Reducible hyperplane sections, II

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Abstract

Let $\widehat{X}$ be a smooth connected subvariety of complex projective space $\mathbb{P}^n$. The question was raised in [2] of how to characterize $\widehat{X}$ if it admits a reducible hyperplane section $\widehat{L}$. In the case in which $\widehat{L}$ is the union of $r \geq 2$ smooth normal crossing divisors, each of sectional genus zero, classification theorems were given for $\dim \widehat{X} \geq 5$ or $\dim X = 4$ and $r = 2$.

This paper restricts attention to the case of two divisors on a threefold, whose sum is ample, and which meet transversely in a smooth curve of genus at least 2. A finiteness theorem and some general results are proven, when the two divisors are in a restricted class including $\mathbb{P}^1$-bundles over curves of genus less than two and surfaces with nef and big anticanonical bundle. Next, we give results on the case of a projective threefold $\widehat{X}$ with hyperplane section $\widehat{L}$ that is the union of two transverse divisors, each of which is either $\mathbb{P}^2$, a Hirzebruch surface $\mathbb{F}_r$, or $\widetilde{\mathbb{F}_2}$.

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Introduction

This paper is a sequel of [2], which initiated the study of a connected submanifold $\widehat{X}$ of complex projective space, which has a reducible hyperplane section $\widehat{L}$. As $\dim \widehat{X}$ increases so does the simplicity of the characterization. In [2] a description is given of $(\widehat{X}, \widehat{L})$ for which $\widehat{L}$ decomposes as $\widehat{A}_1 + \cdots + \widehat{A}_r$ into $r \geq 2$ smooth components with normal crossings under the hypothesis that $h^1(\mathcal{O}_{\hat{A}_i})$ is equal to the sectional genus of $\widehat{A}_i$ for each i. A complete result for the cases $n = 4$ and $r = 2$; and for $n \geq 5$ was obtained. Further, in the case of $n = 3$ and $r = 2$ the situation in which the curve $A_1 \cap A_2$ has genus at most 1 was thoroughly analyzed. Here we investigate the more delicate issues presented by the following specialization of the question.

Problem. *Let $\widehat{L}$ be a very ample line bundle on a projective threefold $\widehat{X}$. Suppose that $\widehat{L}$ decomposes as a divisor into a sum $\widehat{L} = \widehat{A} + \widehat{B}$, where $\widehat{A}$ and $\widehat{B}$ are smooth connected surfaces meeting transversely along a smooth curve $h = \widehat{A} \cap \widehat{B}$. Assume that each of $\widehat{A}$, $\widehat{B}$ is either $\mathbb{P}^2$ or $\mathbb{F}_r$. Then describe $(\widehat{X}, \widehat{L})$.*

The curve h is connected [2, Corollary (2.3)]. We also call h the *hinge curve*.

In this paper we shall deal only with the situation when h has genus $g(h) \geq 2$: we refer to [2, Theorems (3.10), (3.11)] for the cases when $g(h) \leq 1$. We also refer to [2, 5, 7] for related results.

The organization of the paper is as follows. In §2, we present a general finiteness theorem for a threefold $\widehat{X}$ with an ample divisor $\widehat{L}$ of the form $\widehat{A} + \widehat{B}$, where $\widehat{A}$, $\widehat{B}$ are in a restricted class $\mathcal{C}$ of surfaces and meet transversely in a smooth curve of genus ≥ 2. The class $\mathcal{C}$ includes surfaces with nef and big anticanonical bundle; and $\mathbb{P}^1$-bundles over either $\mathbb{P}^1$ or and elliptic curve. The finiteness theorem asserts that there is an $\epsilon > 0$ such that the Kodaira dimension of $K_{\widehat{X}} + \left(\frac{1}{2} + \epsilon\right)\widehat{L}$ is $-\infty$. By a result of Fujita, this implies that $(\widehat{X}, \widehat{L})$ is a birational transform of members of an explicit list of very special pairs.

In §3, it was shown that if the divisors $\widehat{A}$, $\widehat{B}$ are $\mathbb{P}^2$ or scrolls over $\mathbb{P}^1$, then the restriction of the bundle $K_{\widehat{X}} + \widehat{L}$ to the divisors in big.

In §4, the Hodge Index type theorem for reducible divisors leads to the elimination the cases in which both $\widehat{A}$ and $\widehat{B}$ are among $\mathbb{P}^2$ and the singular quadric with an isolated singularity $\widetilde{\mathbb{F}_2}$.

Finally, in §5 we study the case when $\widehat{A}$ is $\mathbb{P}^2$, the Hirzebruch surface $\mathbb{F}_r$, or the singular quadric with isolated singularity $\widetilde{\mathbb{F}_2}$; and $\widehat{B} = \mathbb{F}_s$, under the extra assumption that $(\widehat{A}, \widehat{L}_{\widehat{A}})$, $(\widehat{B}, \widehat{L}_{\widehat{B}})$ are scrolls.

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Contents

1 Background Material

W e work over the complex field $\mathbb{C}$. Throughout the paper we deal with projective varieties V, and follow the usual notation of algebraic geometry. The book [1] is a good reference for standard results and notation of adjunction theory.

For a line bundle L on an irreducible normal variety V of dimension n the *sectional genus*, $g(L) = g(V, L)$, of (V, L) is defined by $2g(L) - 2 = (K_V + (n-1)L) \cdot L^{n-1}$.

