Hilbert curves of polarized varieties, I*

M.C. Beltrametti, A. Lanteri, and A.J. Sommese

August 22, 2008

Contents

Abstract

Let X be a normal Gorenstein complex projective variety. We introduce the Hilbert variety V_X associated to the Hilbert polynomial $\chi(x_1L_1,\dots,x_\rho L_\rho)$, where $L_1,\dots,L_\rho$ is a basis of $\mathrm{Pic}(X)$, ρ being the Picard number of X, and $x_1,\dots,x_\rho$ are complex variables. After studying general properties of V_X we specialize to the Hilbert curve of a polarized variety (X,L), namely the plane curve of degree $\dim(X)$ associated to $\chi(xK_X+yL)$. Special emphasis is given to the case of polarized 3-folds.

Introduction

Let L be an ample line bundle on an irreducible projective manifold X. Let K_X denote the canonical bundle. Associated to the Euler characteristic $\chi(xK_X+yL)$ we define below a plane curve of degree $\dim X$, which we call the Hilbert Curve of the polarized variety (X,L). This article grew out of a study of the special geometry of this curve and the restraints posed on (X,L) by conditions about this curve, e.g., that the curve has a singularity.

Let us start by making everything precise.

Let $\mathrm{Pic}_0(X)\subset\mathrm{Pic}(X)$ denote the topologically trivial line bundles contained in $\mathrm{Pic}(X)$. The function sending $L\in\mathrm{Pic}(X)$ to its Euler characteristic $\chi(L)$ gives rise to polynomial

*2000 *Mathematics Subject Classification.* Primary 14C20, 14N30 ; Secondary 14M99, 14H50.
Keywords and phrases. Polarized variety, fibration, Hilbert polynomial, plane curve

function p from $\mathbf{N}(X) := \mathrm{Pic}(X)/\mathrm{Pic}_0(X) \otimes_{\mathbb{Z}} \mathbb{C}$ to $\mathbb{C}$. This polynomial has degree $\dim X$ and has real coefficients with respect to the natural real structure induced on $\mathbf{N}(X)$. We call the hypersurface V_X defined by setting p to 0, the Hilbert variety of X. Besides being invariant under conjugation, X is invariant under the linear map induced by Serre duality, i.e., $\chi(L) = (-1)^{\dim X}\chi(K_X \otimes L^*)$. We call this latter map, $s : \mathbf{N}(X) \to \mathbf{N}(X)$, the Serre involution.

Given a polarized variety (X, L), we have the vector subspace $< K_X, L > \subset \mathbf{N}(X)$ generated by L and K_X. This subspace is at least one dimensional since L is ample. We assume here that $< K_X, L >$ is isomorphic to $\mathbb{C}^2$, since if this is not true, then we are in the degenerate case when there are integers x, y (not both zero) with $xK_X + yL$ topologically trivial. We denote by $p(x, y)$ the polynomial on $\mathbb{C}^2$ that $\chi(xK_X + yL)$ extends to. We denote the Hilbert curve of the pair by $C_{(X,L)}$, or when no confusion results by C_L.

For for any positive integer, (X, mL), C_L and C_{mL} are biholomorphic. Moreover it is an immediate consequence of the First Lefschetz Theorem on hyperplane sections that given (X, L) with $\dim X \geq 3$ and a smooth $D \in |mL|$, $C_{(X,L)}$ is biholomorphic to $C_{(D,mL_D)}$. On $< K_X, L >$, the fixed point set of this involution s consists of $c := \frac{1}{2}K_X$. The Taylor series expansion of $p(x, y)$ at this point has all coefficients of different parity from $\dim X$ equal to zero. In particular $\left(\frac{1}{2}, 0\right) \in C_L$ if $\dim X$ is odd, and if the point belongs to C_L when $\dim X$ is even, it is a singular point. These and related general facts plus computations of some basic examples are carried out in §2.

