{X,x_i}$, $i = 1, \dots, r$. We say that L is *k-jet ample* if the restriction map $\Gamma(L) \to \Gamma(\mathcal{O}_{\mathcal{Z}}(L))$ is onto for any such 0-cycle $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$.

Note that for $k = 0$, the conditions L is 0-very ample, L is 0-spanned, L is 0-jet ample are all equivalent to L being spanned by $\Gamma(L)$. Furthermore for $k = 1$, the conditions L is 1-very ample, L is 1-spanned, L is 1-jet ample are all equivalent to L being very ample. Note also that k-jet ampleness implies k-very ampleness (see [5, Prop. 2.2]) and, of course, k-very ampleness implies k-spannedness (in fact, the notions of k-very ampleness and k-spannedness coincide for $k \leq 2$).

Let S be a smooth projective surface and let B be a big divisor on S, i.e., the Kodaira dimension $\kappa(B) = 2$. For $n \gg 0$, let $nB = M_n + F_n$ be the decomposition of nB in its moving part M_n and its fixed part F_n. In particular M_n is nef and big and $h^0(M_n) = h^0(nB)$. One has the following result (see [8], [11]).

Lemma 1.3 (Ein-Lazarsfeld-Nakamaye) *Notation as above. Assume that for $n \gg 0$ there is a constant $\beta > 0$ with $h^0(nB) \geq \frac{n^2}{2}\beta + \mathrm{O}(n)$. Then, for $n \gg 0$,*

$$M_n^2 \geq n^2\beta + \mathrm{O}(n).$$

Furthermore, let N be any nef and big divisor on S. Then

$$0 \leq \frac{1}{n} F_n \cdot N \leq B \cdot N - \sqrt{\beta}\sqrt{N^2}.$$

By a *real* divisor we mean an element of $\operatorname{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$. By a nef real divisor R we mean and element $R \in \operatorname{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $R \cdot C \geq 0$ for all effective curves C on S. We will also use the following consequence of the Riemann-Roch theorem (see [8, Lemma 1]).

Lemma 1.4 *Let $\mathcal{E}$ be a Cartier divisor on a smooth projective surface S. Assume that as an element of* $\operatorname{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$*, we have $\mathcal{E} = D - R$ where R is a nef real divisor and D is a positive real multiple of an ample Cartier divisor A. If $D^2 - 2D \cdot R > 0$, then for all $n \gg 0$ we have*

$$h^0(n\mathcal{E}) \geq \frac{n^2}{2}(D^2 - 2D \cdot R) + \mathrm{O}(n).$$

Proof. Note that we have $A \cdot \mathcal{E} = A \cdot D - A \cdot R > A \cdot R \geq 0$. Thus we conclude that $h^2(n\mathcal{E}) = h^0(K_S - n\mathcal{E}) = 0$ for all $n \gg 0$. Therefore from the Riemann-Roch theorem we have for all $n \gg 0$ that

$$h^0(n\mathcal{E}) \geq \mathcal{E}^2 \frac{n^2}{2} + \mathrm{O}(n).$$

Now note that $\mathcal{E}^2 = D \cdot (D - 2R) + R^2 \geq D \cdot (D - 2R)$. Q.E.D.

Definition 1.5 *Let A be an ample Cartier divisor on a smooth surface S. The* nefvalue *of the pair (S, A) is be the real number τ defined by*

$$\tau = \inf\{r \in \mathbb{Q},\ K_S + rA \text{ is nef}\}.$$

2 A sharp Matsusaka-type theorem for k-jet ampleness

The following is a simple consequence of Fernández Del Busto's theorem, but by acknowledging directly the adjunction theory [6], we can use known adjunction theory results to obtain considerably sharper lower bounds for k-jet ampleness.

Theorem 2.1 *Let A be an ample divisor on a nonsingular projective surface S. Let τ be the nefvalue of the pair (S, A), or any real value greater than the nefvalue of the pair (S, A). If*

$$t + \tau > \frac{1}{2}\left(\frac{(A \cdot (K_S + \tau A) + 1)^2}{A^2} + k^2 + 4k + 6\right)$$

then tA is k-jet ample.