By $\mathbb{F}_r$ with $r \geq 0$ we denote the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a section E taking on the minimal self intersection $E^2 = -r$ on the surface. By $\widetilde{\mathbb{F}}_2$ we denote $\mathbb{F}_2$ with the section, which has self intersection -2, blown down. Note that $\widetilde{\mathbb{F}}_2$ is isomorphic to any quadric hypersurface $Q \subset \mathbb{P}^3$, which has a single isolated singularity.

Let V be a normal r-Gorenstein (i.e., rK_V is a Cartier divisor) projective variety of dimension n and let D be a $\mathbb{Q}$-Cartier divisor on V such that $\kappa(D) = n$. We define the *unnormalized spectral value* of the pair (V, D) as

$$u(V, D) := \sup\{t \in \mathbb{Q} \mid \kappa(K_V + tD) = -\infty\}.$$

We refer to [1, §7.1] for details.

The following result follows immediately from [2, §2].

Lemma 1.1 *Let $\widehat{L}$ be an ample line bundle on a smooth projective 3-fold $\widehat{X}$. Assume that there are two smooth connected divisors $\widehat{A}$, $\widehat{B}$ on $\widehat{X}$. Assume that $\widehat{A} + \widehat{B} \in |\widehat{L}|$, that $\widehat{A}$ and $\widehat{B}$ are rational, and that $\widehat{A}$, $\widehat{B}$ intersect transversely in a smooth curve h. Then h is connected, and $h^1(\mathcal{O}_{\widehat{X}}) = h^2(\mathcal{O}_{\widehat{X}}) = 0$.*

2 A Finiteness Theorem

I n this section we prove a general finiteness theorem for pairs $(\widehat{X}, \widehat{L})$ consisting of an ample line bundle on a smooth projective threefold $\widehat{X}$, with $|\widehat{L}|$ containing a divisor $D = \widehat{A} + \widehat{B}$, having two irreducible components from a large class $\mathcal{C}$ of negative Kodaira dimension surfaces. The class $\mathcal{C}$ *consists of the normal connected Gorenstein projective surfaces S with the property, that given any smooth connected Cartier divisor C on S, it follows that either $h^1(\mathcal{O}_C) \leq 1$ or $K_S \cdot C \leq -1$.*

Lemma 2.1 *The class $\mathcal{C}$ includes:*

1. *normal Gorenstein surfaces with $-K_S$ nef and big; or*
2. *$\mathbb{F}_r$, $r \geq 0$, the r-th Hirzebruch surface; or*
3. *a $\mathbb{P}^1$ bundle over an elliptic curve.*

In cases 1) and 2), smooth connected Cartier divisors C with $h^1(\mathcal{O}_C) \geq 2$ satisfy $K_S \cdot C \leq -3$.

Proof. Let C be a smooth connected Cartier divisor of S, i.e., let C be a curve on S with C contained in S_{reg}, the smooth points of S. We assume that we are in the situation that $h^1(\mathcal{O}_C) \geq 2$, since otherwise there is nothing to show.

First assume that $-K_S$ is nef and big, and that the result is false, i.e., that $-K_S \cdot C \leq 2$. We know that $-K_S \cdot C = 0, 1, 2$. If $-K_S \cdot C = 0$, then we conclude, using the Hodge Index Theorem, that $C^2 \leq 0$, which contradicts $h^1(\mathcal{O}_C) \geq 2$. If $-K_S \cdot C = 1$, then we conclude that $C^2 \geq 3$, which contradicts the Hodge Index Theorem, i.e., $C^2 \leq C^2 K_S^2 \leq 1$. If $-K_S \cdot C = 2$, then we conclude that $C^2 \geq 4$, which gives equality in the Hodge Index Theorem, i.e., $4 \leq C^2 \leq C^2 K_S^2 \leq 4$. This implies that numerically $C \sim -K_S$, which implies the contradiction $K_S + C \sim 0$.

For S a Hirzebruch surface the result is a straightforward check.

Assume finally that S is a $\mathbb{P}^1$-bundle over an elliptic curve Y. In this case the section σ of minimal self-intersection satisfies $e := -\sigma^2 \geq -1$, and K_S is numerically equal to $-2\sigma - ef$ for a fiber of the induced projection $\pi : S \to Y$. Since we are assuming that

$h^1(\mathcal{O}_C) \geq 2$, we know that numerically $C = k\sigma + tf$ where $k \geq 2$. Moreover $K_S \cdot C \geq 0$ gives $ke - 2t = 2ke - ek - 2t \geq 0$. Since $C^2 = -ek^2 + 2kt$, we have the absurdity that $2 \leq 2g(C) - 2 = K_S \cdot C + C^2 = (1-k)(ke-2t) \leq 0$. Q.E.D.

One main result of the paper is the Finiteness Theorem (2.2). This theorem shows that, if the hinge curve h has genus $g(h) \geq 2$, the pair $(\widehat{X}, \widehat{L})$ belongs to an explicit list of very special cases described by Fujita (see [3, 4] and also [1, (7.8.1)]).

Note in the following that the hypothesis that h is connected is automatically satisfied if $\widehat{A}$ and $\widehat{B}$ are connected [2, Corollary (2.3)].

