On generation of jets for vector bundles

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Abstract

We introduce and study the k-jet ampleness and the k-jet spannedness for a vector bundle, $\mathcal{E}$, on a projective manifold. We obtain different characterizations of projective space in terms of such positivity properties for $\mathcal{E}$. We compare the 1-jet ampleness with different notions of very ampleness in the literature.
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Introduction

Let $\mathcal{E}$ be a rank r vector bundle on an n-dimensional projective manifold X. If $r = 1$, there is a well understood and accepted notion of very ampleness. If $\mathcal{E}$ is a vector bundle there are a number of notions of very ampleness that turn out to be different. Restricting to notions with good properties, e.g., direct sums of very ample bundles are very ample, we have two main notions. One definition is that $\mathcal{E}$ is very ample if the tautological bundle $\xi_{\mathcal{E}}$ on $\mathbb{P}(\mathcal{E})$ is very ample, and the second stronger definition is based on the existence of enough sections to separate 1-jets.

In §1, we study these notions and define k-jet ampleness of $\mathcal{E}$, the natural generalization of the second notion to the case when global sections separate k-jets (the first notion does not generalize to the case of k-jets with $k > 1$).

In §2, we study the behavior of k-jet ampleness under direct sums, tensor products, and blowing up of finite sets. We also point out the analogue of the Chern class inequalities of [3] in the case of k-jet ample vector bundles.

In §3, we give lower bounds on the number of sections and $\det(\mathcal{E})^n$ for a k-jet vector bundle $\mathcal{E}$, and give characterizations of $(X, \mathcal{E})$ when the lower bounds are taken on. We discuss the behavior of k-jet ampleness under adjunction and make some conjectures of what should optimally be true.

In §4 we compare the different definitions used in the preceding sections.

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1 Background material

Throughout this paper we deal with complex projective manifolds X. We denote by $\mathcal{O}_X$ the structure sheaf of X and by K_X the canonical bundle. By a *vector bundle* we mean a locally free sheaf (of $\mathcal{O}_X$-modules) of finite rank.

1.1 Notation. In this paper, we use the standard notation from algebraic geometry. Let us only fix the following.

$h^i(\mathcal{F})$, the complex dimension of $H^i(X,\mathcal{F})$, for any coherent sheaf $\mathcal{F}$ on X;

$\approx$ denotes linear equivalence of line bundles;

$\Gamma(\mathcal{E}) = H^0(\mathcal{E})$, the space of the global sections of a vector bundle $\mathcal{E}$ on X. We say that $\mathcal{E}$ is spanned if it is spanned at all points of X by $\Gamma(\mathcal{E})$;

$|V|$, the complete linear system associated with a vector subspace, $V \subseteq \Gamma(\mathcal{E})$, for a vector bundle $\mathcal{E}$ on X.

If $\mathcal{J}$ is an ideal sheaf of X and $\mathcal{E}$ is a vector bundle on X, we write $\mathcal{E}/\mathcal{J}$ for $\mathcal{E} \otimes (\mathcal{O}_X/\mathcal{J})$.

Line bundles and divisors are used with little (or no) distinction. Hence we freely use the additive notation.

1.2 k-th order embeddings. Let X be a smooth algebraic variety. Let $\mathcal{E}$ be a rank r vector bundle on X. For each point x on X let $\mathfrak{m}_x$ be the maximal ideal sheaf of x in X, i.e., the stalk of $\mathfrak{m}_x$ at a point $y \neq x$ is $\mathcal{O}_{X,y}$ and at x is the maximal ideal $\mathfrak{m}_x\mathcal{O}_{X,x} \subset \mathcal{O}_{X,x}$. Let $V \subseteq \Gamma(\mathcal{E})$ be a subvector space of $\Gamma(\mathcal{E})$.

We say that $\mathcal{E}$ is *k-jet spanned at x with respect to V* if V gives global sections with arbitrarily prescribed k-jets at x, i.e., if the evaluation map

$$X \times V \to \Gamma(\mathcal{E} \otimes (\mathcal{O}_X/\mathfrak{m}_x^{k+1}))$$

is surjective. We say that $\mathcal{E}$ is *k-jet spanned with respect to V* if $\mathcal{E}$ is k-jet spanned at x with respect to V for each point $x \in X$.

We say that $\mathcal{E}$ is *k-jet spanned at x* (respectively $\mathcal{E}$ is *k-jet spanned*) if $V = \Gamma(\mathcal{E})$ in the above definitions.

Let $x_1, \dots, x_t$ be t distinct points of X. Let $\mathfrak{m}_i$ be the maximal ideal sheaves of the points $x_i \in X$, $i = 1, \dots, t$. Consider the 0-cycle $\mathcal{Z} = x_1 + \cdots + x_t$. We say that $\mathcal{E}$ is *k-jet ample at $\mathcal{Z}$ with respect to V* if for every t-ple $(k_1, \ldots, k_t)$ of positive integers such that $\sum_{i=1}^{t} k_i = k+1$, the evaluation map

$$X \times V \to \Gamma(\mathcal{E} \otimes (\mathcal{O}_X / \otimes_{i=1}^{t} \mathfrak{m}_i^{k_i}))(\cong \oplus_{i=1}^{t} \Gamma(\mathcal{E} \otimes (\mathcal{O}_X/\mathfrak{m}_i^{k_i}))$$

is onto. Here $\mathfrak{m}_i^{k_i}$ denotes the k_i-th tensor power of $\mathfrak{m}_i$. We say that $\mathcal{E}$ is *k-jet ample with respect to V* if, for any $t \geq 1$ and any 0-cycle $\mathcal{Z} = x_1 + \cdots + x_t$, where $x_1, \ldots, x_t$ are t distinct points on X, the vector bundle $\mathcal{E}$ is k-jet ample at $\mathcal{Z}$ with respect to V.

We say that $\mathcal{E}$ is *k-jet ample at $\mathcal{Z}$* (respectively $\mathcal{E}$ is *k-jet ample*) if $V = \Gamma(\mathcal{E})$ in the above definitions.

Hence in particular $\mathcal{E}$ k-jet ample implies that $\mathcal{E}$ is k-jet spanned.

Note that $\mathcal{E}$ is 0-jet ample if and only if $\mathcal{E}$ is 0-jet spanned, if and only if $\mathcal{E}$ is spanned by its global sections. Moreover, for $r = 1$, i.e., $\mathcal{E} = L$ is a line bundle, L is 1-jet ample if and only if L is very ample (see §4 for the case $k = 1$).

