CHERN INEQUALITIES AND SPANNEDNESS OF THE ADJOINT BUNDLE*

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1 Introduction

Let $\mathcal{E}$ be a vector bundle of rank r on an irreducible projective variety X, with Chern classes $c_1(\mathcal{E}), \ldots, c_r(\mathcal{E})$. A weighted homogeneous polynomial, $P(c_1, \ldots, c_r)$, of weighted degree m in variables $c_1, \ldots, c_r$ of weights $1, \ldots, r$ is said to be positive if $P(c_1(\mathcal{E}), \ldots, c_r(\mathcal{E})) \cdot Z > 0$ for any irreducible m-dimensional subvariety Z of X.

A fundamental result of Fulton and Lazarsfeld (see [FL], [F] and §1) is that for ample $\mathcal{E}$ the set of positive polynomials coincides with the set of sums with non-negative coefficients (not all zero) of Schur polynomials. It is an intriguing question what the actual lower bounds are for appropriately positive bundles.

Our interest in this question arose from our work in [BSS]. There we used the criteria of Ein-Lazarsfeld [EL] for spannedness of adjoint bundles on smooth 3-folds to study the spannedness of a variety of adjoint bundles, including bundles of the form $K_X \otimes \det \mathcal{E}$. To use the Ein-Lazarsfeld criteria it was necessary to prove lower bounds for certain expressions in the Chern classes of ample and spanned vector bundles on smooth 3-folds. Ballico ([Ba1] and [Ba2]) makes the very nice conjecture that

*to Professor Hirzebruch on his 65th birthday

$$c_{i_1}(\mathcal{E})\cdots c_{i_k}(\mathcal{E}) \geq \binom{r}{i_1}\cdots\binom{r}{i_k}$$

$$\text{for } 0 \leq i_j \leq r,\ i_1+\cdots+i_k = n$$

with equality ha ppening only when $X \cong \mathbb{P}^n$ and $\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(1)$ In this paper we formulate a conjecture for lower bounds of positive polynomials in Chern classes which includes the inequality part of Ballico's conjecture.

Conjecture *Let X be an n-dimensional smooth projective variety and let $\mathcal{E}$ be an ample and spanned vector bundle on X. Then for any positive polynomial P of weight n in the Chern classes of $\mathcal{E}$ we have*

$$P(c_1(\mathcal{E}),\ldots,c_n(\mathcal{E})) \geq P\left(r, \binom{r}{2}, \ldots, \binom{r}{n}\right).$$

We prove our conjecture for very ample $\mathcal{E}$, i.e. vector bundles such that the tautological bundle $\xi_{\mathcal{E}} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ is very ample.

Theorem *Let X be an n-dimensional smooth projective variety and let $\mathcal{E}$ be a rank r very ample vector bundle on X. Let $P \in \mathbb{Z}[c_1(\mathcal{E}),\cdots,c_r(\mathcal{E})]$ be a positive polynomial of weight n in the Chern classes of $\mathcal{E}$. Then we have the inequality*

$$P(c_1(\mathcal{E}),\ldots,c_n(\mathcal{E})) \geq P\left(r, \binom{r}{2}, \ldots, \binom{r}{n}\right).$$

The proof would work for vector bundles $\mathcal{E}$ which are only ample and spanned if we could prove the natural conjecture that given an ample and spanned vector bundle $\mathcal{E}$ on an irreducible projective variety X, there is at least one fiber F of $\mathbb{P}(\mathcal{E}) \to X$ such that $\pi^*\xi_{\mathcal{E}} - \pi^{-1}(F)$ is nef where $\pi : \widetilde{\mathbb{P}}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ is the blow up of $\mathbb{P}(\mathcal{E})$ along F.

Examples based on an example constructed in [BSS] show that Ballico's conjecture can fail for ample bundles which are not spanned.