Proof. This follows from [8, Theorem*, page 520]. We need only show that

$$\frac{1}{2}\left(\frac{(A \cdot (K_S + \tau A) + 1)^2}{A^2} + k^2 + 4k + 6 - 2\tau\right) > \frac{1}{2}\left(\frac{(A \cdot K_S + 1)^2}{A^2} + k^2 + 4k + 6 - K_S^2\right)$$

or equivalently that

$$(A \cdot (K_S + \tau A) + 1)^2 > (A \cdot K_S + 1)^2 + 2\tau A^2 - A^2 K_S^2.$$

Expanding and simplifying, we see that this is equivalent to $A^2(K_S + \tau A)^2 \geq 0$, which is the case since A is ample and $K_S + \tau A$ is nef. Q.E.D.

Corollary 2.2 *Let A be an ample divisor on a nonsingular projective surface S. Let τ be the nefvalue of the pair (S, A), or any real value greater than the nefvalue of the pair (S, A). If*

$$t + \tau > \frac{1}{2}\left(\frac{(A \cdot (K_S + \tau A) + 1)^2}{A^2} + 11\right)$$

then tA is very ample.

Adjunction theory gives much information on pairs in terms of their nefvalues. Using the classification of Lanteri and Palleschi [10] (see also [12, Corollary 3.4.2]), we see that either $\tau \leq 2$ or (S, A) is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, in which case the estimate is clear.

Corollary 2.3 *Let A be an ample divisor on a nonsingular projective surface. If*

$$t > \frac{1}{2}\left(\frac{(A \cdot (K_S + 2A) + 1)^2}{A^2} + k^2 + 4k + 2\right)$$

then tA is k-jet ample.

Corollary 2.4 *Let A be an ample divisor on a nonsingular projective surface. If*

$$t > \frac{1}{2}\left(\frac{(A \cdot (K_S + 2A) + 1)^2}{A^2} + 7\right)$$

then tA is very ample.

Using again the classification of Lanteri and Palleschi [10] (see also [12, Corollary 3.4.2]), we see that either $\tau \leq 1$ or S is $\mathbb{P}^2$, or a smooth quadric, in which case the estimate is clear, or (S, A) is a scroll.

Corollary 2.5 *Let A be an ample divisor on a nonsingular projective surface, and assume that (S, A) is not a scroll (neither $\mathbb{P}^2$ nor a quadric). If*

$$t > \frac{1}{2}\left(\frac{(A \cdot (K_S + A) + 1)^2}{A^2} + k^2 + 4k + 4\right)$$

then tA is k-jet ample.

Corollary 2.6 *Let A be an ample divisor on a nonsingular projective surface, and assume that (S, A) is not a scroll. If*

$$t > \frac{1}{2}\left(\frac{(A \cdot (K_S + A) + 1)^2}{A^2} + 9\right)$$

then tA is very ample.

Remark 2.7 The above result is sharp, i.e, the assumption that (S, A) is not a scroll is essential. Let $S = \mathbb{P}(\mathcal{E} \oplus \mathcal{L})$ where $\mathcal{E}$ is a degree 1 line bundle on a smooth genus g curve C and $\mathcal{L}$ is a degree δ line bundle on C. Let A denote the tautological line bundle on S. If the corollary was true for scrolls then by restricting to the section corresponding to the quotient morphism $\mathcal{E} \oplus \mathcal{L} \to \mathcal{E} \to 0$ we would have that g equals the sectional genus $g(A) = \dfrac{(K_S + A) \cdot A}{2} + 1$ and hence that $t\mathcal{E}$ is very ample for

$$t > \frac{1}{2}\left(\frac{(2g-1)^2}{2\delta} + 9\right).$$

By choosing $\delta \gg 0$ we would obtain the absurdity that given any degree 1 line bundle $\mathcal{E}$ on an arbitrary curve C, we would have $5\mathcal{E}$ is very ample.

Remark 2.8 It is a theorem of the authors [5] that if a line bundle A is 1-jet ample, then kA is k-jet ample. Asymptotically this implies that if A is ample, then there is a $t_0 > 0$ such that kt_0A is k-jet ample. It is therefore natural to ask the following question.