Note also that for a line bundle L, 1-jet spannedness with respect to V is equivalent to V spanning L and the map given by $|V|$ being an immersion.

We refer to [1] and [4] for more on k-jet ampleness in the case of line bundles. We also refer to [9] for results in the case of surfaces.

1.3 The k-th jet bundle. Let X be a smooth algebraic variety of dimension n and let $\mathcal{E}$ be a rank r vector bundle on X. By the k-th *jet bundle of* $\mathcal{E}$, denoted $J_k(X, \mathcal{E})$ or $J_k(\mathcal{E})$ when no confusion will occur, we mean the vector bundle of rank $r\binom{k+n}{n}$ associated to the sheaf $p^*\mathcal{E}/(p^*\mathcal{E} \otimes \mathcal{J}_\Delta^{k+1})$, where $p : X \times X \to X$ is the projection on the first factor, the tensor product is with respect to $\mathcal{O}_{X \times X}$ and $\mathcal{J}_\Delta$ is the sheaf of ideals of the diagonal, Δ, of $X \times X$. Note that $J_k(\mathcal{E}) = \cup_{x \in X} J_k(\mathcal{E})_x$, where the fiber over each point $x \in X$ is $J_k(\mathcal{E})_x \cong \mathcal{E}_x/\mathfrak{m}_x^{k+1}$. Note also that $J_0(\mathcal{E}) = \mathcal{E}$. Moreover there is a natural map $j_k := j_k^{\mathcal{E}} : \mathcal{E} \to J_k(\mathcal{E})$, defined on the sheaf level and which is not a bundle map. It sends a germ of a section to its k-jet.

Interpreting $J_k(\mathcal{E})$ as the bundle of k-jets of $\mathcal{E}$, i.e., Taylor expansions of holomorphic sections of $\mathcal{E}$ truncated after the k-th term, one has an exact sequence

$$0 \to {T_X^*}^{(k)} \otimes \mathcal{E} \to J_k(\mathcal{E}) \to J_{k-1}(\mathcal{E}) \to 0, \tag{1}$$

where ${T_X^*}^{(k)} := S^k(T_X^*)$ denotes the k-th symmetric power of the cotangent bundle T_X^*. In particular there is a surjective natural map $J_k(\mathcal{E}) \to J_{k'}(\mathcal{E}) \to 0$ for $k' \leq k$. Then it follows from (1) that

- If $J_k(\mathcal{E})$ is spanned by its sections for some k, then $J_{k'}(\mathcal{E})$ is spanned by its sections when $k' \leq k$.

Note that j_k gives rise to a natural evaluation map, which we denote by the same symbol, $j_k : X \times \Gamma(\mathcal{E}) \to J_k(\mathcal{E})$. This map takes $(x, s) \in X \times \Gamma(\mathcal{E})$ to the k-th jet $j_k(s(x)) \in J_k(\mathcal{E})$. Since for each point $x \in X$ there is a canonical isomorphism $\mathcal{E} \otimes (\mathcal{O}_X/\mathfrak{m}_x^{k+1}) \cong J_k(\mathcal{E})_x$, it follows that

- $\mathcal{E}$ is k-jet spanned if and only if the evaluation map j_k is surjective.

We have the following general result.

Lemma 1.4 *Let $\mathcal{E}$ be a vector bundle on a smooth projective variety X. Let $p : \mathbb{P}(\mathcal{E}) \to X$ be the bundle projection and let $\xi_{\mathcal{E}}$ be the tautological line bundle of $\mathbb{P}(\mathcal{E})$. Then $J_1(\mathcal{E}) \cong p_* J_1(\xi_{\mathcal{E}})$.*

Proof. One has a natural surjective map $p^*\mathcal{E} \to \xi_{\mathcal{E}} \to 0$ which induces a surjective map $\alpha : J_1(p^*\mathcal{E}) \to J_1(\xi_{\mathcal{E}}) \to 0$. We also have a canonical exact sequence

$$0 \to p^* J_1(\mathcal{E}) \to J_1(p^*\mathcal{E}) \to T^*_{\mathbb{P}(\mathcal{E})/X} \otimes p^*\mathcal{E} \to 0,$$

where $T^*_{\mathbb{P}(\mathcal{E})/X}$ denotes the relative cotangent bundle. Applying p_* to this sequence and noting that $p_*(T^*_{\mathbb{P}(\mathcal{E})/X})$ is the zero sheaf we get $J_1(\mathcal{E}) \cong p_* J_1(p^*\mathcal{E})$. Thus, it suffices to show that

$$p_* J_1(p^*\mathcal{E}) \cong p_* J_1(\xi_{\mathcal{E}}). \tag{2}$$

To this purpose consider the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T^*_{\mathbb{P}(\mathcal{E})} \otimes T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}} \arrow[r] \arrow[d] & T^*_{\mathbb{P}(\mathcal{E})} \otimes p^*\mathcal{E} \arrow[r] \arrow[d] & T^*_{\mathbb{P}(\mathcal{E})} \otimes \xi_{\mathcal{E}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} = \ker(\alpha) \arrow[r] \arrow[d] & J_1(p^*\mathcal{E}) \arrow[r] \arrow[d] & J_1(\xi_{\mathcal{E}}) \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}} \arrow[r] \arrow[d] & p^*\mathcal{E} \arrow[r] \arrow[d] & \xi_{\mathcal{E}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{3}$$

The isomorphism (2) will follow if we show

1. that $p_*\mathcal{K} = 0$; and
2. that there is an injection $0 \to p_{(1)}\mathcal{K} \to p_{(1)} J_1(p^*\mathcal{E})$.

Since the derived functors $p_{(i)}(T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}})$ are zero for all $i \geq 0$, we conclude that $p_*\mathcal{K} \cong p_*(T^*_{\mathbb{P}(\mathcal{E})} \otimes T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}})$. To see that this sheaf is zero, consider the exact sequence

$$0 \to p^*T^*_X \to T^*_{\mathbb{P}(\mathcal{E})} \to T^*_{\mathbb{P}(\mathcal{E})/X} \to 0 \tag{4}$$

tensored with $T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}}$. Since letting r denote the rank of $\mathcal{E}$ we have

$$H^0(T^*_{\mathbb{P}^{r-1}}(1)) = 0$$

and

$$H^0(T^*_{\mathbb{P}^{r-1}} \otimes T^*_{\mathbb{P}^{r-1}}(1)) = 0,$$

we conclude that

$$p_*(p^*T^*_X \otimes T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}}) \cong T^*_X \otimes p_*(T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}}) = 0$$

and

$$p_*(T^*_{\mathbb{P}(\mathcal{E})/X} \otimes T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}}) = 0$$

respectively. Thus using again sequence (4) tensored with $T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}}$ we have $p_*(T^*_{\mathbb{P}(\mathcal{E})} \otimes T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}})$ squeezed between two zero sheaves and thus zero. Thus $p_*\mathcal{K}$ is zero.