Since the first lower bounds that we obtained in [BSS] for certain polynomials in the Chern classes of vector bundles $\mathcal{E}$ on 3-folds X arose in the investigation of when $K_X \otimes \det(\mathcal{E})$ was spanned, it is interesting to us what we can say about the spannedness of $K_X \otimes \det(\mathcal{E})$ for general smooth X using the information obtained in the course of studying our conjectured lower bound. Using this new information and some work [BS] on varieties with many linear subspaces, we show that given a very ample vector bundle $\mathcal{E}$, then $K_X \otimes \det(\mathcal{E})$ is spanned when $\mathrm{rank}(\mathcal{E}) \geq \dim X + 1$. We also show that if this fails for $\mathrm{rank}(\mathcal{E}) = \dim X$, then the lower bound in the conjecture of Ballico is taken on. The proof of this result leads to a spannedness result for adjoint bundles on varieties with many linear subspaces, e.g. scrolls, quadric fibrations, and non-degenerate projective submanifolds of $\mathbb{P}_n$ with dual varieties of codimension at least $\dim X/3 + 1$.

Throughout the paper we use standard notation in algebraic geometry. Let us recall the following. For any sheaf $\mathcal{F}$ on a complex variety X, we denote by $h^i(X, \mathcal{F})$ the complex dimension of the vector space $H^i(X, \mathcal{F})$. For a line bundle $\mathcal{L}$ we denote by $|\mathcal{L}|$ the complete linear system associated to $\mathcal{L}$ and by $\Gamma(\mathcal{L})$ the space of the global sections of $\mathcal{L}$. We denote by " $\approx$ " the linear equivalence of line bundles. As usual, line bundles and divisors are used with little (or no) distinction. Hence we shall freely switch from the multiplicative to the additive notation and vice versa.

Let us recall a few definitions we use. Let $\mathcal{L}$ be a line bundle on a n-dimensional variety, X. We say that $\mathcal{L}$ is *numerically effective*, or *nef* for short, if $\mathcal{L} \cdot C \geq 0$ for all effective curves C on X, and in this case $\mathcal{L}$ is said to be *big* if $c_1(\mathcal{L})^n > 0$ where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$. We say that $\mathcal{L}$ is *spanned* if it is spanned by $\Gamma(\mathcal{L})$ at all points of X. A vector bundle $\mathcal{E}$ is said to be *numerically effective*, or *nef* for short (respectively *ample*; respectively *very ample*) if the tautological bundle $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ is nef (respectively ample; respectively very ample).

Let $p \in \mathbb{Z}[X_1, \ldots, X_r]$ be a homogeneous polynomial in r variables. The *weight* of a monomial $X_1^{\gamma_1} \cdots X_s^{\gamma_s}$ is defined to be $\gamma_1 + 2\gamma_2 + \cdots + s\gamma_s$.

Let $\mathcal{E}$ be a vector bundle of rank r on a variety X. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of an integer m in integers $\leq r$:

$$r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0, \quad \sum_{i=1}^{k} \lambda_i = m.$$

We define the Schur polynomial $S_\lambda(\mathcal{E}) \in \mathbb{Z}[c_1(\mathcal{E}), \ldots, c_r(\mathcal{E})]$ in the Chern classes of $\mathcal{E}$ of weight m as the $m \times m$ determinant

$$S_\lambda(\mathcal{E}) = \det(c_{\lambda_i + j - i}(\mathcal{E}))_{1 \leq i,j \leq m}.$$

Examples include $c_m(\mathcal{E})$ for $\lambda = (m, 0, \ldots, 0)$ and $s_m(\mathcal{E})$ for $\lambda = (1, \ldots, 1)$. Here s_m is the m-th Segre class, i.e. the m-th component of the total Segre class

$$s(\mathcal{E}) = c(\mathcal{E}^*)^{-1}.$$

Note that, since $S_\lambda(\mathcal{E})$ has weight m, if a Chern class $c_i(\mathcal{E})$ occurs in a monomial of $S_\lambda(\mathcal{E})$, then $i \leq m$. Moreover, by convention, $c_0(\mathcal{E}) = 1$, $c_i(\mathcal{E}) = 0$ for either $i < 0$ or $i > r$. A theorem of Fulton-Lazarsfeld says that for $\mathcal{E}$ ample $S_\lambda(\mathcal{E})$ is a *positive polynomial*, i.e. $S_\lambda(\mathcal{E}) \cdot Z > 0$ for all $Z \subset X$ integral subschemes of dimension m (see [F], 12.1.7).

