TWO RESULTS ON BRANCHED COVERINGS OF GRASSMANNIANS

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Introduction. In this paper we present two results on branched coverings of Grassmannians. Throughout this introduction, let $G := \mathrm{Gr}\,(r,n)$ denote the Grassmannian of r-dimensional complex vector subspaces of $\mathbb{C}^n$. Further, assume that $n \geq r+2 \geq 4$, and let $\mathfrak{p} : G \to \mathbb{P}^{\binom{n}{r}-1}$ denote the Plücker embedding.

First we give some examples of branched covers of G that are not the pullbacks under $\mathfrak{p}$ of branched covers from a manifold to $\mathbb{P}^{\binom{n}{r}-1}$. Our interest in such examples grew out of the paper [7] of the first author. The main result of that paper, which generalized a theorem of Lazarsfeld [9, 10] for branched covers of projective space, showed that a manifold, which is a branched cover of G, has the same complex cohomology groups as the Grassmannian in an appropriate range of dimensions depending only on G and the degree of the covering. Our construction technique is to consider divisors on a $\mathbb{P}^1$-bundle $\mathcal{P}$ over the Grassmannian G. By the "principle of counting constants" [6], it is known that the general divisor contained in the linear system of a sufficiently high power of a very ample line bundle on $\mathcal{P}$ is a branched cover of G under the bundle projection. It is straightforward to see that we can choose this cover to have degree $\dim G$. If the branched cover $X \to G$ extends to a branched covering from a projective manifold X' to $\mathbb{P}^{\binom{n}{r}-1}$, then we can use the estimate of the covering degree δ and Lazarsfeld's theorem [9, 10] to conclude that the second homotopy group of X' is one dimensional. The Lefschetz hyperplane section theorem lets us conclude that the second homotopy group of X is two dimensional. Finally we use a theorem of Fulton and Lazarsfeld [4] to conclude that this is impossible.

Our second result about the geometry of Grassmannians, Theorem 2.3, shows that an obvious natural way to construct branched covers of Grassmannians only leads to trivial examples. Let G, r, n be as above. For $r < n' < n$, let $G' \subset G$ be the subgrassmannian $\mathrm{Gr}\,(r,n')$ of G for a fixed embedding of $\mathbb{C}^{n'} \to \mathbb{C}^n$. If X is a subvariety of G whose homology class is a positive multiple of the homology class of G', then after moving X by an automorphism of G there is a branched cover of X onto G'. Motivated by the analogous result for surfaces in $\mathrm{Gr}\,(2,4)$ (see [3] and [1]), we show in §2 that such X's are subgrassmannians. This result is equivalent to the statement that, if $\mathcal{F}$ is the rank $n-r$ tautological quotient vector bundle of $G \times \mathbb{C}^n$ and if $f : X \to G$ is any branched cover, then every section of $f^*\mathcal{F}$ arises as the pullback of a section of $\mathcal{F}$. This result is reminiscent of Gieseker's examples [5, Chapter 3].

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1. Some branched covers of Grassmannians. By a variety we always mean an irreducible and reduced quasiprojective complex algebraic set.

Given a coherent sheaf $\mathcal{S}$ on an algebraic set Z, we denote the i-th sheaf cohomology group by $H^i(Z,\mathcal{S})$, or by $H^i(\mathcal{S})$ when Z is clear from context. Similarly, we denote $\dim H^i(Z,\mathcal{S})$ by $h^i(Z,\mathcal{S})$ or by $h^i(\mathcal{S})$.

Let $\mathcal{F}$ be a rank 2 vector bundle on a connected complex projective manifold X. Let $\mathbb{P}(\mathcal{F})$ denote the $\mathbb{P}^1$-bundle of one dimensional vector space quotients of fibers of $\mathcal{F}$. We do not distinguish between vector bundles and locally free sheaves. Thus letting $\pi : \mathbb{P}(\mathcal{F}) \to X$ denote the induced projection and letting $\xi_{\mathcal{F}}$ be the tautological line bundle on $\mathbb{P}(\mathcal{F})$, we have that $\pi_*\xi_{\mathcal{F}} \cong \mathcal{F}$. We say that $\mathcal{F}$ is *big* if there is some integer $N > 0$ such that the mapping given by $|N\xi_{\mathcal{F}}|$ has a $(\dim \mathbb{P}(\mathcal{F}))$-dimensional image. If $\mathcal{F}$ is spanned (or even nef) this is equivalent to the condition that $c_1(\xi_{\mathcal{F}})^{\dim \mathbb{P}(\mathcal{F})} > 0$.