Theorem 2.2 (Finiteness theorem) *Let $\widehat{L}$ be an ample line bundle on a smooth projective 3-fold $\widehat{X}$. Assume that there are two divisors $\widehat{A}$, $\widehat{B}$ on $\widehat{X}$ from the class $\mathcal{C}$. Assume that $\widehat{A} + \widehat{B} \in |\widehat{L}|$ and that $\widehat{A}$, $\widehat{B}$ intersect transversely in a smooth connected curve h of genus $g(h) \geq 2$. Then $u(\widehat{X}, \widehat{L}) > \frac{1}{2}$. In particular, $\widehat{X}$ is of Kodaira dimension $-\infty$, and thus satisfies $h^3(\mathcal{O}_{\widehat{X}}) = 0$.*

Proof. For simplicity of notation, we omit $\hat{\ }$'s in this proof. The genus formula yields

$$(K_X + L) \cdot h = (K_X + A + B) \cdot A \cdot B = 2g(h) - 2, \tag{1}$$

or $(K_A + B_A) \cdot B_A = 2g(h) - 2$, and therefore, by definition of class $\mathcal{C}$, one has $B_A \cdot B_A \geq 2g(h) - 1$, and similarly $A_B \cdot A_B \geq 2g(h) - 1$. Then (1) gives

$$K_X \cdot h \leq -2g(h). \tag{2}$$

Now compute, for any real number ε, $0 < \varepsilon < \dfrac{1}{4g(h) - 2}$,

$$\begin{aligned}
\left(K_X + \left(\frac{1}{2} + \varepsilon\right) L\right) \cdot h &= \left(K_X + L - \left(\frac{1}{2} - \varepsilon\right) L\right) \cdot h \\
&= 2g(h) - 2 - \left(\frac{1}{2} - \varepsilon\right) \cdot L \cdot h \\
&\leq 2g(h) - 2 - \left(\frac{1}{2} - \varepsilon\right)(4g(h) - 2) \\
&= -1 + \varepsilon(4g(h) - 2) < 0.
\end{aligned}$$

Finally, for $h = A \cap B$ on X, we have the normal bundle decomposition $N_{h/X} = N_{h/A} \oplus N_{h/B}$ and $\deg(N_{h/A}) = B^2 \cdot A = B_A \cdot B_A \geq 2g(h) - 1$ by the above. It follows that $h^1(N_{h/A}) = 0$ and $N_{h/A}$ has not identically zero sections. Similarly for $N_{h/B}$. Then $N_{h/X}$ is generically spanned by its global sections and $h^1(N_{h/X}) = 0$. Thus general deformation theory implies that the union of the deformations of h on X contains an open set. Therefore the inequality $(K_X + (\frac{1}{2} + \varepsilon)L) \cdot h < 0$ proved above, shows that $u(X, L) > \dfrac{1}{2}$, cf., [1, (7.6.4)]. Q.E.D.

A little more can be said on the case of $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ or surfaces with nef and big anticanonical bundle.

Proposition 2.3 *Let $\widehat{L}$ be an ample line bundle on a smooth projective 3-fold $\widehat{X}$. Assume that there are two smooth divisors $\widehat{A}$, $\widehat{B}$ on $\widehat{X}$ which are either $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ or surfaces with nef and big anticanonical bundle. Assume that $\widehat{A}+\widehat{B} \in |\widehat{L}|$ and that $\widehat{A}$, $\widehat{B}$ intersect transversely in a smooth connected curve h. Then $H^0(K_{\widehat{X}}+\widehat{L}) \to H^0(K_h) \to 0$.*

Proof. Tensor the Koszul complex

$$0 \to \mathcal{O}_{\widehat{X}} \to \widehat{A} \oplus \widehat{B} \to \widehat{L} \to \widehat{L}_h \to 0$$

with $K_{\widehat{X}}$. Using the hypercohomology spectral sequence, we see that the desired result will follow if we show that $H^2(K_{\widehat{X}}) = H^1(K_{\widehat{X}}+\widehat{A}) = H^1(K_{\widehat{X}}+\widehat{B}) = 0$.

The assertion $H^2(K_{\widehat{X}}) = 0$ follows from Lemma (1.1). To see that $H^1(K_{\widehat{X}}+\widehat{A}) = 0$ consider the exact sequence

$$0 \to K_{\widehat{X}} \to K_{\widehat{X}}+\widehat{A} \to K_{\widehat{A}} \to 0.$$

Now use Lemma (1.1) and the fact that $\widehat{A}$ is rational. The argument for $H^1(K_{\widehat{X}}+\widehat{B}) = 0$ is identical. Q.E.D.

One consequence of Proposition (2.3) is that under the same hypotheses with the added assumption that $g(h) \geq 2$, it follows that the Kodaira dimension of $K_{\widehat{X}}+\widehat{L}$ is at least one. This implies that the Kodaira dimension of $K_{\widehat{X}}+2\widehat{L}$ is three, and also that the restriction of $K_{\widehat{X}}+2\widehat{L}$ to $\widehat{A}$ (or $\widehat{B}$) is nontrivial. Therefore [2, Theorems (3.6), (3.8)] specialize to the following result.