Let X be a smooth connected n-dimensional variety and let L be a very ample line bundle on X. We say that (X, L) is a *scroll* (respectively a *quadric fibration*) over a variety Y of dimension m if there exists a surjective morphism with connected fibers $\varphi : X \to Y$ such that $K_X + (n-m+1)L \approx \varphi^*\mathcal{L}$ (respectively $K_X + (n-m)L \approx \varphi^*\mathcal{L}$) for an ample line bundle $\mathcal{L}$ on Y. We say that (X, L) is a $\mathbb{P}^d$-*bundle* over Y if there is a surjective morphism $\varphi : X \to Y$ such that each fiber F is isomorphic to $\mathbb{P}^d$ and $L_F \cong \mathcal{O}_{\mathbb{P}^d}(1)$. Let $\mathcal{D}$ be the *discriminant locus* of (X, L), i.e.

$$\mathcal{D} := \{H \in |L|\,,\ H \text{ singular}\}.$$

$\mathcal{D}$ is irreducible and the codimension, $k+1$, of $\mathcal{D}$ in $\mathbb{P}(\Gamma(L))$ depends only on (X, L). The *defect*, $\operatorname{def}(X, L)$, of the pair (X, L) is defined to be this integer k (see e.g. [BFS]).

We would like to thank the University of Bayreuth and the DFG-Schwerpunktprogramm “Komplexe Mannigfaltigkeiten” for making our work possible. The first and the third author would also like to thank the Max-Planck-Institut für Mathematik in Bonn for its support during the 1992–93 academic year. The third author also thanks the National Science Foundation (DMS 93–02121) for their support.

2 Inequalities for Chern classes.

First, let us recall the following well known lemma due to Castelnuovo and Mumford [Mu]. In fact we need only a very special case of it.

Lemma 1 *Let L be a very ample line bundle on an irreducible projective variety X. Let $Y \subset X$ be an irreducible subvariety of degree d relative to L, i.e. $d = L^{\dim Y} \cdot Y$. If either Y is smooth or $Y \subset reg(X)$ and $codim_X Y = 1$,* *then $L^d \otimes \mathcal{I}_Y$ is spanned by global sections, where $\mathcal{I}_Y$ denotes the ideal sheaf of Y in X.*

Proof. See [Mu] or [BS]. We will apply the lemma only in the case that Y is a linear subspace of X with respect to L. In this case the assertion follows immediately from the Koszul resolution of $\mathcal{I}_Y$

$$0 \longrightarrow L^{-k} \longrightarrow \cdots \longrightarrow (L^{-1})^{\oplus k} \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

Here k denotes the codimension of Y in X. Exactly the same proof works also in case Y is a complete intersection of higher degree. Q.E.D.

The following consequence of the Lemma above was the starting observation for the paper

Corollary 1 *Let $\mathcal{E}$ be a rank r very ample vector bundle on an irreducible projective variety X. Let us denote by $\pi : \widetilde{X} \to X$ the blow up of X at a smooth point x. Let $E := \pi^{-1}(x)$ denote the exceptional divisor. Then $\pi^*\mathcal{E} \otimes \mathcal{I}_E$ is a spanned vector bundle on $\widetilde{X}$, where $\mathcal{I}_E$ denotes the ideal sheaf of E in $\widetilde{X}$.*

Proof. We have a commutative square

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \stackrel{\tilde{\pi}}{\longleftarrow} & \mathbb{P}(\pi^*\mathcal{E}) \\ p\downarrow & & \downarrow\tilde{p} \\ X & \stackrel{\pi}{\longleftarrow} & \widetilde{X} \end{array}$$

where p, $\tilde{p}$ are the bundle projections and $\tilde{\pi}$ is the blow up of $\mathbb{P}(\mathcal{E})$ along the fiber $F := p^{-1}(x)$. Let $\widetilde{E} := \tilde{\pi}^{-1}(F)$ and let ξ be the tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ of $\mathbb{P}(\mathcal{E})$. Then $\tilde{\xi} := \tilde{\pi}^*\xi$ is the tautological bundle $\mathcal{O}_{\mathbb{P}(\pi^*\mathcal{E})}(1)$ on $\mathbb{P}(\pi^*\mathcal{E})$. Since F is a linear $\mathbb{P}^{r-1}$ relative to ξ, by Lemma 1 we conclude that $\xi \otimes \mathcal{I}_F$ is spanned, $\mathcal{I}_F$ denoting the ideal sheaf of F in $\mathbb{P}(\mathcal{E})$. Since $\tilde{\xi} \otimes \mathcal{I}_{\widetilde{E}} \cong \tilde{\pi}^*(\xi \otimes \mathcal{I}_{\widetilde{F}})$, we conclude that $\tilde{\xi} \otimes \mathcal{I}_{\widetilde{E}}$ is spanned. It thus follows that $\tilde{p}_*(\tilde{\xi} \otimes \mathcal{I}_{\widetilde{E}})$ is spanned. Since