Theorem 1.1 *Let $\mathcal{F}$ be a big and spanned rank 2 vector bundle on a connected complex projective variety X. Let $\pi : \mathbb{P}(\mathcal{F}) \to X$ and $\xi_{\mathcal{F}}$ be as above. For any $d \geq \dim X$, let D be a generic element of $|d\xi_{\mathcal{F}}|$. It follows that D is a variety and $\pi_D : D \to X$ is a degree d branched cover.*

Proof. Since $\xi_{\mathcal{F}}$ is spanned we see that given any fiber $F \cong \mathbb{P}^1$ of π, the map $H^0(\xi_{\mathcal{F}}^{\otimes d}) \to H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ is onto for all $d \geq 0$. Thus, letting $\mathcal{J}_F$ denote the ideal sheaf of F, it follows that for any $d \geq 0$ we have $h^0(\xi_{\mathcal{F}}^{\otimes d}) - h^0(\xi_{\mathcal{F}}^{\otimes d} \otimes \mathcal{J}_F) = h^0(\mathcal{O}_{\mathbb{P}^1}(d)) = d + 1$. Thus if $d \geq \dim X$ a generic $D \in |d\xi_{\mathcal{F}}|$ contains no fiber F.

It remains to show that D is irreducible. This is a simple consequence of the fact that $\xi_{\mathcal{F}}$ is big and spanned, e.g., [12, Theorem (3.42)]. $\square$

Corollary 1.2 *Let $\mathfrak{p} : G \to \mathbb{P}^{\binom{n}{r}-1}$ denote the Plücker embedding of $G := \mathrm{Gr}\,(r, n)$. Assume that $n \geq r+2 \geq 4$. Then there exists a projective manifold Y and a branched covering $f : Y \to G$ of degree $\dim G$ such that there is no branched covering $f' : Y' \to \mathbb{P}^{\binom{n}{r}-1}$ of a projective manifold Y' with f the pullback of f' under $\mathfrak{p}$.*

Proof. We will need a few simple estimates that follow from the inequalities, $n \geq r + 2 \geq 4$. First note that $\dim G = r(n - r)$ and that

$$\binom{n}{r} \geq \binom{n}{2} \geq r(n-r) + 2 \geq 6. \tag{1}$$

Take an ample and spanned vector bundle $\mathcal{F}$ of rank 2 on G, e.g., $\mathcal{O}_G(1) \oplus \mathcal{O}_G(1)$ with $\mathcal{O}_G(1)$ the very ample line bundle whose sections give the Plücker embedding. By Theorem 1.1, it follows that by choosing a generic $Y \in |(\dim G)\xi_{\mathcal{F}}|$ we obtain a degree $\dim G = r(n - r)$ branched covering $\pi_Y : Y \to G$. Note that since Y is generic it is smooth by Bertini's theorem. Let $f = \pi_Y$.

We proceed by contradiction. Let $\mathbb{P} := \mathbb{P}^{\binom{n}{r}-1}$. Assume that the covering $f : Y \to G$ is the pullback of a branched covering $f' : Y' \to \mathbb{P}$ of degree $\dim G$, where Y' is a projective manifold. Using equation 1, it follows from the first Lefschetz hyperplane section theorem that $h_2(Y, \mathbb{C}) = h_2(\mathbb{P}(\mathcal{F}), \mathbb{C}) = 2$. By Lazarsfeld's Barth type theorem for branched coverings of projective spaces [9, 10], $\pi_i(Y') \cong \pi_i(\mathbb{P})$ for

$$i \leq \dim \mathbb{P} - \deg f' + 1 = \left(\binom{n}{r} - 1\right) - r(n-r) + 1 = \binom{n}{r} - r(n-r).$$

Thus by equation 1, $\pi_2(Y') = \pi_2(\mathbb{P}) = \mathbb{Z}$.