Question 2.9 *Let $t(k)$ be the minimum k such that $t(k)A$ is k-jet ample. Then compute the lim inf and the lim sup of $\dfrac{t(k)}{k}$ as $k \to \infty$.*

3 A Matsusaka-type theorem for k-spannedness

In what follows $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ will denote a curvilinear 0-cycle of length $k+1$ on a smooth projective surface S, and L an ample line bundle on S.

3.1 Blowing up of curvilinear 0-cycles. Here we only carry out from [2, §2] the case when $\mathrm{Supp}(\mathcal{Z})$ is a single point z. The discussion in the case $\mathrm{Supp}(\mathcal{Z}) = \{z_1, \ldots, z_r\}$, $1 \leq r \leq k+1$, is a straightforward modification of it.

Let $\pi_\alpha : S_\alpha \to S_{\alpha-1}$ be the blowing up of $S_{\alpha-1}$ at a point $p_{\alpha-1}$, $E_\alpha = \pi_\alpha^{-1}(p_{\alpha-1})$ the exceptional divisors, $\alpha = 1, \ldots, k+1$, $\overline{E}_\alpha$ the proper transforms of E_α under $\pi_{k+1} \circ \cdots \circ \pi_{\alpha+1}$, $\alpha = 1, \ldots, k$, and $\overline{E}_{k+1} = E_{k+1}$. We have

$$\overline{E}_\alpha \cdot \overline{E}_\beta = 1 \text{ if } |\alpha - \beta| = 1; \quad \overline{E}_\alpha \cdot \overline{E}_\beta = 0 \text{ if } |\alpha - \beta| > 1;$$

and

$$\overline{E}_\alpha \cdot \overline{E}_\alpha = -2 \text{ if } \alpha \neq k+1; \quad \overline{E}_{k+1} \cdot \overline{E}_{k+1} = -1.$$

Let $\overline{S} = S_{k+1}$, $S_0 = S$, $p_0 = z$ and denote by $\pi : \overline{S} \to S$ the composition of the π_α's. Let $\mathcal{J}_{\mathcal{Z}}$ be the ideal sheaf of $\mathcal{Z}$ in S. By a suitable choice of the points p_α's it is easy to check that

$$\pi^*\mathcal{J}_{\mathcal{Z}} \approx -\overline{E}_1 - 2\overline{E}_2 - \cdots - (k+1)\overline{E}_{k+1},$$

and

$$K_{\overline{S}} \approx \pi^*K_S + \overline{E}_1 + 2\overline{E}_2 + \cdots + (k+1)\overline{E}_{k+1}.$$

In the general case, let k_i be the length of $\mathcal{Z}$ at x_i, $i = 1, \ldots, t$. That is $\mathcal{J}_{\mathcal{Z}}\mathcal{O}_{S,x_i}$ is generated by $(u_i, v_i^{k_i})$, where u_i, v_i are local coordinates at x_i on S, $i = 1, \ldots, t$. Then the exceptional divisor of the blowing up along $\mathcal{Z}$ is $\overline{E} = \sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} \alpha\overline{E}_\alpha^{(i)}$, where the $\overline{E}_\alpha^{(i)}$'s are the proper transforms of the exceptional divisors $E_\alpha^{(i)}$'s constructed as above and corresponding to each point z_i, $i = 1, \ldots, r$.

We need the following improvement of [8, Lemma 3]. We will use the notation $\widetilde{\beta(t)}$ to keep the analogy with [8] clear.

Lemma 3.2 *Let t, d, d_1, σ be integers and let x be a real number. Assume that $d \geq 1$, $t \geq 0$, and $\sigma \geq 4$ and define $\widetilde{\beta(t)} := (t+x)\left((t-x)d - 2d_1\right)$. If*

$$t > \frac{1}{2}\left(\frac{(d_1 + xd + 1)^2}{d} + \sigma - 2x\right)$$

then

1. $\widetilde{\beta(t)} > \sigma$*; and*
2. $td - d_1 - \sqrt{d\left(\widetilde{\beta(t)} - \sigma\right)} < 1.$