$$\tilde{p}_*(\tilde{\xi} \otimes \mathcal{I}_{\widetilde{E}}) \cong \tilde{p}_*(\tilde{\xi} \otimes \tilde{p}^*\mathcal{O}(-E)) \cong \tilde{p}_*\tilde{\xi} \otimes \mathcal{I}_E \cong \pi^*\mathcal{E} \otimes \mathcal{I}_E$$

we are done. Q.E.D.

The following general fact is the second step to prove the main Theorem below.

Proposition 1 *Let X be an n-dimensional irreducible projective variety and let $\mathcal{E}$ be a vector bundle of rank r on Y. Assume that there exists a smooth point $x_0 \in X$ such that $\pi^*\mathcal{E} \otimes \mathcal{O}(-E)$ is nef where $\pi : \widetilde{X} \to X$ is the blowing up of X at x_0 and $E = \pi^{-1}(x_0)$ denotes the exceptional divisor. Let $P \in \mathbb{Z}[c_1(\mathcal{E}), \dots, c_r(\mathcal{E})]$ be a positive polynomial of weight n in the Chern classes of $\mathcal{E}$. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n, $r \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$, $\sum_{i=1}^k \lambda_i = n$. Then we have the inequality*

$$P(c_1(\mathcal{E}), c_2(\mathcal{E}), \dots, c_n(\mathcal{E})) \geq P\left(r, \binom{r}{2}, \dots, \binom{r}{n}\right).$$

Proof. By taking a resolution of singularities we may assume that X is smooth. Since $P = \sum_\lambda a_\lambda S_\lambda$, it is enough to show that

$$S_\lambda(\mathcal{E}) = S_\lambda(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) \geq S_\lambda\left(r, \binom{r}{2}, \dots, \binom{r}{n}\right),$$

where $S_\lambda(\mathcal{E})$ is the Schur polynomial corresponding to the partition $\lambda = (\lambda_1, \dots, \lambda_n)$. Let $\mathcal{F} := \pi^*\mathcal{E} \otimes \mathcal{O}_{\widetilde{X}}(-E)$. We have for $k > 0$,

(1) $$c_k(\mathcal{F}) = \sum_{j=0}^{k} \binom{r-j}{k-j} c_j(\pi^*\mathcal{E}) \cdot c_1(-E)^{k-j}.$$

Since

(2) $$c_j(\pi^*\mathcal{E}) \cdot c_1(-E)^i = 0$$

for $j > 0, i > 0$, we get for $k > 0$,

(3) $$c_k(\mathcal{F}) = c_k(\pi^*\mathcal{E}) + \binom{r}{k} c_1(-E)^k.$$

By definition we have

$$S_\lambda\left(c_1(\mathcal{F}), \dots, c_n(\mathcal{F})\right) = \det \begin{pmatrix} c_{\lambda_1}(\mathcal{F}) & \cdots & c_{\lambda_1+n-1}(\mathcal{F}) \\ \vdots & & \vdots \\ c_{\lambda_n-n+1}(\mathcal{F}) & \cdots & c_{\lambda_n}(\mathcal{F}) \end{pmatrix}$$

Inserting (3) into the determinant and using equation (2) for $j > 0, i > 0$, we find

$$S_\lambda(c_1(\mathcal{F}), \ldots, c_n(\mathcal{F})) =$$

$$\det \begin{pmatrix} c_{\lambda_1}(\pi * \mathcal{E}) & \cdots & c_{\lambda_1+n-1}(\pi^*\mathcal{E}) \\ \vdots & & \vdots \\ c_{\lambda_n-n+1}(\pi^*\mathcal{E}) & \cdots & c_{\lambda_n}(\pi^*\mathcal{E}) \end{pmatrix}$$

$$+(-E)^n \det \begin{pmatrix} \binom{r}{\lambda_1} & \cdots & \binom{r}{\lambda_1+n-1} \\ \vdots & & \vdots \\ \binom{r}{\lambda_n-n+1} & \cdots & \binom{r}{\lambda_n} \end{pmatrix}$$