On the other hand, a result of Fulton and Lazarsfeld [4, Theorem 9.6] gives the following isomorphisms between two relative homotopy groups:

$$f'_* : \pi_j(Y', Y, y) \xrightarrow{\cong} \pi_j(\mathbb{P}, G, f(y)) \quad \text{where} \quad y \in Y,$$

if $j \leq \dim Y' - \mathrm{cod} G = r(n - r)$, i.e., for $j \leq 4$ by equation 1. Now consider the homotopy sequences of the pairs (Y', Y) and $(\mathbb{P}, G)$:

$$\begin{array}{ccccccccc}
\pi_3(Y') & \to & \pi_3(Y', Y) & \to & \pi_2(Y) & \to & \pi_2(Y') & \to & \pi_2(Y', Y) \\
\tilde{f}_* \downarrow & & \tilde{f}_* \downarrow & & \downarrow & & \tilde{f}_* \downarrow & & \tilde{f}_* \downarrow \\
\pi_3(\mathbb{P}) & \to & \pi_3(\mathbb{P}, G) & \to & \pi_2(G) & \to & \pi_2(\mathbb{P}) & \to & \pi_3(\mathbb{P}, G).
\end{array}$$

Therefore by the four lemma, $\pi_2(Y) \cong \pi_2(G) \cong \mathbb{Z}$, which, by Hurewicz's theorem, implies the contradiction that $h_2(Y, \mathbb{C}) = 1$. $\square$

Remark 1.3 Take an ample and spanned vector bundle $\mathcal{F}$ of rank 2 on a projective manifold X. For a branched covering $f : Y \to X$ constructed as above with degree $d \geq \dim X$, let $\mathcal{E}$ be the vector bundle $(f_*\mathcal{O}_Y/\mathcal{O}_X)^*$ (see Lazarsfeld [9, 10]). The ampleness and spannedness of $\mathcal{F}$ gives the same positivity condition on $\mathcal{E}$. To see this, note that the exact sequence $0 \to \xi_{\mathcal{F}}^{-d} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})} \to \mathcal{O}_Y \to 0$ gives

$$\mathcal{E} \cong \left(\pi_*\mathcal{O}_Y/\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}\right)^* \cong \left(\pi_{(1)}\xi_{\mathcal{F}}^{-d}\right)^* \cong \pi_*\left(\xi_{\mathcal{F}}^{d-2} \otimes \pi^* \det \mathcal{F}\right) \cong S^{d-2}\mathcal{F} \otimes \det \mathcal{F}$$

by using relative duality (cf., [2]) and the canonical bundle formula.

2. A result on the geometry of Grassmannians. We first need a characterization of projective space in terms of the sizes of the families of linear subspaces. This result should be well known, but we do not know a reference.

Let $G := \mathrm{Gr}\,(r+1, n+1)$ denote the Grassmannian of $(r+1)$-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$. Let $\mathcal{P} \subset G \times \mathbb{P}^n$ denote the tautological family of linear $\mathbb{P}^r$'s in $\mathbb{P}^n$. Let $p : \mathcal{P} \to G$ and $q : \mathcal{P} \to \mathbb{P}^n$ denote the two morphisms of $\mathcal{P}$ induced by the product projections.

Proposition 2.1 *Let $\mathbb{P}^n$, G, $\mathcal{P}$, p, and q be as in the preceding paragraph. Let $Z \subset G$ be a subvariety. Let $X := q(p^{-1}(Z))$ with $\delta = \dim X$. Then $\dim Z \leq (r+1)(\delta - r)$ with equality only if X is a linear $\mathbb{P}^\delta$ and Z is a subgrassmannian $\mathrm{Gr}\,(r+1, \delta+1) \subset G$.*

Proof. The proposition is clear if $\dim Z = 0$. We proceed by induction assuming that the result is true when $\dim Z < d$ for some positive integer d. Thus let $d := \dim Z > 0$.