Note that $p_*(T^*_{\mathbb{P}(\mathcal{E})} \otimes p^*\mathcal{E}) \to p_*(T^*_{\mathbb{P}(\mathcal{E})} \otimes \xi_{\mathcal{E}})$ is an isomorphism (both are isomorphic to $T^*_X \otimes \mathcal{E}$). Using this and the fact that $p_{(1)}(T^*_{\mathbb{P}(\mathcal{E})/X} \otimes \xi_{\mathcal{E}}) = 0$, a diagram chase shows that there is an injection $0 \to p_{(1)}\mathcal{K} \to p_{(1)}J_1(p^*\mathcal{E})$ if there is an injection

$$0 \to p_{(1)}(T^*_{\mathbb{P}(\mathcal{E})} \otimes p^*\mathcal{E}) \to p_{(1)}J_1(p^*\mathcal{E}). \tag{5}$$

To show (5), consider the diagram with an exact row and column

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & T^*_{\mathbb{P}(\mathcal{E})/X} \otimes p^*\mathcal{E} \arrow[d] & \\
0 \arrow[r] & p^*(T_X \otimes \mathcal{E}) \arrow[r] & J_1(p^*\mathcal{E}) \arrow[r] & J_{1/X}(p^*\mathcal{E}) \arrow[r] \arrow[d] & 0, \\
 & & & p^*\mathcal{E} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

where $J_{1/X}(p^*\mathcal{E})$ denotes the relative 1-jet bundle (defined by the horizontal exact sequence above). Since $p_{(1)}(p^*\,(T_X^*\otimes\mathcal{E}))\cong p_{(1)}(\mathcal{O}_{\mathbb{P}(\mathcal{E})})\otimes T_X^*\otimes\mathcal{E}=0$, we have a surjection

$$p_*J_1(p^*\mathcal{E})\to\mathcal{E}(\cong p_*p^*\mathcal{E})\to 0, \tag{6}$$

if we have a surjection $p_*J_{1/X}(p^*\mathcal{E})\to\mathcal{E}\to 0$. Letting $p^{-1}\mathcal{O}_X$ denote the topological inverse image of $\mathcal{O}_X$, and noting that j_1 gives a $p^{-1}\mathcal{O}_X$ splitting of the vertical sequence, the last surjection follows from the decomposition, $p_{(i)}J_{1/X}(p^*\mathcal{E})\cong p_{(i)}(p^*\mathcal{E})\oplus p_{(i)}(T^*_{\mathbb{P}(\mathcal{E})/X}\otimes p^*\mathcal{E})$, $i\geq 0$. By looking at diagram (3), we see that surjection (6) implies (5), which in turn gives the desired isomorphism (2). Q.E.D.

For further general properties of jet bundles we refer to [8] and [13].

2 Some general properties

In this section we will examine various functorial properties of the definitions given in §1. In this section X always denotes a smooth projective variety.

Proposition 2.1 *Let $\mathcal{E}_j$ be a b_j-jet ample vector bundle on X, $j=1,\ldots,m$. Then $\mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_m$ is a $\min_{j=1,\ldots,m}\{b_j\}$-jet ample vector bundle on X.*

Proof. Assume e.g., that $b_1=\min_{j=1,\ldots,m}\{b_j\}$. Let $\{x_1,\ldots,x_t\}$ be a collection of t distinct points of X and let $(k_1,\ldots,k_t)\in\mathbb{Z}_+^t$ such that $\sum_{i=1}^t k_i=b_1+1$. Then, by assumption, we have surjective maps $\sigma_j: H^0(\mathcal{E}_j)\to H^0(\mathcal{E}_j/\otimes_{i=1}^t\mathfrak{m}_{x_i}^{k_i})$, $j=1,\ldots,m$. Thus, by composition, we get a surjective map

$$H^0(\oplus_{j=1}^m\mathcal{E}_j)=\oplus_{j=1}^m H^0(\mathcal{E}_j)\xrightarrow{\oplus_{j=1}^m grs_j}\oplus_{j=1}^m H^0(\mathcal{E}_j/\otimes_{i=1}^t\mathfrak{m}_{x_i}^{k_i})\cong H^0((\oplus_{j=1}^m\mathcal{E}_j)/\otimes_{i=1}^t\mathfrak{m}_{x_i}^{k_i}).$$

Q.E.D.

Proposition 2.2 *Let $\mathcal{E}$ be a k-jet ample vector bundle on X. Then $\mathcal{E}/\mathcal{F}$ is a k-jet ample vector bundle for each subvector bundle $\mathcal{F}$ of $\mathcal{E}$.*

Proof. Let $\{x_1,\ldots,x_t\}$ be a collection of t distinct points of X and let $(k_1,\ldots,k_t)\in\mathbb{Z}_+^t$ such that $\sum_{i=1}^t k_i=k+1$. Note that, by tensoring with $\mathcal{O}_X/\otimes_{i=1}^t\mathfrak{m}_{x_i}^{k_i}$, the surjection $\mathcal{E}\to\mathcal{E}/\mathcal{F}\to 0$ induces a surjective map $\mathcal{E}/\otimes_{i=1}^t\mathfrak{m}_{x_i}^{k_i}\to(\mathcal{E}/\mathcal{F})/\otimes_{i=1}^t\mathfrak{m}_{x_i}^{k_i}$. Thus the assertion follows from the commutativity of the diagram

$$\begin{array}{ccc} H^0(\mathcal{E}) & \longrightarrow & H^0(\mathcal{E}/\otimes_{i=1}^t\mathfrak{m}_{x_i}^{k_i}) \\ \downarrow & & \downarrow \\ H^0(\mathcal{E}/\mathcal{F}) & \longrightarrow & H^0((\mathcal{E}/\mathcal{F})/\otimes_{i=1}^t\mathfrak{m}_{x_i}^{k_i}) \end{array}$$

where the upper horizontal arrow is surjective by assumption and the right vertical arrow is surjective by construction. Q.E.D.