Since $S_\lambda(c_1(\pi^*\mathcal{E}), \ldots, c_n(\pi^*\mathcal{E})) = S_\lambda(c_1(\mathcal{E}), \ldots, c_n(\mathcal{E}))$ and using $(-E)^n = (-1)^n(E)^n = (-1)^n(-1)^{n-1} = -1$, we finally get

$$S_\lambda(c_1(\mathcal{F}), \ldots, c_n(\mathcal{F}) = S_\lambda(c_1(\mathcal{E}), \ldots, c_n(\mathcal{E})) - S_\lambda\left(r, \binom{r}{2}, \ldots, \binom{r}{n}\right).$$

Since $\mathcal{F}$ is a nef vector bundle we have by [DPS], (2.5) that

$$S_\lambda(c_1(\mathcal{F}), \ldots, c_n(\mathcal{F})) \geq 0.$$

This concludes the proof. Q.E.D.

Theorem 1 *Let X be an n-dimensional smooth projective variety and let $\mathcal{E}$ be a rank r very ample vector bundle on X. Let $P \in \mathbb{Z}[c_1(\mathcal{E}), \cdots, c_r(\mathcal{E})]$ be a positive polynomial of weight n in the Chern classes of $\mathcal{E}$. Then we have the inequality*

$$P(c_1(\mathcal{E}), \ldots, c_n(\mathcal{E})) \geq P\left(r, \binom{r}{2}, \ldots, \binom{r}{n}\right).$$

Proof. This follows from Corollary 1 and Proposition 1. Q.E.D.

Corollary 2 *Let X be an n-dimensional smooth projective variety and let $\mathcal{E}$ be a rank r very ample vector bundle on X. Then*

(1)

$$c_{i_1}(\mathcal{E})\cdots c_{i_k}(\mathcal{E}) \geq \binom{r}{i_1}\cdots\binom{r}{i_k}$$

$$\text{for}\ \ 0 \leq i_j \leq r,\ \ i_1+\cdots+i_k = n$$

(2)

$$c_1(\mathcal{E})^n \geq r^n$$

(3)

$$c_n(\mathcal{E}) \geq \binom{r}{n}$$

(4)

$$s_{i_1}(\mathcal{E})\cdots s_{i_k}(\mathcal{E}) \geq \binom{r+i_1-1}{i_1}\cdots\cdots\binom{r+i_k-1}{i_k}$$

$$\text{for}\ \ 0 \leq i_j \leq r,\ \ i_1+\cdots+i_k = n$$

(5)

$$s_n(\mathcal{E}) \geq \binom{r+n-1}{n}.$$

Proof. To prove (1), apply Theorem 1 with the weight n positive polynomial $P = c_{i_1}(\mathcal{E})\cdots c_{i_k}(\mathcal{E})$, $0 \leq i_j \leq r,\ i_1+\cdots+i_k = n$. Then (2) and (3) follow from (1) by taking $i_1 = \cdots = i_n = 1$ and $i_1 = \cdots = i_{n-1} = 0, i_n = n$ respectively. To show (5) we use first Theorem 1 to get the estimate

$$s_n(\mathcal{E}) \geq \det\begin{pmatrix} \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{n} \\ \binom{r}{0} & \binom{r}{1} & \cdots & \binom{r}{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \binom{r}{1} \end{pmatrix}.$$

But the value of this determinant is $\binom{r+n-1}{n}$, a fact well known in classical algebraic geometry, c.f. [S] or [ACGH], pp. 94–95. Finally (4) follows in a similar way as (5). Q.E.D.

The inequality (1) in Corollary 2 was conjectured by E. Ballico in [Ba1] in the more general case that $\mathcal{E}$ is ample and spanned. He proved this conjecture in the special case $n = 2, r = 3$.

It would be interesting to study the boundary cases (see also [Ba 2] for the case $n = 2, r = 3$).

Question 1 *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank r on a projective variety X of dimension n. If there is equality*

$$S_\lambda(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) = S_\lambda\left(r, \binom{r}{2}, \dots, \binom{r}{n}\right)$$

for some partition λ of n, does this imply $X \cong \mathbb{P}^n$, $\mathcal{E} \cong \mathcal{O}^n_{\mathbb{P}}(1)^{\oplus r}$?