Theorem 2.4 *Let $\widehat{L}$ be an ample line bundle on a smooth projective 3-fold $\widehat{X}$. Assume that there are two smooth divisors $\widehat{A}$, $\widehat{B}$ on $\widehat{X}$ which are either $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ or surfaces with nef and big anticanonical bundle. Assume that $\widehat{A}+\widehat{B} \in |\widehat{L}|$ and that $\widehat{A}$, $\widehat{B}$ intersect transversely in a smooth connected curve h of genus ≥ 2. Then there is a surjective morphism $\phi : \widehat{X} \to X$, where X is a smooth projective 3-fold, such that:*

1. *ϕ expresses $\widehat{X}$ as the blowup of X at a finite set $\mathcal{F}$, and there is an ample line bundle L on X such that $\widehat{L} \cong \phi^*L - \phi^{-1}(\mathcal{F})$;*
2. *$K_{\widehat{X}}+2\widehat{L} \cong \phi^*(K_X+2L)$ where K_X+2L is ample;*
3. *K_X+L is either nef and big, or (X, L) is a conic fibration over a surface Y in the sense of adjunction theory [1], i.e., there exists a morphism $\nu : X \to Y$ with $K_X+L \cong \nu^*H$ for an ample line bundle H on a normal surface Y;*
4. *ϕ is an embedding in a neighborhood of h; and*
5. *$L = A+B$ where $A := \phi(\widehat{A})$ and $B := \phi(\widehat{B})$ are Cartier divisors meeting transversely in $\phi(h)$ and each having at most one point contained in the set $\mathcal{F}$.*

From now on we usually abuse notation, and let h to denote $\phi(h)$. We also write h_A (respectively h_B) to emphasize that we view h as a curve on A (respectively on B).

Lemma 2.5 *Let $(\widehat{X}, \widehat{L})$, (X, L), $\widehat{A}$, $\widehat{B}$, A, B be as in (2.4). Then*

1. $h^{i,0}(X) = 0$, $i = 1, 2, 3$;
2. $h^i(K_X + A) = h^i(K_X + B) = 0$ *for all* $i \geq 0$; *and*
3. *the restriction map gives the following isomorphisms*

$$H^0(K_X + L) \cong H^0(K_A + h_A) \cong H^0(K_B + h_B) \cong H^0(K_h).$$

Proof. Noting that the first reduction morphism, ϕ, of Lemma (2.4) is birational, the first assertion follows immediately from Lemma (1.1) and Theorem (2.2).

To prove 2), consider the exact sequence

$$0 \to K_X + B \to K_X + L \to K_A \to 0.$$

By the assumption on A, $h^0(K_A) = h^1(K_A) = 0$, $h^2(K_A) = 1$, $h^3(K_A) = 0$. Thus from the cohomology sequence associated to the sequence above we infer that $h^i(K_X + B) = 0$ (and by symmetry $h^i(K_X + A) = 0$) for all $i \geq 0$.

Item 3) follows immediately from the first two assertions. Q.E.D.

Theorem 2.6 *Let* $\widehat{L}$ *be an ample line bundle on a smooth projective* 3*-fold* $\widehat{X}$. *Assume that there are two smooth divisors* $\widehat{A}$, $\widehat{B}$ *on* $\widehat{X}$ *which are either* $\mathbb{P}^1$*-bundles over* $\mathbb{P}^1$ *or surfaces with nef and big anticanonical bundle. Assume that* $\widehat{A} + \widehat{B} \in |\widehat{L}|$ *and that* $\widehat{A}, \widehat{B}$ *intersect transversely in a smooth connected curve* h *of genus* $g(h) \geq 2$. *Let* X, A, B, L *be as in* (2.4). *Then* $H^0(K_X + L)$ *spans* $K_X + L$ *in a neighborhood of* $A + B$.

Proof. By Lemma (2.5), the desired spannedness of $K_X + L$ will follow from the spannedness of $K_A + h_A$ and $K_B + h_B$. From Theorem (2.4) we know that $K_X + L$ is nef (and hence $K_A + h_A$ and $K_B + h_B$ are also).

First assume that $\widehat{A}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Either the map ϕ of Theorem (2.4) is an isomorphism on $\widehat{A}$, in which case A is also a $\mathbb{P}^1$-bundle, or, by [2, Theorem (3.6), 2)], $\phi_{\widehat{A}}$ expresses $\widehat{A}$ as the blowup of A at one point. In this latter case, $\widehat{A}$ is the Hirzebruch surface $\mathbb{F}_1$, and $A := \phi(\widehat{A}) = \mathbb{P}^2$ (note that $\mathbb{F}_1$ is the only Hirzebruch surface with a -1 curve). Since $K_X + L$ is nef, $K_A + h_A$ is nef, and for either $\mathbb{P}^2$ or $\mathbb{P}^1$-bundles over $\mathbb{P}^1$, nef line bundles are spanned.

Now assume that $-K_{\widehat{A}}$ is nef and big. Note that $-K_A$ is also nef and big. Indeed, going to the first reduction map we have a birational morphism $\phi_{\widehat{A}} : \widehat{A} \to A$ where some disjoint -1 curves are collapsed. Writing $-K_{\widehat{A}} = K_{\widehat{A}} + 2(-K_{\widehat{A}})$, we see from the basepoint free theorem that $-NK_{\widehat{A}}$ is spanned for $N \gg 0$. Thus $-NK_A$ is spanned off the finite set equal to the image of the exceptional curves. This implies $-K_A$ is nef. Since $K_A^2 > K_{\widehat{A}}^2$, bigness is clear.