Proposition 2.3 *Let $\mathcal{E}$ be an a-jet ample vector bundle on X and let $\mathcal{F}$ be a b-jet ample vector bundle on X. Then $\mathcal{E} \otimes \mathcal{F}$ is an $(a+b)$-jet ample vector bundle on X.*

Proof. It is a straightforward modification of the proof of [4, Lemma (2.2)]. Q.E.D.

As an immediate consequence of (2.2) and (2.3) we have the following.

Corollary 2.4 *Let $\mathcal{E}$ be a k-jet ample vector bundle of rank r on X. Then $\wedge^m \mathcal{E}$ and $S^m(\mathcal{E})$ are mk-jet ample vector bundles. In particular* $\det(\mathcal{E})$ *is a rk-jet ample line bundle on X.*

Remark 2.5 Note that if in (2.1) we assume that the vector bundles $\mathcal{E}_j$ are b_j-jet spanned for each j and similarly in (2.2), (2.3), (2.4) we assume that the vector bundle $\mathcal{E}$ is k-jet spanned, then all the corresponding conclusions hold true for jet spannedness as well. This is an immediate consequence of the above proofs, by noting also that the proof of the key-lemma [4, (2.2)] makes only use of jet spannedness.

Proposition 2.6 *Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of vector bundles on X. Assume that $\mathcal{E}$ is k-jet ample and $\mathcal{G}$ is k-jet ample with respect to $V := \mathrm{Im}(\Gamma(\mathcal{F}) \to \Gamma(\mathcal{G}))$. Then $\mathcal{F}$ is k-jet ample.*

Proof. Let $x_1, \dots, x_t$ be t distinct points and let $k_1, \dots, k_t$ be positive integers such that $\sum_{i=1}^t k_i = k+1$. Let $\mathfrak{m}_i$ be the ideal sheaves of the points x_i, $i = 1, \dots, t$. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X \times \Gamma(\mathcal{E}) & \longrightarrow & X \times \Gamma(\mathcal{F}) & \longrightarrow & X \times V & \longrightarrow & 0 \\
 & & \downarrow e_{\mathcal{E}} & & \downarrow e_{\mathcal{F}} & & \downarrow e_{\mathcal{G}} & & \\
0 & \longrightarrow & \Gamma(\mathcal{E}/\otimes_{i=1}^t \mathfrak{m}_i^{k_i}) & \longrightarrow & \Gamma(\mathcal{F}/\otimes_{i=1}^t \mathfrak{m}_i^{k_i}) & \longrightarrow & \Gamma(\mathcal{G}/\otimes_{i=1}^t \mathfrak{m}_i^{k_i}) & \longrightarrow & 0,
\end{array}$$

where the vertical arrows are the evaluation maps. By hypothesis $e_{\mathcal{E}}, e_{\mathcal{G}}$ are surjective. By a diagram chase we see that $e_{\mathcal{F}}$ is surjective, so we are done.Q.E.D.

The following is the rank > 1 version of [6, Proposition (3.5)] (compare also with [3, Corollary (1.1)] and [4, Lemma (3.1)]).

Proposition 2.7 *Let $\mathcal{E}$ be a k-jet ample vector bundle on X. Let $x_1, \dots, x_t$ be t distinct points on X and let $a_1, \dots, a_t$ be t positive integers. Let $\pi : \widetilde{X} \to X$ be the blowing up of X at $x_1, \dots, x_t$. Let $E_i := \pi^{-1}(x_i)$ be the exceptional divisors, $i = 1, \dots, t$. Then $\pi^*\mathcal{E} \otimes \mathcal{O}_{\widetilde{X}}(-\sum_{i=1}^t a_i E_i)$ is $\mu := \min\{k - \sum_{i=1}^t a_i, a_1, \dots, a_t\}$-jet ample.*

Proof. It is a straightforward modification of the proof of [6, (3.5)]. Q.E.D.

In the setting of Proposition (2.7), if we blow up a single point $x \in X$, $\pi : \widetilde{X} \to X$, we have that $\pi^*\mathcal{E} \otimes \mathcal{O}_{\widetilde{X}}(-kE)$ is spanned by global sections only assuming that $\mathcal{E}$ is k-jet spanned. Therefore, comparing with Corollary (1.5) of [3], we see that a formal extension of the arguments of [3] gives us the following lower bounds for the Chern classes and the Segre classes of $\mathcal{E}$.

Corollary 2.8 *Let X be an n-dimensional smooth projective variety and let $\mathcal{E}$ be a k-jet spanned vector bundle of rank r on X. Then*

1. $c_{i_1}(\mathcal{E}) \cdots c_{i_t}(\mathcal{E}) \geq k^n \binom{r}{i_1} \cdots \binom{r}{i_t}$ *for* $0 \leq i_j \leq r$, $i_1 + \cdots + i_t = n$;
2. $s_{i_1}(\mathcal{E}) \cdots s_{i_t}(\mathcal{E}) \geq k^n \binom{r+i_1-1}{i_1} \cdots \binom{r+i_t-1}{i_t}$ *for* $0 \leq i_j \leq r$, $i_1 + \cdots + i_t = n$.

In particular $c_n(\mathcal{E}) \geq k^n \binom{r}{n}$ *and* $s_n(\mathcal{E}) \geq k^n \binom{r+n-1}{n}$.

Remark 2.9 It is worthwhile to note that the same argument as in the proof of Lemmas (3.2), (3.3) in [6] gives us the following result.

Let $X_1, \ldots, X_t$ be t smooth projective varieties and let $\mathcal{E}_1, \ldots, \mathcal{E}_t$ be t vector bundles on $X_1, \ldots, X_t$ respectively. Let $p_i : X_1 \times \cdots \times X_t \to X_i$ be the projections on each factor, $i = 1, \ldots, t$. For $i = 1, \ldots, t$ assume that $\mathcal{E}_i$ is k_i-jet ample and let $k := \min\{k_1, \ldots, k_t\}$. Then $p_1^*\mathcal{E}_1 \otimes \cdots \otimes p_t^*\mathcal{E}_t$ is a k-jet vector bundle on the product $X_1 \times \cdots \times X_t$.