Note that $c_1(\mathcal{E})^n = r^n$ implies equality for all S_λ. Let us explicitly note that Theorem 1 is false for a merely ample vector bundle $\mathcal{E}$. The example below was actually constructed in [BSS], Ex.13

in another context.

Example 1 *Let C be a smooth curve of genus ≥ 2. From [BSS], Lemma 12, we know that there exists for $r \geq 1$ an ample, stable, rank r vector bundle E on C with* $\det(E) \cong \mathcal{O}_C(x)$ *where $x \in C$. Let $X := \mathbb{P}^1 \times C$ with product projections $p : X \to \mathbb{P}^1, q : X \to C$. Let $\mathcal{E} := p^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes q^*E$. Then $\mathcal{E}$ is ample, not spanned for $r \geq 2$ (since E is not spanned) and $c_1(\mathcal{E})$ is a sum of vertical $\mathbb{P}^{1's}$ plus r vertical $C^{'s}$. Therefore $c_1^2(\mathcal{E}) = 2r$. Hence for $r \geq 3$ we cannot have $c_1^2(\mathcal{E}) \geq r^2$, showing that the inequalities (4) of Corollary 2 are not true.*

The above results suggest the following

Conjecture 1 *Let X be an n-dimensional smooth projective variety and let $\mathcal{E}$ be an ample and spanned vector bundle on X. Then for any positive polynomial P of weight n in the Chern classes of $\mathcal{E}$ we have*

$$P(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) \geq P\left(r, \binom{r}{2}, \dots, \binom{r}{n}\right).$$

3 Spannedness of adjoint bundles

In this section we prove the existence of global sections of adjunction bundles that span the adjunction bundles on linear subspaces.

Proposition 2 *Let L be a very ample line bundle on an n-dimensional, normal, Cohen-Macaulay projective variety X. Assume that the locus, $Irr(X)$, of nonrational singularities of X is at most 0-dimensional. Given a k-dimensional linear subspace $P \subset reg(X)$, it follows that the restriction map*

$$\Gamma(K_X \otimes L^t) \to \Gamma((K_X \otimes L^t)_P) \to 0$$

is onto if $t \geq n-k+1$. If the rational map associated to $|L-P|$ has an n-dimensional image, then the conclusion holds with $t \geq n-k$.

Proof. It is enough to show that $H^1(X, K_X \otimes L^t \otimes \mathcal{I}_P) = 0$ for $t \geq n-k+1$, where $\mathcal{I}_P$ is the ideal sheaf

defining P in X.

Let $\psi : \widetilde{X} \to X$ be the blow up of X along P. Then

$$K_{\widetilde{X}} \cong \psi^* K_X + (n-k-1)\mathcal{P},$$

where $\mathcal{P} := \psi^{-1}(P)$. Since P is a linear $\mathbb{P}^k$ relative to L, Lemma 1 applies to say that $L \otimes \mathcal{I}_P$ is spanned by global sections and hence it follows that $\mathcal{L} := \psi^* L - \mathcal{P}$ is spanned by global sections. Thus by the Kawamata - Viehweg vanishing theorem we get

$$H^1(\widetilde{X}, K_{\widetilde{X}} \otimes \mathcal{L}^s \otimes \psi^* L^{t-s}) = 0$$

for all $t \geq s \geq 1$ if $\mathcal{L}$ is big, and for all $t > s \geq 0$ in general. Therefore by the Leray spectral sequence and the projection formula, we conclude that

$$H^1(X, K_X \otimes L^t \otimes \psi_*((n-k-1-s)\mathcal{P})) = 0$$

for all $t \geq s \geq 1$ if $\mathcal{L}$ is big, and for all $t > s \geq 0$ in general. Note that $\psi_* \mathcal{O}_{\widetilde{X}}(-\mathcal{P}) = \mathcal{I}_P$. Then by choosing $s = n-k$ we are done. Q.E.D.