Proof. Note that using the hypothesized lower bound we have

$$t + x > \sigma/2 \geq 2. \tag{1}$$

Assume that $\widetilde{\beta(t)} \leq \sigma$, i.e., that $(t+x)\left((t-x)d - 2d_1\right) \leq \sigma$. Using the hypothesized lower bound for t we have:

$$(t+x)\left(\frac{1}{2}\left[(d_1 + xd + 1)^2 + d\sigma - 2dx\right] - xd - 2d_1\right) \leq \sigma.$$

Noting that

$$\frac{1}{2}\left[(d_1 + xd + 1)^2 + d\sigma - 2dx\right] - xd - 2d_1 = \frac{1}{2}\left[(-d_1 - xd + 1)^2 + d\sigma\right]$$

we have

$$\frac{(t+x)d\sigma}{2} \leq (t+x)\left(\frac{1}{2}\left[(-d_1 - xd + 1)^2 + d\sigma\right]\right) \leq \sigma.$$

Thus we must have $d(t+x) \leq 2$, which contradicts equation (1). Therefore we have that $\widetilde{\beta(t)} > \sigma$.

Now assume that the second inequality we want to prove is false, i.e., that $td - d_1 - \sqrt{d\left(\widetilde{\beta(t)} - \sigma\right)} \geq 1$. We then have

$$td - d_1 - 1 \geq \sqrt{d\left(\widetilde{\beta(t)} - \sigma\right)}.$$

Squaring both sides and substituting for $\widetilde{\beta(t)}$ we have

$$(td - d_1 - 1)^2 \geq d\left((t+x)\left((t-x)d - 2d_1\right) - \sigma\right).$$

Simplifying and dividing by $2d$ we have

$$t \leq \frac{1}{2}\left(\frac{(d_1 + xd + 1)^2}{d} + \sigma - 2x\right)$$

which contradicts the hypothesized lower bound for t. Q.E.D.

We can prove now an effective version of Matsusaka's Big Theorem in case of k-spannedness. The proof of the theorem below is due to Fernández Del Busto. The argument we give is a modification of his proof. The difference is that we deal with curvilinear 0-cycles of given length. This allows us to obtain a linear bound in k instead of a quadratic bound as in the corresponding result for k-jet ampleness proved in [8] (compare with [8, §1] and Corollary (3.4) below).

Theorem 3.3 *Let L be an ample line bundle on a nonsingular projective surface S. Let τ be the nefvalue of the pair (S, L) or any real value grater than the nefvalue of the pair (S, L). If*

$$t + \tau > \frac{1}{2}\left(\frac{(L\cdot(K_S+\tau L)+1)^2}{L^2}\right) + 2k + 3,$$

then tL is k-spanned.

Proof. Through the proof we will use Lemma (3.2) with $d := L^2$, $d_1 := L \cdot K_S$ and $x = \tau$. Let $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ be a curvilinear 0-cycle on S of length $k+1$, with $\mathrm{Supp}(\mathcal{Z}) = \{z_1, \ldots, z_r\}$, and let $k_1, \ldots, k_r$ be r positive integers such that $\sum_{i=1}^{r} k_i = k+1$, with k_i the local length of $\mathcal{Z}$ at z_i, $i = 1, \ldots, r$, as in (1.2). Let $\mathcal{J}_{\mathcal{Z}}$ be the ideal sheaf of $\mathcal{Z}$ in S. Let π be the blowing up of S along the 0-cycle $\mathcal{Z}$. Then to show that tL is k-spanned, it suffices to show that, with the same notation as in (3.1), and by using Leray's spectral sequence,

$$H^1(S, tL \otimes \mathcal{J}_{\mathcal{Z}}) \cong H^1(\overline{S}, \pi^*(tL) - \sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} \alpha \overline{E}_{\alpha}^{(i)}) = (0). \tag{2}$$

Since $K_S + \tau L$ is nef, we can write $tL - K_S$, t a positive integer, as the difference of an ample real divisor and a nef real divisor

$$tL - K_S = (t+\tau)L - (K_S + \tau L). \tag{3}$$

Consider the divisor

$$B := \pi^*(tL - K_S) - 2\sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} \alpha \overline{E}_{\alpha}^{(i)}.$$