3 Characterizations of projective space

Let $\mathcal{E}$ be a k-jet ample or k-jet spanned vector bundle on a smooth projective variety X of dimension n. In this section we give a lower bound for $h^0(\mathcal{E})$ and the degree of $\det(\mathcal{E})$. We also study the "boundary cases" when either $h^0(\mathcal{E})$ or $(\det(\mathcal{E}))^n$ reaches the the lowest possible value. For any merely ample vector bundle $\mathcal{E}$ we also consider the special case when the k-th jet bundle $J_k(\mathcal{E})$ is trivial for a given k (compare with [13] for the line bundle case). This gives different characterizations of projective space. As a consequence of these results we obtain an adjunction type result concerning the jet spannedness of $K_X + \det(\mathcal{E})$ for a k-jet spanned vector bundle $\mathcal{E}$.

Theorem 3.1 *Let $\mathcal{E}$ be an ample vector bundle of rank r on a smooth projective variety X of dimension n. Assume that $J_k(\mathcal{E})$ is trivial for some $k > 0$. Then $(X, \mathcal{E}) \cong (\mathbb{P}^n, \oplus^r \mathcal{O}_{\mathbb{P}^n}(k))$ (hence in particular $\mathcal{E}$ is k-jet ample).*

Proof. By dualizing and by tensoring with $\mathcal{E}$ the exact sequence (1) we get a surjective map $J_k(\mathcal{E})^* \otimes \mathcal{E} \to T_X^{(k)} \otimes \mathcal{E}^* \otimes \mathcal{E} \to 0$.

Notice that we have an injection $\mathcal{O}_X \to \mathcal{E}^* \otimes \mathcal{E}$. By dualizing and by tensoring with $T_X^{(k)}$ we obtain a surjective map $T_X^{(k)} \otimes \mathcal{E}^* \otimes \mathcal{E} \to T_X^{(k)} \to 0$.

Thus by composition we get a surjective map $J_k(\mathcal{E})^* \otimes \mathcal{E} \to T_X^{(k)} \to 0$. Since $J_k(\mathcal{E})$ is trivial and $\mathcal{E}$ is ample, we conclude that $T_X^{(k)}$ is ample. It thus follows that $-K_X = \det(T_X)$ is ample, so that X is a Fano manifold. Therefore Mori's theory applies to say that X contains an extremal ray $R = \mathbb{R}_+[\gamma]$, where γ is a possibly singular reduced irreducible rational curve satisfying the numerical condition $-K_X \cdot \gamma \leq n+1$. Let $f : \mathbb{P}^1 \to C \subset X$ be the normalization of any irreducible reduced rational curve $C \subset X$, in particular any deformation of γ. Note that $f^*(T_X^{(k)}) \cong (f^*T_X)^{(k)}$ is ample since $T_X^{(k)}$ is ample. Thus f^*T_X is ample on $\mathbb{P}^1$. Therefore, by using Mori's proof of the Hartshorne conjecture (see e.g., [11, §4] for a discussion) we conclude that $X \cong \mathbb{P}^n$. By dualizing the exact sequence (1) we get a surjective map $J_k(\mathcal{E})^* \to T_X^{(k)} \otimes \mathcal{E}^* \to 0$. Since $J_k(\mathcal{E})$ is trivial it follows that $T_X^{(k)} \otimes \mathcal{E}^*$ is spanned. Take a line $\ell \subset \mathbb{P}^n \cong X$. From the normal bundle sequence of ℓ in $\mathbb{P}^n$,

$$0 \to T_\ell \cong \mathcal{O}_{\mathbb{P}^1}(2) \to T_{X|\ell} \to N_{\ell/X} \cong \oplus^{n-1}\mathcal{O}_{\mathbb{P}^1}(1) \to 0,$$

we get $T_X^{(k)}{}_{|\ell} \cong (T_{X|\ell})^{(k)} \cong (\oplus^{n-1}\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))^{(k)}$. Therefore we see that $\mathcal{O}_{\mathbb{P}^1}(k)$ occurs as a direct summand of $T_X^{(k)}{}_{|\ell}$.

The restriction of $\mathcal{E}$ to ℓ is $\mathcal{E}_\ell \cong \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some positive integers a_i, $i = 1, \ldots, r$. Since, by the above, the restriction $(T_X^{(k)} \otimes \mathcal{E}^*)_\ell$ is spanned and contains $\mathcal{O}_{\mathbb{P}^1}(k - a_i)$ as direct summands we conclude that $a_i \leq k$, $i = 1, \ldots, r$.

For a rank r vector bundle $\mathcal{E}$ on X it follows that

$$\det(J_k(\mathcal{E})) \approx \frac{1}{k}\binom{k+n}{n+1}(krK_X + (n+1)\det(\mathcal{E})).$$

Since the assumption on $J_k(\mathcal{E})$ implies that $\det(J_k(\mathcal{E}))$ is trivial and $X \cong \mathbb{P}^n$, we get $\det(\mathcal{E}) \approx \mathcal{O}_{\mathbb{P}^n}(kr)$. On the other hand, $\det(\mathcal{E}) \approx \mathcal{O}_{\mathbb{P}^n}(a_1 + \cdots + a_r)$. Then we infer that $a_1 + \cdots + a_r = kr$. Since $a_i \leq k$, $i = 1, \ldots, r$, we then conclude that $a_i = k$ for $i = 1, \ldots, r$.

Therefore $(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(-k))_\ell \cong \oplus^r \mathcal{O}_\ell$. From [12, (3.2.1), p. 51] it follows that $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(-k) \cong \oplus^r \mathcal{O}_{\mathbb{P}^n}$, i.e., $\mathcal{E} \cong \oplus^r \mathcal{O}_{\mathbb{P}^n}(k)$. Q.E.D.

The result above has the following consequences.

Proposition 3.2 *Let $\mathcal{E}$ be an ample vector bundle of rank r on a smooth projective variety X of dimension n. We have:*

1. *If $\mathcal{E}$ is k-jet spanned at a given point $x \in X$, then $h^0(\mathcal{E}) \geq r\binom{k+n}{n}$;*
2. *If equality happens and $\mathcal{E}$ is k-jet spanned, then $(X, \mathcal{E}) \cong (\mathbb{P}^n, \oplus^r \mathcal{O}_{\mathbb{P}^n}(k))$.*

Proof. For a given point $x \in X$, the map $H^0(\mathcal{E}) \to H^0(\mathcal{E} \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1})$ is onto and therefore

$$h^0(\mathcal{E}) \geq h^0(\mathcal{E} \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1}) = rh^0(\mathcal{O}_X/\mathfrak{m}_x^{k+1}) = r\binom{k+n}{n}.$$

Since $\mathcal{E}$ is k-jet spanned, the evaluation map $j_k : X \times \Gamma(\mathcal{E}) \to J_k(\mathcal{E})$ is surjective. If $h^0(\mathcal{E}) = r\binom{k+n}{n} = \operatorname{rank}(J_k(\mathcal{E}))$ it thus follows that j_k is in fact an isomorphism and therefore $J_k(\mathcal{E})$ is trivial. Then Theorem (3.1) applies to give the result. Q.E.D.