Remark 1 *Letting a be such that $\det N_P^X \cong \mathcal{O}_{\mathbb{P}^k}(a)$ in Proposition 2, we see that global sections of $\Gamma(K_X \otimes L^t)$ span $K_X \otimes L^t$ in a neighborhood of P if $t \geq \max\{k+1+a, n-k+1\}$. In fact*

$$(K_X + tL)_P \cong K_P - \det N_P^X + tL_P \cong \mathcal{O}_{\mathbb{P}^k}(-k-1-a+t)$$

is spanned if $t \geq k+1+a$, so that sections lift by Proposition 2 as soon as in addition $t \geq n-k+1$.

If the rational map associated to $|L-P|$ has an n-dimensional image, then we get the spannedness of $K_X \otimes L^t$ with $t \geq$ max $\{k+1+a, n-k\}$.

Let us discuss now some applications of the above results.

Corollary 3 *Let X be an n-dimensional smooth connected variety. Let L be a very ample line bundle on X. Assume that (X, L) is a scroll, $p : X \to Y$, over a variety Y of dimension m. Let $k = n - m$. Then $K_X + (k+1)L$ is spanned by global sections if $k \geq n/2$. The same is true if (X, L) is a $\mathbb{P}^k$-bundle.*

Proof. If (X, L) is a $\mathbb{P}^k$-bundle, then X is covered by linear $\mathbb{P}^k$'s, so we conclude by Proposition 2 and Remark 1 that $K_X + (k+1)L$ is spanned if $k+1 \geq n-k+1$ or $k \geq n/2$.

Let (X, L) be a scroll. Fix a point $x \in X$. We can choose a sequence F_i of general fibers of p such that the point x is contained in the limit of the F_i's. Let P be the limit of the F_i's in the Hilbert scheme of the F_i's. The fibers F_i are linear $\mathbb{P}^k$'s, so P is a linear $\mathbb{P}^k$ and $\det N_P^X \cong \mathcal{O}_P$. Thus Proposition 2 and Remark 1 apply again to give the result. Q.E.D.

Corollary 4 *Let $\mathcal{E}$ be a very ample vector bundle on a smooth projective manifold X, of dimension n. Then $K_X \otimes \det \mathcal{E}$ is spanned by global sections if $\operatorname{rank}(\mathcal{E}) \geq n+1$, or if $\operatorname{rank}(\mathcal{E}) = n$ and $c_1(\mathcal{E})^n > \operatorname{rank}(\mathcal{E})^n$.*

Proof. Let $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the tautological bundle of $\mathbb{P}(\mathcal{E})$. Let $r = \operatorname{rank}(\mathcal{E})$. Then $\mathbb{P}(\mathcal{E})$ is covered by linear $\mathbb{P}^{r-1}$'s , such that $N_{\mathbb{P}^{r-1}}^{\mathbb{P}(\mathcal{E})} \cong \mathcal{O}_{\mathbb{P}^{r-1}}$. Let $r \geq n+1$. By Proposition 2 and Remark 1 applied to $(\mathbb{P}(\mathcal{E}), \xi)$ and $P = \mathbb{P}^{r-1}$, it follows that $K_{\mathbb{P}(\mathcal{E})} \otimes \xi^r$ is spanned by global sections if $r \geq \dim \mathbb{P}(\mathcal{E}) - (r-1) = n+1$. Let $p : \mathbb{P}(\mathcal{E}) \to X$ be the bundle projection. Since

$$K_{\mathbb{P}(\mathcal{E})} \otimes \xi^r \cong p^*(K_X \otimes \det(\mathcal{E})),$$

the first part of the statement is proved.

Assume now $r = n$. Let $\pi : \widetilde{X} \to X$ be the blow up of X at a point $x \in X$. Let $\mathcal{I}_P$ be the ideal sheaf of $P := \pi^{-1}(x)$ in $\widetilde{X}$. By Corollary 1 we know that $\pi^*\mathcal{E} \otimes \mathcal{I}_P$ is spanned and therefore $\det(\pi^*(\mathcal{E}) \otimes \mathcal{I}_P) \cong \pi^* \det(\mathcal{E}) - nP$ is spanned. By Leray's spectral sequence and the projection formula we have

$$H^1(\widetilde{X}, K_{\widetilde{X}} \otimes \pi^* \det(\mathcal{E}) \otimes \mathcal{O}(-nP)) \cong H^1(\widetilde{X}, \pi^* K_X \otimes \pi^* \det(\mathcal{E}) \otimes \mathcal{O}(-P))$$