Since p and q are fiber bundles with connected fibers we have that X is a variety. Let $f := \dim p^{-1}(Z) - \dim X$. Then

$$r + d = \dim p^{-1}(Z) = f + \delta \tag{2}$$

Let $G' := \mathrm{Gr}\,(r, \delta)$ denote the Grassmannian of linear $\mathbb{P}^{r-1}$'s in the $(\delta-1)$-dimensional projective space $\mathbb{P}(T^*_{X,x})$ for a general point $x \in X$. Let $\mathcal{P}' \subset G' \times \mathbb{P}(T^*_{X,x})$ denote the tautological family of linear $\mathbb{P}^{r-1}$'s in $\mathbb{P}(T^*_{X,x})$. Let $p' : \mathcal{P}' \to G'$ and $q' : \mathcal{P}' \to \mathbb{P}(T^*_{X,x})$ denote the two morphisms of $\mathcal{P}'$ induced by the product projections.

Every linear $\mathbb{P}^r$ in X containing x gives rise to a linear $\mathbb{P}^{r-1} \subset \mathbb{P}(T^*_{X,x})$. This gives rise to a morphism $i : q^{-1}(x) \to G'$. Since distinct linear $\mathbb{P}^r$'s containing x go to distinct linear $\mathbb{P}^{r-1}$'s under i, we have that i is one-to-one. Let Z' denote $i(q^{-1}(x))$. Note that since x is a general point, Z' is the fiber $q^{-1}_{p^{-1}(Z)}(x)$ and we have that $\dim Z' = f$. Let $X' := q'(p'^{-1}(Z'))$ and $\delta' = \dim X'$. By the induction hypothesis we have $\dim Z' \leq r(\delta' + 1 - r)$. Using equation 2 and noting that $\delta' \leq \delta - 1$ we have that

$$d \leq r(\delta - r) + \delta - r = (\delta - r)(r+1).$$

This gives the desired inequality. If we have equality here, then $\delta' = \delta - 1$ which implies that $X' = \mathbb{P}(T^*_{X,x})$. Thus the projection from $x \in \mathbb{P}^n$ expresses X as a cone. Since x is a general point of X and hence smooth we conclude from [11, Theorem 5.11] that X is a linear $\mathbb{P}^\delta$. In this case we have that $Z \subset \mathrm{Gr}\,(r, \delta)$. Since $\dim Z = d = (\delta - r)(r+1) = \dim \mathrm{Gr}\,(r, \delta)$, Z must be this subgrassmannian of G. $\square$

We need the fact that certain subgrassmannians of a Grassmannian have vector bundle neighborhoods. On $G := \mathrm{Gr}\,(r+1, n+1)$ we have the tautological sequence of vector bundles, $0 \to \mathcal{F}^* \to G \times \mathbb{C}^{n+1} \xrightarrow{\phi} \mathcal{E} \to 0$, with rank $\mathcal{F} = r+1$ and rank $\mathcal{E} = n - r$. Given any integer $t \geq 1$, we have an embedding $G \to \mathrm{Gr}\,(r+1, n+1+t)$ given by sending $x \in G$ to the point in $\mathrm{Gr}\,(r+1, n+1+t)$ corresponding to the subspace $\mathcal{F}^*_x \oplus \{0\}$ of $\mathbb{C}^{n+1+t} \cong \mathbb{C}^{n+1} \oplus \mathbb{C}^t$. We can extend this map to an embedding of $\mathcal{F}^{\oplus t}$ into $\mathrm{Gr}\,(r+1,n+1+t)$. Send $f:=(f_1,\ldots,f_t)\in\mathcal{F}_x^{\oplus t}$ to the point in $\mathrm{Gr}\,(r+1,n+1+t)$ corresponding to the image of $\mathcal{F}_x^*$ in $\mathbb{C}^{n+1}\oplus\mathbb{C}^t$ under the map sending $v\in\mathcal{F}_x^*$ to $(v,f(v))\in\mathbb{C}^{n+1}\oplus\mathbb{C}^t$.

We need an easy consequence of a well known theorem of Kleiman [8].