Proposition 3.3 *Let $\mathcal{E}$ be an ample k-jet spanned vector bundle of rank r on a smooth projective variety X of dimension n. Then*

1. $\det(\mathcal{E})^n = (rk)^n$ *if and only if* $(X, \mathcal{E}) \cong (\mathbb{P}^n, \oplus^r \mathcal{O}_{\mathbb{P}^n}(k))$*;*
2. $\det(\mathcal{E})^n \geq (rk)^{n-1}(rk+1)$ *otherwise.*

Proof. Since $\det(\mathcal{E})$ is rk-jet ample by Corollary (2.4) and Remark (2.5), we have $(\det(\mathcal{E}))^n = (rk)^n$ if and only if $(X, \det(\mathcal{E})) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(rk))$ by [4, Corollary (3.1)]; note in fact that the proof of Theorem (3.1) and Corollary (3.1) in [4] makes only use of the jet spannedness. Now, the same argument as in the proof of (3.1) shows that $\mathcal{E} \cong \oplus^r \mathcal{O}_{\mathbb{P}^n}(k)$ in this case. This proves 1).

Thus we can assume $(X, \det(\mathcal{E})) \not\cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(rk))$, so that 2) follows again from [4, Theorem (3.1)]. Q.E.D.

Building up on the argument of [10, §1], we can now prove the following result for jets supported on a single point (compare with [3, (2.4)] and [14] for related results in the case of a very ample vector bundle).

Theorem 3.4 *Let $\mathcal{E}$ be an ample k-jet spanned vector bundle of rank r on a smooth projective variety of dimension n. Assume $kr \geq n$. Then $K_X + \det(\mathcal{E})$ is $(kr-n)$-jet spanned unless $(X, \mathcal{E}) \cong (\mathbb{P}^n, \oplus^r \mathcal{O}_{\mathbb{P}^n}(k))$.*

Proof. We have to show that for every $x \in X$, the map

$$H^0(K_X \otimes \det(\mathcal{E})) \to H^0((K_X \otimes \det(\mathcal{E}))/\mathfrak{m}_x^{rk-n+1})$$

is surjective. Let $\pi : Y \to X$ be the blowing up of X at x with $E = \mathbb{P}^{n-1}$ the exceptional divisor. Notice that $\det(\mathcal{E})$ is rk-jet spanned by (2.4) and Remark (2.5). Hence we can apply [4, Lemma (3.1)], whose proof only uses jet spannedness, to conclude that $\pi^* \det(\mathcal{E}) - rkE$ is spanned. Moreover, by (3.3), one has $(\pi^* \det(\mathcal{E}) - rkE)^n = (\det(\mathcal{E}))^n - (rk)^n > 0$ unless $(X, \mathcal{E}) \cong (\mathbb{P}^n, \oplus^r \mathcal{O}_{\mathbb{P}^n}(k))$.

Thus by Leray's spectral sequence and the Kawamata-Viehweg vanishing theorem (see e.g., [7]), we have

$$\begin{aligned} H^1(K_X \otimes \det(\mathcal{E}) \otimes \mathfrak{m}_x^{rk-n+1}) &= H^1(\pi^*(K_X + \det(\mathcal{E})) - (rk-n+1)E) \\ &= H^1(K_Y + \pi^* \det(\mathcal{E}) - rkE) = 0 \end{aligned}$$

Q.E.D.

The above result gives some evidence to expect the following conjectures to be true in the line bundle case.

Conjecture 3.5 *Let $\mathcal{L}$ be a k-jet ample line bundle on a smooth n-dimensional projective variety X, $n \geq 2$. Assume $k \geq n$. Then $K_X + \mathcal{L}$ is $(k-n)$-jet ample unless $(X, \mathcal{L}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$.*

Note that the following conjecture is true for surfaces in the classical case $k = 1$.

Conjecture 3.6 *Let $\mathcal{L}$ be a k-jet ample line bundle on a smooth n-dimensional projective variety X, $n \geq 2$. Assume $k \geq n-1$. Then $K_X + \mathcal{L}$ is $(k-n+1)$-jet spanned (perhaps even $(k-n+1)$-jet ample) unless either $(X, \mathcal{L}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a))$, with $a = k, k+1$, or $(X, \mathcal{L}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(k))$, $\mathcal{Q}$ hyperquadric in $\mathbb{P}^{n+1}$, or X is a $\mathbb{P}^{n-1}$-bundle, $p : X \to C$, over a smooth curve C and $\mathcal{L}_F \approx \mathcal{O}_{\mathbb{P}^{n-1}}(k)$ for any fiber $F \cong \mathbb{P}^{n-1}$ of p.*

Remark 3.7 Note that if $\mathcal{L}$ is a k-jet ample line bundle on a smooth surface X with $k \geq n+1 = 3$, then $K_X + \mathcal{L}$ is very ample by Reider's theorem. Indeed, from [4, Theorem (3.1)] we have $\mathcal{L}^2 \geq (n+1)^2 + (n+1) = 12$. If $K_X + \mathcal{L}$ is not very ample we know from Reider's theorem (see e.g., [5, (8.5.1)]) that there exists an effective curve C on X such that

$$\mathcal{L} \cdot C - 2 \leq C \cdot C < \mathcal{L} \cdot C/2 < 2.$$

Thus $C \cdot C = 1$ and $\mathcal{L} \cdot C \leq 3$. This contradicts the Hodge inequality $(C \cdot C)(\mathcal{L} \cdot \mathcal{L}) \leq (\mathcal{L} \cdot C)^2$.

4 Comparing definitions

In this section we compare the case $k = 1$, i.e., the 1-jet ampleness, with some other positivity properties given in the literature. The following two notions of "very ample" vector bundles have been introduced for "ad hoc" settings and, as we will see, are not equivalent. We heard the second notion to be attributed to Lazarsfeld (see e.g., [11, §3]).

Let X be a smooth projective variety. A vector bundle $\mathcal{E}$ on X is said to be *very ample* if the tautological line bundle $\xi_{\mathcal{E}}$ of $\mathbb{P}(\mathcal{E})$ is very ample.