Consider the line bundle h_A. We would like to show by Reider's Theorem [6] that $K_A + h_A$ is spanned. Note that $h_A^2 = 2g(h_A) - 2 - K_A \cdot h_A \geq 2 + 3 = 5$ by the hypothesis $g(h_A) \geq 2$ and Lemma (2.1). Since h_A is a smooth curve of positive genus, and $K_A \cdot h_A < 0$, we conclude that h_A is nef and big. Therefore by Reider's Theorem, either $K_A + h_A$ is spanned, or there exists an effective Cartier divisor $\ell \subset A$ such that either $h_A \cdot \ell = 0$ with $\ell^2 = -1$, or $h_A \cdot \ell = 1$ with $\ell^2 = 0$.

In the former case, $K_A \cdot \ell < 0$, since $K_A \cdot \ell \leq 0$ and $K_A \cdot \ell + \ell^2$ is even. This contradicts the nefness of $K_A + h_A$.

Finally consider the case $h_A \cdot \ell = 1$ with $\ell^2 = 0$. Note that since ℓ is effective, we cannot have $-K_A \cdot \ell = 0$ by the usual Hodge index relation. Thus we have $K_A \cdot \ell < 0$. Since $K_A \cdot \ell + \ell^2$ is even, we have that $K_A \cdot \ell \leq -2$. This implies that $(K_A + h_A) \cdot \ell \leq -1$, which contradicts nefness of $K_A + h_A$. Q.E.D.

3 Some Birationality Results

3.1 (Working assumptions) Let $\widehat{L}$ be a very ample line bundle on a 3-fold $\widehat{X}$. Assume that there are two smooth transverse divisors $\widehat{A}$, $\widehat{B}$ on $\widehat{X}$ with $\widehat{A} + \widehat{B} \in |L|$ and $\widehat{A}, \widehat{B} \in \{\mathbb{P}^2, \mathbb{F}_r\}$. Assume that the hinge curve $h = \widehat{A} \cap \widehat{B}$ has genus $g(h) \geq 2$.

From Theorem (2.4), we know that there exists the first reduction (X, L), $\phi : \widehat{X} \to X$, with $K_X + 2L$ ample and $K_X + L$ nef. If $A = \phi(\widehat{A})$, $B = \phi(\widehat{B})$, then $A + B \in |L|$ and $A, B \in \{\mathbb{P}^2, \mathbb{F}_r\}$. Furthermore we know by 5) of Theorem (2.4), that neither $\widehat{A}$ nor $\widehat{B}$ is a fiber of ϕ and that A, B meet transversely along the curve $\phi(h)$ isomorphic to h.

Lemma 3.2 *Assumptions and notation as in* (3.1). *The complete linear systems* $|K_A + h_A|$ *and* $|K_B + h_B|$ *maps* h *generically one-to-one. In particular,* $|K_A + h_A|$, $|K_B + h_B|$, *and* $K_X + L$ *are nef and big.*

Proof. By Lemma (2.5) we see that showing complete linear systems $|K_A + h_A|$ and $|K_B + h_B|$ maps h generically one-to-one, shows that $|K_X + L|$ maps h generically one-to-one.

We know from 3) of Theorem (2.4) that the restriction of $K_X + L$ of one of the divisors A, B is nef and big (by ampleness of $A + B$ either A or B surjects on the base). Assume for simplicity, that $K_B + h_B \approx (K + L)_B$ is nef and big. If $B = \mathbb{P}^2$ or $\mathbb{F}_0$, the line bundle $K_B + h_B$ is ample, and indeed very ample.

Thus we may assume $B = \mathbb{F}_r$, $r \geq 1$. Let $\mathcal{E} := E + rf$. Then either $K_B + h_B = a\mathcal{E} + bf$ is very ample or $b = 0$ and $K_B + h_B = a\mathcal{E}$. Thus $|K_B + h_B|$ maps h generically one-to-one.

Now looking at conclusion 3) of Theorem (2.4), we see that if $K + L$ fails to be nef and big, then using the notation of Theorem (2.4), that ν maps h two-to-one onto a curve $\nu(h)$ with all restrictions of elements of $H^0(K_X + L)$ to h the pullbacks of restrictions of sections of $H_{\nu(h)}$. This is a contradiction to the assertion that $|K_X + L|$ maps h generically one-to-one onto its image.

Assume that $(K_X + L)_A \approx K_A + h_A$ is not nef and big. Since the genus of the curve h_A is not 0, the map given by $|K_A + h_A|$ cannot be generically one-to-one on h_A. Q.E.D.

The following is a corollary of the preceding lemma.

Lemma 3.3 *Assumptions and notation as in* (3.1). *Assume* $A = \mathbb{F}_r$ *and let* $h = aE + bf$ *on* A. *Then* $a \geq 3$.

Proof. Note that $a = h \cdot f \geq 0$, and $a \neq 1$ since $g(h) > 0$. Assume $a = 2$. Then $(K_A + h_A) \cdot f = -2 + 2 = 0$ and hence $|(K_X + L)_A| = |(K_A + h_A)|$ collapses A along the ruling f. This contradicts Lemma (3.2). Q.E.D.

4 The Cone Cases

The main result in this section is to showing that in the situation when we have a reducible ample divisor $L = A + B$ with both of A and B are in $\{\mathbb{P}^2, \widetilde{\mathbb{F}_2}\}$ is very restricted. The proof of this is based on the usual Hodge Index type theorem for ample divisors, which yields in our case

$$[(A+B)\cdot A\cdot A][(A+B)\cdot B\cdot B] \leq [(A+B)\cdot A\cdot B]^2 \tag{3}$$

with equality if and only if A is a rational multiple of B as homology class.