For a positive integer n, consider the exact sequence

$$0 \to nB \to \pi^*(n(tL - K_S)) \to Q \to 0,$$

where Q denotes the quotient sheaf. Since

$$h^0(Q) = \sum_{i=1}^{r} h^0(\mathcal{O}_S/(x_i, y_i^{k_i})^{2n}) = \sum_{i=1}^{r} \ell(\mathcal{O}_S/(x_i, y_i^{k_i})^{2n}),$$

we have that

$$h^0(nB) \geq h^0(n(tL - K_S)) - \sum_{i=1}^{r} \ell(\mathcal{O}_S/(x_i, y_i^{k_i})^{2n}), \tag{4}$$

where (x_i, y_i) are local coordinates at the points z_i, $i = 1, \ldots, r$. Now, let $\sigma := 4(k+1)$ and let $\beta(t)$ be as in (3.2). Let us assume

$$t > \frac{1}{2}\left(\frac{(L\cdot(K_S+\tau L)+1)^2}{L^2} + 4(k+1) - 2\tau\right). \tag{5}$$

Thus by Lemma (3.2) we conclude that

$$\widetilde{\beta(t)} - 4(k+1) > 0. \tag{6}$$

Therefore, recalling (3), Lemma (1.4) applies (with $D = (t+\tau)L$, $R = K_S + \tau L$, so that $\widetilde{\beta(t)} = D \cdot (D - 2R)$) to say that, for $n \gg 0$,

$$h^0(n(tL - K_S)) \geq \frac{n^2}{2}\widetilde{\beta(t)} + \mathrm{O}(n).$$

On the other hand,

$$\ell(\mathcal{O}_S/(x_i, y_i^{k_i})^{2n}) = k_i\frac{2n(2n+1)}{2} = k_i n(2n+1), \quad i = 1, \ldots, r.$$

Thus we infer from (4) that, for $n \gg 0$,

$$\begin{aligned} h^0(nB) &\geq \frac{n^2}{2}\widetilde{\beta(t)} - n(2n+1)\sum_{i=1}^{r} k_i + \mathrm{O}(n) \\ &= \frac{n^2}{2}\widetilde{\beta(t)} - n(2n+1)(k+1) + \mathrm{O}(n) \\ &= \frac{n^2}{2}(\widetilde{\beta(t)} - 4(k+1)) + \mathrm{O}(n) \end{aligned}$$

Thus, recalling (6), we conclude that B is big.

Let $nB = M_n + F_n$ be the decomposition of nB in its moving part M_n and its fixed part F_n. Then by Lemma (1.3) we have

$$\begin{aligned} \frac{1}{n}F_n \cdot \pi^* L &\leq B \cdot \pi^* L - \sqrt{\widetilde{\beta(t)} - 4(k+1)}\sqrt{(\pi^* L)^2} \\ &= (tL - K_S)\cdot L - \sqrt{\widetilde{\beta(t)} - 4(k+1)}\sqrt{L^2}. \end{aligned}$$

Therefore, by Lemma (3.2),

$$\frac{1}{n}F_n \cdot \pi^* L < 1$$

and hence the irreducible components of the divisor $\left\{\frac{1}{n}F_n\right\}$ are exceptional, that is

$$\left\{\frac{1}{n}F_n\right\} = \sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} \eta_\alpha \overline{E}_\alpha^{(i)}, \quad \eta_\alpha \geq 0. \tag{7}$$

Fix $n \gg 0$. By Bertini's theorem we can choose a general divisor D in $|nB|$ such that if $D = M_n + F_n$, with F_n the fixed part of nB, then M_n is reduced. Consider the divisor

$$\mathcal{D} := \pi^*((t+1)L - K_S) - 2\sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} \alpha \overline{E}_\alpha^{(i)} - \frac{1}{n}D.$$

Since $n\mathcal{D} = n\pi^* L$, the divisor $\mathcal{D}$ is numerically equivalent to $\pi^* L$ and hence $\mathcal{D}$ is nef and big. Since M_n is reduced, one has $\left\{\frac{M_n}{n}\right\} = 0$, so that $\left\{\frac{D}{n}\right\} = \left\{\frac{F_n}{n}\right\}$, and therefore

$$\left\lceil \frac{-D}{n} \right\rceil = -\left\{\frac{D}{n}\right\} = -\sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} \eta_\alpha \overline{E}_\alpha^{(i)}.$$