A vector bundle $\mathcal{E}$ on X is said to be *strongly very ample* if there exists a very ample line bundle $\mathcal{L}$ on X such that $\mathcal{E} \otimes \mathcal{L}^{-1}$ is generated by its global sections. For example, the tangent bundle of $\mathbb{P}^n$ is strongly very ample.

Note that for a line bundle, $\mathcal{M}$, very ampleness and strong very ampleness coincide since $\mathcal{M} \otimes \mathcal{M}^{-1} = \mathcal{O}_X$ is spanned (see also •) below).

We want to show that 1-jet ampleness is a "middle point" definition between the previous two, i.e.,

- strongly very ample $\Longrightarrow$ 1-jet ampleness $\Longrightarrow$ very ampleness.

Proposition 4.1 *Let $\mathcal{E}$ be a strongly very ample vector bundle on a smooth projective variety X. Then $\mathcal{E}$ is 1-jet ample.*

Proof. Since $\mathcal{E}$ is strongly very ample, there exists a very ample line bundle $\mathcal{L}$ such that $\mathcal{E} \otimes \mathcal{L}^{-1}$ is spanned. Therefore Proposition (2.3) implies that $\mathcal{E} = (\mathcal{E} \otimes \mathcal{L}^{-1}) \otimes \mathcal{L}$ is 1-jet ample (recall that for line bundles, 1-jet ampleness is equivalent to very ampleness). Q.E.D.

Proposition 4.2 *Let $\mathcal{E}$ be a 1-jet ample vector bundle on a smooth projective variety X. Then $\mathcal{E}$ is very ample.*

Proof. Let $r := \operatorname{rank}(\mathcal{E})$, $n := \dim X$ and let $\xi_{\mathcal{E}}$ be the tautological line bundle of the $\mathbb{P}^{r-1}$-bundle $p : \mathbb{P}(\mathcal{E}) \to X$. We have to show that $\xi_{\mathcal{E}}$ is very ample.

Since $\mathcal{E}$ is spanned by global sections, it follows that $\xi_{\mathcal{E}}$ is spanned by global sections. Since no less than all the sections of $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ span $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$, we see that the mapping $\phi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}_{\mathbb{C}}$, associated to $|\xi_{\mathcal{E}}|$, embeds all fibers of p. Thus we must show that

1. given two distinct points x, y of $\mathbb{P}(\mathcal{E})$ with $x' := p(x) \neq y' := p(y)$, it follows that $\phi(x) \neq \phi(y)$;
2. given a point $x \in \mathbb{P}(\mathcal{E})$ and a nonzero tangent vector τ_x at x, then the differential $d\phi(\tau_x)$ is not zero.

Let x, y be two distinct points of $\mathbb{P}(\mathcal{E})$ with $x' := p(x) \neq y' := p(y)$, and note that by the 1-jet ampleness of $\mathcal{E}$ the map $H^0(\mathcal{E}) \to \mathcal{E}_{x'} \oplus \mathcal{E}_{y'}$ is onto. Thus we can choose sections s_1, s_2 of $H^0(\mathcal{E})$ such that, regarding s_1, s_2 as sections of $H^0(\xi_{\mathcal{E}})$ under the isomorphism $H^0(\mathcal{E}) \cong H^0(p_*\xi_{\mathcal{E}}) \cong H^0(\xi_{\mathcal{E}})$, one has $s_1(x) = 0 \neq s_2(x)$ and $s_2(y) = 0 \neq s_1(y)$. To see this recall that x corresponds to a one dimensional vector subspace of $\mathcal{E}^*_{x'}$ and y corresponds to a one dimensional vector subspace of $\mathcal{E}^*_{y'}$, and under the isomorphism $H^0(\xi_{\mathcal{E}}) \cong H^0(\mathcal{E})$ sections of $\mathcal{E}$ are identified with linear forms on the fibers of $\mathcal{E}^*$. Thus $\phi(x) \neq \phi(y)$.

Now take any point $x \in \mathbb{P}(\mathcal{E})$. Using the fact that $\mathcal{E}$ is 1-jet ample, we have that the map $H^0(\mathcal{E}) \to \mathcal{E}/m_x^2$ is onto. Thus we can find global sections of $\mathcal{E}$ $e_i, e_{i,j}$ with $i = 1, \ldots, r$ and $j = 1, \ldots, n$ such that

1. the e_i span $\mathcal{E}_{x'}$, with $e_i(x) = 0$ for $i > 1$, $e_1(x) \neq 0$;
2. $e_{i,j}(x') = 0$ for all i, j, and
3. with respect to a local trivialization of $\mathcal{E}$ and local coordinates z_j in a neighborhood of x' with x' as origin, we have

$$\frac{\partial e_{i,j}}{\partial z_k}(x') = \delta_k^j e_i(x')$$

   where δ_k^j is 0 unless $k = j$, in which case it is 1.

Using these sections it is straightforward to check that the differential of ϕ at x has rank $n + r - 1$. Therefore the differential of ϕ maps the tangent space of $\mathbb{P}(\mathcal{E})$ at x isomorphically onto its image. We thus conclude that $\mathcal{E}$ is very ample. Q.E.D.

We provide examples to show that the three "very ampleness" notions considered above are in fact not equivalent.

Let us start showing that the 1-jet ampleness is stronger than very ampleness.

Example 4.3 There are examples of scrolls $V = \mathbb{P}(\mathcal{E})$ in $\mathbb{P}^5$, where $\mathcal{E}$ is a rank 2 vector bundle on a surface X, such that the tautological line bundle $\xi_{\mathcal{E}}$ is very ample, and embeds V in $\mathbb{P}^5$, but $\mathcal{E}$ is not 1-jet ample.

From the lists of low degree threefolds in $\mathbb{P}^5$ (see e.g., [2, Chapter 6]), consider the following examples.

1. $V = \mathbb{P}(\mathcal{E})$ is of degree 6 and $\mathcal{E}$ is a rank 2 vector bundle over $X = \mathbb{P}^2$ given by the exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to \mathcal{J}_{\mathcal{Z}}(4) \to 0,$$
where $\mathcal{Z} = x_1 + \cdots + x_{10}$ is a reduced 0-cycle of $\mathbb{P}^2$ supported on 10 distinct points in general position;
2. $V = \mathbb{P}(\mathcal{E})$ is of degree 7 and $\mathcal{E}$ is a rank 2 vector bundle over the blowing up $X = \mathbb{P}^2(x_1, \ldots, x_6)$ of $\mathbb{P}^2$ at six points $x_1, \ldots, x_6$ in general position;
3. $V = \mathbb{P}(\mathcal{E})$ is a $\mathbb{P}^1$-bundle of degree 9 over a minimal $K3$ surface X.