Lemma 4.1 *Let L be an ample line bundle on a smooth connected projective threefold X. Assume that A, B are two reduced divisors on X which meet transversely in a smooth curve h of genus $g(h)$. Assume that $A + B \in |L|$, and that $A, B \in \{\mathbb{P}^2, \widetilde{\mathbb{F}_2}\}$. Then $g(h) \leq 1$.*

Proof. Assume without loss of generality that $g := g(h) \geq 2$. Note that in this case the degree of h on A, B are uniquely determined by g.

First let us do the case of $A = B = \mathbb{P}^2$. Then $h_A \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ and $h_B \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ where $2g-2 = d(d-3)$. Note that $d^2 = h_A^2 = B\cdot B\cdot A = h_B \cdot N_{B/X}$. Thus $N_{B/X} = \mathcal{O}_{\mathbb{P}^2}(d)$, and similarly $N_{A/X} = \mathcal{O}_{\mathbb{P}^2}(d)$. Plugging into Equation (3), we get equality. Thus $A = \lambda B$ as homology classes for some $\lambda \in \mathbb{Q}$. Since $A^2\cdot B = d^2 = B^2\cdot A$, we see that $\lambda = 1$. Thus since L is ample and since $L = 2A = 2B$ in homology, it follows that A, B are ample. The Lefschetz theorem yields $\mathrm{Pic}(X) = \mathrm{Pic}(A) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}^2}(1)]$. Therefore $K_X \approx \mathcal{O}_X(c)$, $\mathcal{O}_X(A) \approx \mathcal{O}_X(a)$, where $a \geq 1$ by ampleness. Then $(K_X + A)_A \approx K_A \approx \mathcal{O}_{\mathbb{P}^2}(-3)$ gives $K_X + A \approx \mathcal{O}_X(c+a) \approx \mathcal{O}_X(-3)$. Therefore $1 + c \leq a + c = -3$, or $c \leq -4$. Thus $X = \mathbb{P}^3$ by the Kobayashi-Ochiai Theorem [1, (3.1.6)]) and $g = 0$, contradicting the assumption $g > 0$.

The case of $A = B = \widetilde{\mathbb{F}_2}$ proceeds in the same way, except that one of the possibilities allowed by the Kobayashi-Ochiai Theorem [1, (3.1.6)]) is (X, L) is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$. In this case $g = 1$.

Finally, consider the case when one of A, B is $\mathbb{P}^2$ and the other is $\widetilde{\mathbb{F}_2}$. By renaming if necessary we can assume that $A = \mathbb{P}^2$ and $B = \widetilde{\mathbb{F}_2}$. Letting $h_B = A_B = \mathcal{O}_B(\delta)$ and $h_A = B_A = \mathcal{O}_A(d)$, we have that $A^2 \cdot B = 2\delta^2$, $B^2 \cdot A = d^2$. Also from $d^2 = h_A^2 = B \cdot B \cdot A = h_B \cdot N_{B/X}$ we conclude that $N_{B/X} = \mathcal{O}_B(d^2/2\delta)$. Similarly we conclude that $N_{A/X} = \mathcal{O}_A(2\delta^2/d)$. Thus $A^3 = \dfrac{4\delta^4}{d^2}$ and $B^3 = \dfrac{d^4}{4\delta^2}$. Plugging into Equation (3), we conclude again that A, B are positive multiples of L in homology and hence ample. Using the argument from the case when both are $\mathbb{P}^2$, we see that $X = \mathbb{P}^3$. In this case $g = 0$. Q.E.D.

5 The Cone and Scroll Cases

We keep again our working assumption as in (3.1). In this section we consider the remaining case when both A and B are Hirzebruch surfaces, under the *extra assumption that* $(\widehat{A}, \widehat{L}_{\widehat{A}})$, $(\widehat{B}, \widehat{L}_{\widehat{B}})$ *are scrolls*, i.e., $\widehat{A}$, $\widehat{B}$ are both scrolls with respect to $\widehat{L}$.

We start with the following general lemma.

Lemma 5.1 *Let $\widehat{L}$ be a very ample line bundle on a 3-fold $\widehat{X}$. Let $\widehat{A}+\widehat{B} \in |L|$, where $\widehat{A}$, $\widehat{B}$ are two smooth divisors on $\widehat{X}$ meeting transversely in a smooth curve h of genus $g(h) > 0$. Assume that $(\widehat{A}, \widehat{L}_{\widehat{A}})$ and $(\widehat{B}, \widehat{L}_{\widehat{B}})$ are scrolls or cones (from a vertex not contained in h) over smooth curves $C_{\widehat{A}}$, $C_{\widehat{B}}$ with scroll (or cone) projections $p_{\widehat{A}} : \widehat{A} \to C_{\widehat{A}}$, $p_{\widehat{B}} : \widehat{B} \to C_{\widehat{B}}$ respectively. Then $\pi = (p_{\widehat{A}}, p_{\widehat{B}}) : h \to C_{\widehat{A}} \times C_{\widehat{B}}$ maps h isomorphically onto a smooth curve.*

Proof. Let $\xi \subset h$ be a subscheme of degree 2 (i.e., a pair of distinct points, or a tangent subscheme supported at a single point). We show that π separates ξ. If not, ξ is contained in a fiber of π. Hence ξ belongs to a fiber of $p_{\widehat{A}}$ and one of $p_{\widehat{B}}$, in which case the same holds true for the line ℓ spanned by ξ. But since the intersection $\widehat{A} \cap \widehat{B}$ is transverse and connected, it follows that $\widehat{A} \cap \widehat{B} = \ell$. This contradicts the hypothesis that $g(h) > 0$. Q.E.D.