By using (7), the Kawamata-Viehweg vanishing theorem thus implies the vanishing of the higher cohomology groups of the divisor

$$\begin{aligned} \mathcal{F} &:= K_{\overline{S}} + \left\lceil \pi^*((t+1)L - K_S) - 2\sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} \alpha \overline{E}_\alpha^{(i)} - \frac{D}{n} \right\rceil && (8) \\ &= (t+1)\pi^* L - \sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} (\alpha + \eta_\alpha)\overline{E}_\alpha^{(i)}. && (9) \end{aligned}$$

In particular $H^1(\overline{S}, \mathcal{F}) = (0)$. This implies that

$$H^1(\overline{S}, (t+1)\pi^* L - \sum_{i=1}^{r}\sum_{\alpha=1}^{k_i} \alpha \overline{E}_\alpha^{(i)} = (0)$$

and hence that $(t+1)L$ is k-spanned. Recalling the assumption (5), we are done. Q.E.D.

For $k = 1$, the result gives a slightly sharper result for very ampleness than the result of the last section (compare also with [7, (13.10)]).

Corollary 3.4 *Let L be an ample line bundle on a nonsingular projective surface S, then tL is very ample if $t + \tau > \dfrac{(L \cdot (K_S + \tau L) + 1)^2}{2L^2} + 5$.*

Proof. Note that $\tau \leq 3$ (see e.g., [10]). Then use (3.3) with $k = 1$. Q.E.D.

In particular using adjunction theory as before (to obtain Corollaries (2.4), (2.6)) we have the following.

Corollary 3.5 *Let L be an ample line bundle on a nonsingular projective surface S. Then*

1. *tL is very ample if $t > \dfrac{(L \cdot (K_S + 2L) + 1)^2}{2L^2} + 3$;*
2. *If (X, L) is not a scroll, then tL is very ample if $t > \dfrac{(L \cdot (K_S + L) + 1)^2}{2L^2} + 4$.*

Remark 3.6 (The Del Pezzo surface case (cf., [2, (2.6)] and [5, (5.2)]) Let (S, L) be a Del Pezzo surface, i.e., $L = -K_S$ is ample and $\tau = 1$ is the nefvalue. The result (2.6) of [2] gives in the worst case that $-tK_S$ is k-spanned if $t \geq k + 2$. Theorem (3.3) gives

- Assume $t > \dfrac{1}{2}\left(\dfrac{1}{K_S^2} + 4k + 4\right)$, i.e., $t \geq 2k + 3$. Then $-tK_S$ is k-spanned.

4 An asymptotic result for k-very ampleness

In this section we show an asymptotic result for k-very ampleness and a few consequences of it.

Theorem 4.1 *Let L be an ample line bundle on a smooth surface S. Let k_0 be an integer such that $A := (k_0 - 1)L - K_S$ is nef and big and such that $A \cdot L \geq 2$. Then for $t \geq k + k_0$, tL is k-very ample and therefore k-spanned.*

Proof. Note that

$$\begin{aligned} ((k+k_0)L-K_S)^2 &= ((k+1)L+A)^2 \\ &\geq (k+1)^2+4(k+1)+1 \geq k^2+6k+6. \end{aligned} \tag{10}$$

Using the main theorem of [4] it follows that $tL = K_S + (tL - K_S)$ is k-very ample if $tL - K_S$ is nef and big; $(tL-K_S)^2 \geq 4k+5$; and there exist no effective divisors C on S with

$$(tL-K_S)\cdot C - k - 1 \leq C^2 < \frac{(tL-K_S)\cdot C}{2} < k+1. \tag{11}$$

For $t \geq k+k_0$ we have $tL - K_S = (t-k_0+1)L + A$ is ample and $(tL-K_S)^2 \geq k^2+6k+6 > 4k+5$. Assume that there was an effective C on S with

$$(tL-K_S)\cdot C - k - 1 \leq C^2 < \frac{(tL-K_S)\cdot C}{2} < k+1.$$

Note that

$$(tL-K_S)\cdot C \geq k+2. \tag{12}$$

To see this note that $(tL-K_S)\cdot C = (t-k_0+1)L\cdot C + A\cdot C \geq (k+1)L\cdot C + A\cdot C$. Thus $(tL-K_S)\cdot C \geq k+1$. If $(tL-K_S)\cdot C < k+2$ we have that $A\cdot C = 0$. Since C is effective and A is nef and big we conclude by the Hodge index theorem that $C^2 < 0$. But this contradicts $0 \leq (tL-K_S)\cdot C - k - 1 \leq C^2$. Thus $(tL-K_S)\cdot C \geq k+2$. Therefore $C^2 \geq 1$ by (11). Note that $C^2 \geq 1$ also implies that $A \cdot C \geq 1$.