$$\cong H^1(X, K_X \otimes \det(\mathcal{E}) \otimes \mathcal{M}_x),$$

where $\mathcal{M}_x$ is the ideal sheaf of x in X. Thus by the Kawamata-Viehweg vanishing theorem, we conclude that

$$H^1(X, K_X \otimes \det(\mathcal{E}) \otimes \mathcal{M}_x) = 0 \tag{4}$$

as soon as $\pi^* \det(\mathcal{E}) - nP$ is nef and big. By the above, this is the case unless $(\pi^* \det(\mathcal{E}) - nP)^n = 0$, which is equivalent to $c_1(\mathcal{E})^n = r^n$. From (4) follows that the restriction map

$$\Gamma(X, K_X \otimes \det(\mathcal{E})) \to (K_X \otimes \det(\mathcal{E}))_x \cong \mathbb{C}$$

is onto and hence $K_X \otimes \det(\mathcal{E})$ is spanned in a neighborhood of x. Since this is true for all $x \in X$, $K_X \otimes \det(\mathcal{E})$ is spanned. Q.E.D.

Corollary 5 *Let L be a very ample line bundle on a projective manifold X, of dimension n. Let $\varphi : X \to Y$ be a quadric fibration. Let $m = \dim Y, f = n - m$. Then $K_X \otimes L^{n-m}$ is spanned for $f \geq \frac{2n}{3} + 1$.*

Proof. Let F be a general fiber of φ. Then either $\dim F = f = 2k$ or $\dim F = f = 2k+1$ and in both cases F contains a linear $\mathbb{P}^k$. We consider first the case $f = 2k$. By the adjunction formula we get

$$\det N^X_{\mathbb{P}^k} \cong K_{\mathbb{P}^k} - K_{X|\mathbb{P}^k} \cong K_{\mathbb{P}^k} - K_{F|\mathbb{P}^k} \cong \mathcal{O}_{\mathbb{P}^k}(k-1).$$

Fix a point $x \in X$. We can choose a sequence of such $\mathbb{P}^k$'s, contained in general fibers of φ, such that the point x is contained in the limit P of the $\mathbb{P}^k$'s in the Hilbert scheme of the $\mathbb{P}^k$'s. Then $\det N^X_P \cong \mathcal{O}_P(k-1)$. By Proposition 2 and Remark 1 we know that $K_Y + fL$ is spanned if $f = 2k \geq \max\{n-k+1, k-1\}$, which is equivalent to $2k \geq n-k+1$, or $k \geq (n+1)/3$, that is $f \geq 2(n-1)/3$.

In the case $f = 2k+1$ we have $\det N^X_{\mathbb{P}^k} \cong \mathcal{O}_{\mathbb{P}^k}(k)$ and exactly the same argument as above shows that $K_X + fL$ is spanned if $k \geq n/3$ or $f \geq (2n/3)+1$. Q.E.D.

For the definition of extremal ray and contraction of an extremal ray occurring in the Corollary below, we refer to [Mo].

Corollary 6 *Let L be a very ample line bundle on a smooth projective manifold X, of dimension n. If the defect $def(X,L) := k$ of (X,L) is $k \geq n/3$, then $K_X \otimes L^{\frac{n+k}{2}+1}$ is spanned by global sections and defines a morphism which is a contraction of an extremal ray.*

Proof. Since $k \geq 1$ it is a classical fact that there is a linear $\mathbb{P}^k$ passing through a general point of X and $K_{X|\mathbb{P}^k} \cong \mathcal{O}_{\mathbb{P}^k}((n-k-2)/2)$ or $\det N^X_{\mathbb{P}^k} \cong \mathcal{O}_{\mathbb{P}^k}((n-k)/2)$ (see [BFS], (0.3), [E], (2.4)). By a standard limit argument, as in the proof of Corollary 5, we see that there is a linear $\mathbb{P}^k$ through each point of X and the determinant of the normal bundle of this $\mathbb{P}^k$ is $\mathcal{O}_{\mathbb{P}^k}((n-k)/2))$. Thus Proposition 2 and Remark 1 apply to say that $K_X \otimes L^{\frac{n+k}{2}+1}$ is spanned by global sections if $((n+k)/2)+1 \geq n-k+1$, or $k \geq n/3$.

The last part of the statement follows now from [BFS] (0.12). Q.E.D.

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