Theorem 5.2 *Let $\widehat{L}$ be a very ample line bundle on a smooth projective threefold $\widehat{X}$. Assume that there exists two irreducible divisors $\widehat{A}$, $\widehat{B}$ on $\widehat{X}$, which meet transversely in a smooth curve h, and such that $\widehat{A}+\widehat{B} \in |\widehat{L}|$. Assume further that*

1. *$(\widehat{A}, \widehat{L}_{\widehat{A}})$ is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, or $(Q, \mathcal{O}_{\mathbb{P}^3}(1)_Q)$ with $Q \subset \mathbb{P}^3$ the singular quadric $\widetilde{\mathbb{F}}_2$; and*

2. *$(\widehat{B}, \widehat{L}_{\widehat{B}})$ is a scroll over $\mathbb{P}^1$.*

Then $g(h) = 0$.

Proof. We do the case where $\widehat{A} = \mathbb{P}^2$. The case of $\widehat{A} = \widetilde{\mathbb{F}}_2$ is proved in a completely similar way. Let $N_{\widehat{A}/\widehat{X}} = \widehat{A}_{\widehat{A}} \cong \mathcal{O}_{\mathbb{P}^2}(d)$ denote the normal bundle of $\widehat{A}$ in $\widehat{X}$. Let $h_{\widehat{A}} = \widehat{B}_{\widehat{A}} = \mathcal{O}_{\mathbb{P}^2}(\delta)$. Letting E denote a section of $\widehat{B}$ with $E^2 = -r \le 0$, and letting $\mathcal{E} = E + rf$ for a fiber f of the scroll projection, we have $\widehat{B}_{\widehat{B}} = M\mathcal{E} + Nf$ and $h_{\widehat{B}} = \widehat{A}_{\widehat{B}} = a\mathcal{E} + bf$ for integers M, N, a, b. Note that by Lemma (5.1) we have $g := g(h) = (a-1)(\delta-1)$. Note by the formulae for the genus on $\widehat{A}$ and $\widehat{B}$ we have the formulae $2g = (\delta-1)(\delta-2)$ and $2g = (a-1)(ar+2b-2)$. Assuming that $g \ge 1$, and hence that $\delta \ge 3$, $a \ge 2$, immediately gives $\delta = 2a$ and $4a = ar + 2b$.

Note that $d\delta = \widehat{A}_{\widehat{A}} \cdot \widehat{B}_{\widehat{A}} = \widehat{B} \cdot \widehat{A}^2 = \widehat{A}^2_{\widehat{B}} = a(ar+2b)$. Combined with $\delta = 2a$ and $4a = ar + 2b$, we conclude that $d = \delta$. Since $\widehat{L}_{\widehat{A}} = \mathcal{O}_{\mathbb{P}^2}(d+\delta)$, we get a contradiction to $\widehat{L}_{\widehat{A}} \cong \mathcal{O}_{\mathbb{P}^2}(1)$. Q.E.D.

Now let us specialize Lemma (5.1) to the case when $\widehat{A} = \mathbb{F}_r$, $\widehat{B} = \mathbb{F}_s$. Denote by $\mathcal{E}_{\widehat{A}} = E_{\widehat{A}} + rf_{\widehat{A}}$, $E^2_{\widehat{A}} = -r$, $f_{\widehat{A}}$ a fiber of the ruling $\widehat{A} = \mathbb{F}_r \to \mathbb{P}^1$; and similarly for $\widehat{B}$. Write

$$h_{\widehat{A}} = a(E_{\widehat{A}} + rf_{\widehat{A}}) + bf_{\widehat{A}}; \quad h_{\widehat{B}} = \alpha(E_{\widehat{B}} + sf_{\widehat{B}}) + \beta f_{\widehat{B}}, \tag{4}$$

on $\widehat{A}$, $\widehat{B}$, respectively,

Note that by Lemma (3.3) we may assume $a \ge 3$, $\alpha \ge 3$. Furthermore, since h is a positive genus curve on the Hirzebruch surfaces $\widehat{A}$, $\widehat{B}$, we may also assume $b \ge 0$, $\beta \ge 0$.

By Lemma (5.1), $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\pi(h)) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,\alpha)$, and hence

$$g(h) = (a-1)(\alpha-1). \tag{5}$$

The genus formula also yields

$$2g(h) - 2 \;=\; (a-1)(ar+2b-2).$$

Therefore, by using (5), we deduce that

$$2\alpha = ar + 2b. \tag{6}$$

Similarly we find

$$2a = \alpha s + 2\beta. \tag{7}$$

Combining (6) and (7), we have $(4-rs)a\alpha = 4b\beta + 2(b\alpha s + \beta a r)$.

$$2a\alpha = a(ar+2b) = \alpha(\alpha s + 2\beta). \tag{8}$$

Write

$$N_{\widehat{A}/\widehat{X}} = -\lambda(E_{\widehat{A}} + rf_{\widehat{A}}) + \rho f_{\widehat{A}};\;\; N_{\widehat{B}/\widehat{X}} = -\mu(E_{\widehat{B}} + sf_{\widehat{B}}) + \sigma f_{\widehat{B}},$$

for integers λ, μ, ρ, σ.