Note that in each case $\mathcal{E}$ is not 1-jet ample. Assume otherwise. Then in particular $\mathcal{E}$ is very ample, and hence ample, by (4.2). Therefore $h^0(X, \mathcal{E}) \geq 6$ by (3.2), 1). Since $h^0(\mathbb{P}(\mathcal{E}), \xi_{\mathcal{E}}) = h^0(X, \mathcal{E}) = 6$ in each of the examples 1), 2), 3), we would have $X \cong \mathbb{P}^2$ and $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ by (3.2), 2), which is not the case.

More generally, $\mathbb{P}^d$-bundles of "middle dimension" in $\mathbb{P}^n$ are other examples.

Example 4.4 Consider a $\mathbb{P}^d$-bundle $V = \mathbb{P}(\mathcal{E})$, $\mathcal{E}$ a rank $d+1$ vector bundle on a smooth projective variety X, embedded in $\mathbb{P}^n$ with either $\dim V = \frac{n}{2}$ if n is even or $\dim V = \frac{n+1}{2}$ if n is odd. Thus $\mathcal{E}$ is very ample but is not 1-jet ample. Indeed, otherwise, we would have by (3.2), 1), the numerical contradictions
$$n+1 = h^0(V, \xi_{\mathcal{E}}) = h^0(X, \mathcal{E}) \geq (d+1)\left(\frac{n}{2}+1\right) \geq n+2, \quad \text{for } n \text{ even, or}$$
$$n+1 = h^0(V, \xi_{\mathcal{E}}) = h^0(X, \mathcal{E}) \geq (d+1)\left(\frac{n+1}{2}+1\right) \geq n+3, \quad \text{for } n \text{ odd.}$$

Let us show that strong very ampleness is stronger than 1-jet ampleness.

Example 4.5 We construct here an example of 1-jet ample vector bundle which is not strongly very ample. Let C be a nonhyperelliptic curve of genus $g = 5$ and let D be a non-effective divisor of degree 3 on C. Consider the vector bundle $\mathcal{E} := K_C \oplus K_C(D)$.

Note that K_C and $K_C(D)$ are very ample line bundles on C and hence they are 1-jet ample (see (1.2)). From Proposition (2.1) it thus follows that $\mathcal{E}$ is a 1-jet ample rank two vector bundle on C. Take a very ample line bundle $\mathcal{L}$ on C. Since there are no smooth curves of genus 5 in $\mathbb{P}^2$, we must have that $\Gamma(\mathcal{L})$ embeds C in $\mathbb{P}^N$ with $N \geq 3$. Notice that, by Castelnuovo's bound on the genus, $\mathcal{L}$ must be of degree ≥ 7.

If $\mathcal{E} \otimes \mathcal{L}^{-1} = (K_C \otimes \mathcal{L}^{-1}) \oplus (K_C(D) \otimes \mathcal{L}^{-1})$ is spanned then, since $\deg(K_C \otimes \mathcal{L}^{-1}) \leq 1$, we conclude that $\mathcal{L} \approx K_C$. Thus $\mathcal{E} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_C \oplus \mathcal{O}_C(D)$ is not spanned since D is not effective. Therefore $\mathcal{E}$ is not strongly very ample.

Remark 4.6 The previous example shows that strong very ampleness is not a functorial property. In fact $\mathcal{E} = K_C \oplus K_C(D)$ as in (4.5) is not strongly very ample, but it is the direct sum of two very ample, i.e., strongly very ample, line bundles.

Note also that if $\mathcal{E}$, $\mathcal{F}$ are strongly very ample vector bundles on a variety X, then, as it clearly follows from the definition, $\mathcal{E} \otimes \mathcal{F}$ is strongly very ample.

References

[1] M.C. Beltrametti, P. Francia and A.J. Sommese, On Reider's method and higher order embeddings, Duke Math. J. 58 (1989), 425–439.

[2] M.C. Beltrametti, M. Schneider and A.J. Sommese, *Some special properties of the adjunction theory for* 3*-folds in* $\mathbb{P}^5$, Memoirs of A.M.S., v. 116, n. 554 (1995).

[3] M.C. Beltrametti, M. Schneider and A.J. Sommese, Chern inequalities and spannedness of adjoint bundles, Israel Mathematical Conference Proceedings, Bar-Ilan, 1993 (ed. by M. Teicher), v. 9, A.M.S. (1996), 97–107.

[4] M.C. Beltrametti and A.J. Sommese, On k-jet ampleness, *Complex Analysis and Geometry*, ed. by V. Ancona and A. Silva, The University Series in Mathematics, Plenum Press, (1993), 355–376.

[5] M.C. Beltrametti and A.J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, Expositions in Mathematics 16 (1995), W. de Gruyter, Berlin.

[6] M.C. Beltrametti and A.J. Sommese, Notes on embeddings of blowups, Journal of Algebra 186 (1996), 861–871.

[7] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the Minimal Model Problem, in *Algebraic Geometry, Sendai 1985*, Adv. Stud. Pure Math. 10 (1987), 283–360.

[8] A. Kumpera and D. Spencer, *Lie Equations; Vol. I: General Theory*, Ann. of Math. Stud. 73 (1972), Princeton University Press, Princeton, N.J.

[9] A. Langer, A note on k-jet ampleness on surfaces, preprint.

[10] A. Lanteri, M. Palleschi and A.J. Sommese, Very ampleness of $K_X \otimes \mathcal{L}^{\dim X}$ for ample and spanned line bundles $\mathcal{L}$, Osaka J. Math. 26 (1989), 647–664.

[11] R. Lazarsfeld, Some applications of the theory of positive vector bundles, *Complete Intersections*, CIME course 1983, Acireale (Catania) ed. by S. Greco and R. Strano, Lecture Notes in Math. 1092 (1984), 29–61, Springer-Verlag, New York.

[12] C. Okonek, M. Schneider, and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Progr. Math. 3 (1980), Birkhäuser, Boston.

[13] A.J. Sommese, Compact complex manifolds possessing a line bundle with a trivial jet bundle, Abh. Math. Sem. Univ. Hamburg 47 (1978), 79–91.

[14] Y.-G. Ye and Q. Zhang, On ample vector bundles whose adjunction bundles are not numerically effective, Duke Math. J. 60 (1990), 671–687.

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