From $(k+1)L\cdot C + A\cdot C \leq (tL-K_S)\cdot C \leq 2k+1$ we conclude that $L\cdot C = 1$. Thus, since $C^2 \geq 1$, we conclude that

$$L^2 = C^2 = 1 \tag{13}$$

and therefore by the Hodge index theorem $L \sim C$. Thus $A\cdot C = A \cdot L \geq 2$.

Therefore $(tL-K_S)\cdot C \geq (t-k_0+1)L\cdot C + A\cdot C \geq k+1+A\cdot C \geq k+3$. Thus we have the contradiction $C^2 \geq (tL-K_S)\cdot C - k - 1 \geq 2$ to equation (13). Q.E.D.

Corollary 4.2 *Assume that A is a nef and big line bundle on a smooth surface S such that $K_S + A$ is ample. Then $(k+2)(K_S+A)$ is k-very ample and therefore k-spanned.*

Proof. We take $L := K_S + A$ and $k_0 = 2$ in Theorem (4.1). Indeed $A = (k_0-1)(K_S + A) - K_S$ is nef and big. Also $A\cdot L = A\cdot(K_S+A)$ is positive since both bundles are nef and big, and is even by the usual parity relation. Q.E.D.

Corollary 4.3 *Let L be an ample line bundle on a smooth surface S. There exists an integer $k_0 > 0$ such that for $t \geq k + k_0$, tL is k-very ample and therefore k-spanned.*

Proof. Choose a $k_0 > 0$ such that $A := (k_0-1)L - K_S$ is ample and $A\cdot L \geq 2$. Now apply the above theorem. Q.E.D.

In the case of surfaces with ample canonical bundle we have the following (compare with [1, (5.4)]).

Corollary 4.4 *Assume that L is an ample line bundle on a smooth surface S and $K_S \sim rL$ for some $r > 0$. Then tL is k-very ample and therefore k-spanned for $t \geq k+r+3$. If further $L^2 \geq 2$, then tL is k-very ample and therefore k-spanned for $t \geq k+r+2$.*

Proof. First assume that $L^2 \geq 2$. We can take $k_0 = r+2$ in Theorem (4.1) since $(k_0-1)L - K_S \sim (k_0-r-1)L$ is ample and $((k_0-1)L-K_S)\cdot L \geq 2$ in this case. If $L^2 = 1$, then take $k_0 = r+3$. Q.E.D.

Taking $r = 1$ we have the following result.

Corollary 4.5 *Assume that K_S is ample on a smooth surface S. Then tK_S is k-very ample and therefore k-spanned for $t \geq k+4$. If further $K_S^2 \geq 2$, then tK_S is k-very ample and therefore k-spanned for $t \geq k+3$.*

Remark 4.6 To write down an explicit k_0 in Theorem (4.1) is straightforward. First use Fernández Del Busto's result to choose an explicit k' such that $k'L$ is very ample. Now by the usual formula for the doublepoint divisor [9] we have that $(k'^2L^2-4)L-K_S$ is spanned. Thus $(k'^2L^2-2)L-K_S$ is ample and satisfies

$$L\cdot((k'^2L^2-2)L-K_S) = L\cdot(((k'^2L^2-4)L-K_S)+2L) \geq 2L^2 \geq 2.$$

Thus we can take $k_0 = k'^2L^2-1$.

It would be interesting to know what the best choice of k_0 is in Theorem (4.1). as $k\to\infty$, i.e., letting $t(k)$ be the minimal integer such that $t(k)L$ is k-very ample, find the the lim inf and the lim sup of $t(k)-k$ as $k\to\infty$.

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