Note that on $\widehat{A}$ one has $h_{\widehat{B}}^2 = \widehat{A}_{\widehat{B}}^2 = \widehat{A}^2 \cdot \widehat{B} = \widehat{A}_{\widehat{A}} \cdot \widehat{B}_{\widehat{A}} = N_{\widehat{A}/\widehat{X}} \cdot h_{\widehat{A}}$. Since

$$h_{\widehat{B}}^2 = \alpha^2 s + 2\alpha\beta = \alpha(\alpha s + 2\beta) \quad\text{and}\quad N_{\widehat{A}/\widehat{X}} \cdot h_{\widehat{A}} = -a\lambda r - \lambda b + \rho a,$$

we find $\alpha(\alpha s + 2\beta) = -a\lambda r - \lambda b + \rho a$. Similarly, we find $a(ar+2b) = -\alpha\mu s - \mu\beta + \sigma\alpha$. Then by (8) we have

$$-a\lambda r - \lambda b + \rho a = -\alpha\mu s - \mu\beta + \sigma\alpha. \tag{9}$$

Since $(\widehat{A}, \widehat{L}_{\widehat{A}})$ is a scroll, we also have $\widehat{L}_{\widehat{A}}(= \widehat{A}_{\widehat{A}} + \widehat{B}_{\widehat{A}} = N_{\widehat{A}/\widehat{X}} + h_{\widehat{A}}) = E_{\widehat{A}} + jf_{\widehat{A}}$. On the other hand, the coefficient of $E_{\widehat{A}}$ in the expression for $N_{\widehat{A}/\widehat{X}} + h_{\widehat{A}}$ is $-\lambda + a$. Therefore the last equality for $\widehat{L}_{\widehat{A}}$ implies $a - \lambda = 1$. Similarly the scroll condition for $(\widehat{B}, \widehat{L}_{\widehat{B}})$ gives $\alpha - \mu = 1$. So from (9) we have

$$-a(a-1)r - (a-1)b + \rho a = -\alpha(\alpha-1)s - (\alpha-1)\beta + \sigma\alpha.$$

Then in particular $\rho = 2\alpha + (a-1)r + b + \dfrac{b}{a}$. This implies $\rho \geq 6$ (with $\rho = 6$ giving $r = b = 0$), as well as a divides b, say, $b = ab'$.

In the same way, we find $\sigma = 2a + (a-1)s + \beta + \dfrac{\beta}{\alpha}$. So $\sigma \geq 6$ (with $\sigma = 6$ giving $s = \alpha = 0$), as well as $\beta = \alpha\beta'$.

Thus formulas (6) and (7) become $2\alpha = a(r+2b')$ and $2a = \alpha(s+2\beta')$. From this we also find

$$4 = (r+2b')(s+2\beta'). \tag{10}$$

Since $b', \beta' \geq 0$ it follows that $rs \leq 4$ and hence $r, s \in \{0,1,2,3,4\}$.

The following theorem summarizes the discussion above.

Theorem 5.3 *Let $\widehat{L}$ be a very ample line bundle on a 3-fold $\widehat{X}$. Let $\widehat{A}+\widehat{B} \in |L|$, where $\widehat{A}$, $\widehat{B}$ are two smooth divisors on $\widehat{X}$ meeting transversely in a smooth curve h of genus $g(h) > 0$. Assume that $\widehat{A} = \mathbb{F}_r$, $\widehat{B} = \mathbb{F}_s$ are Hirzebruch surfaces. Further assume that $(\widehat{A}, \widehat{L}_{\widehat{A}})$ and $(\widehat{B}, \widehat{L}_{\widehat{B}})$ are scrolls over smooth curves. Then $r, s \in \{0,1,2,4\}$ and the possible values of the coefficients $b = ab'$, $\beta = \alpha\beta'$ as in the expressions (4) of h as a curve of $\widehat{A}$, $\widehat{B}$ respectively are listed in the table below.*

$s \diagdown r$	0	1	2	3	4
0	$b' = \beta' = 1$	$b' = 0,\ \beta' = 2$	$b' = 0,\ \beta' = 1$	$\surd$	$\surd$
1	$b' = 2,\ \beta' = 0$	$\surd$	$b' = 1,\ \beta' = 0$	$\surd$	$b' = \beta' = 0$
2	$b' = 1,\ \beta' = 0$	$b' = 0,\ \beta' = 1$	$b' = \beta' = 0$	$\surd$	$\surd$
3	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
4	$\surd$	$b' = \beta' = 0$	$\surd$	$\surd$	$\surd$

Proof. A simply numerical check, by using (10) and the symmetry between r and s, gives the possible values for the integers r, s, b', β' in the table (the symbol "$\surd$" means that the corresponding case does not occur). E.g., if $r = 0$, equality (10) gives $2 = b'(s + 2\beta')$. This leads to the cases $(s, b', \beta') = (0,1,1), (1,2,0), (2,1,0)$ as in the first column. Thus we may assume $r, s \geq 0$. E.g., if $r = 3$, (10) gives $4 = 3(s + 2\beta') + 2b'(s + 2\beta')$, so that $b' \neq 0$ and hence $b' > 0$, this giving again a numerical contradiction. Q.E.D.

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