Lemma 2.2 *Any variety X in $\mathrm{Gr}\,(r+1,n+1+t)$ homologous to a multiple of $G:=\mathrm{Gr}\,(r+1,n+1)$ can be translated so that it is in the above vector bundle neighborhood.*

Proof. Let U denote the vector bundle neighborhood of $G':=\mathrm{Gr}\,(r+1,n+1+t)$. Note that the complement of U is a union $\cup_{i\in I}Z_i$ of a finite number of irreducible subvarieties Z_i of G'. Kleiman's theorem [8] guarantees that we can find a translate gX of X that for each $i\in I$ meets Z_i in either the empty set or a set of dimension $\dim Z_i+\dim X-\dim \mathrm{Gr}\,(r+1,n+1+t)$. If $Z_i\cap gX\neq\emptyset$, then because the dimension is the dimension of the homology class h representing $Z_i\cap gX$, the intersection $Z_i\cap gX$ with multiplicities represents h. Since $G\cap Z_i=\emptyset$ and since the homology class representing h is a multiple of the homology class $G\cap Z_i$, we are done. $\square$

For a subvariety V in $\mathrm{Gr}\,(r+1,n+1)$, let $\mathcal{R}_V$ be the union of $\mathbb{P}^r$'s in $\mathbb{P}^n$ represented by V.

Theorem 2.3 *Let $n\geq n'>r\geq 1$. If a subvariety X of $\mathrm{Gr}\,(r+1,n+1)$ is homologous to a nontrivial multiple $d[G']$ of a subgrassmannian $G':=\mathrm{Gr}\,(r+1,n'+1)$ in $\mathrm{Gr}\,(r+1,n+1)$, then X is a translate gG' of the subgrassmannian G' by an automorphism of $\mathrm{Gr}\,(r+1,n+1)$.*

Proof. Since the truth or falsity of the theorem is not affected by replacing X by a translate of X by an automorphism of $\mathrm{Gr}\,(r+1,n+1)$, we will simply rename as X, the different translates that arise of X by automorphisms of $\mathrm{Gr}\,(r+1,n+1)$.

Let U denote the vector bundle neighborhood of G' constructed in the paragraph before Lemma 2.2. By Lemma 2.2 we can assume that some translate of X, which we rename X, is in U.

Note that $\mathcal{R}_{G'}$ is a linear $\mathbb{P}^{n'}$. By the construction before Lemma 2.2 there is a vector bundle neighborhood V of $\mathbb{P}^{n'}$ in $\mathbb{P}^n$.

Let $\pi:U\to G'$ be the bundle projection. For a general fiber f of π, $X\cdot f=dG'\cdot f=d$. Therefore $\pi_X:X\to G'$ is generically d to 1. Since π is affine, we conclude that $\pi_X:X\to G'$ is a finite morphism of degree d. By using multiplication in U by an element of $\mathbb{C}^*$ that is sufficiently small in absolute value, X can be assumed to be as close as we want to the zero section, G' in U. In particular, we can assume that the union of the $\mathbb{P}^r$'s in $\mathbb{P}^n$ represented by X, $\mathcal{R}_X$, is a compact connected subvariety contained in V. Therefore since the vector bundle projection $V\to\mathbb{P}^{n'}$ is affine $\mathcal{R}_X$ is at most n'-dimensional. By Proposition 2.1, $\mathcal{R}_X$ must be $\mathbb{P}^{n'}$ and X must be isomorphic to some subgrassmannian $\mathrm{Gr}\,(r,n-1)$ and $d=1$. $\square$

The above result can be rephrased as saying that given a Grassmannian $G:=\mathrm{Gr}\,(n-r,n)$ with a rank $r\geq 2$ tautological quotient bundle $\mathcal{E}$ of the trivial bundle $G\times\mathbb{C}^n$ and given an irreducible subvariety X representing a nonzero multiple of $c_r(\mathcal{E})^t$ for some $t>0$ it follows that X is a subgrassmannian of G defined by t sections of $\mathcal{E}$. It is natural to ask the following question.

Question 2.4 Which homology classes on a Grassmannian have no irreducible subvariety representing them?

The following more restricted version of this question is very natural in light of Theorem 2.3.

Question 2.5 Given positive integers r,r',n,n' with $r'\leq r$ and $n'-r'\leq n-r$, there are embeddings of $G':=\mathrm{Gr}\,(r'+1,n'+1)$ into $G:=\mathrm{Gr}\,(r+1,n+1)$. For which quadruples, r',n',r,n, is it true that any irreducible subvariety of G homologous to a multiple of G' is a translate of G' by an element of the automorphism group of G.

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