In §3 more detailed information on the Hilbert curve is presented. In Theorem 3.4, it is shown that if the closure of the Hilbert curve in $\mathbb{P}^2$ is smooth, then the Hilbert curve intersect with the line at infinity consists of $\dim X$ distinct points. Also in this section is a characterization of polarized surfaces whose Hilbert curve is a double line.

In §4, a detailed study is made of the case when $\dim X = 3$. In this case, the Hilbert curve is a cubic curve. Example 4.10 shows that different smooth threefolds may lead to smooth, but nonisomorphic plane curves. Numerical characterizations are given of polarized threefolds, whose Hilbert curves satisfy various singularity conditions, e.g., having a singularity on the line at infinity.

In §5, an analysis is made of the quotients of Hilbert curves under the Serre involution. It is shown that the quotient has a natural map into $\mathbb{P}^3$, with image a Castelnuovo curve.

In §6 an analysis of the Hilbert curve of fibrations is made.

In §7, plane curves invariant under the Serre involution are characterized.

We thank the Duncan Chair of the University of Notre Dame and the University of Milan (xxxxx) for making our collaboration possible.

1 Conventions and basic notation

We work over the complex field $\mathbb{C}$. Throughout the paper we deal with projective varieties X. We denote by $\mathcal{O}_X$ the structure sheaf of X. For any coherent sheaf $\mathcal{F}$ on X, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(X, \mathcal{F})$.

Let L be a line bundle on X. The line bundle L is said to be *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on X. L is said to be *big* if $\kappa(L) = \dim(X)$, where $\kappa(L)$ denotes the Kodaira dimension of L. If L is nef then this is equivalent to $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L and $n = \dim(X)$. The pull-back ι^*L

of a line bundle L on X by an embedding $\iota : W \hookrightarrow X$ is denoted by L_W. We denote by K_X the canonical bundle of a Gorenstein variety X.

1.1 Notation. We use standard notation from algebraic geometry, among which we recall the following ones:

$\approx$, the linear equivalence of line bundles; $\equiv$, the numerical equivalence of line bundles;

$\chi(L) = \sum_i (-1)^i h^i(L)$, the Euler characteristic of a line bundle L;

$|L|$, the complete linear system associated to a line bundle L;

$\kappa(D)$, the Kodaira dimension of the line bundle associated to a $\mathbb{Q}$-Cartier divisor D on X; and $\kappa(X) := \kappa(K_X)$, the Kodaira dimension of X, for X smooth.

Line bundles and divisors are used with little (or no) distinction. We almost always use the additive notation. We say that a line bundle L is *spanned* if it is spanned, i.e., globally generated, at all points of X by $H^0(X, L)$.

1.2 Let L be a line bundle on an irreducible, normal, Gorenstein n-dimensional projective variety X. For $j = 0, \ldots, n$, define the j-th *pluridegree of the pair* (X, L) as

$$d_j(L) := K_X^j \cdot L^{n-j}.$$

If no confusion will arise, we simply write $d_j = d_j(L)$. We also set $d := d_0$.

Note that, if L and K_X are nef, then one has $d_j^2 \geq d_{j+1}d_{j-1}$ for $j = 1, \ldots, n-1$ by the Hodge index theorem (see e.g., [1, (2.5.1)]).

1.3 Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two projective plane curves. Then we denote by $m_P(\mathcal{C}_1, \mathcal{C}_2)$ the intersection multiplicity of $\mathcal{C}_1$, $\mathcal{C}_2$ at a point $P \in \mathcal{C}_1 \cap \mathcal{C}_2$, defined by the formula

$$m_P(\mathcal{C}_1, \mathcal{C}_2) = \dim_{\mathbb{C}} \left(\mathcal{O}_{\mathbb{P}^2, P}/(f_1, f_2) \right),$$

where f_1 and f_2 are local equations of $\mathcal{C}_1$ and $\mathcal{C}_2$ around P.

It would be easy to prove that if P is s_1-fold for one of the two curves and s_2-fold for the other, the intersection multiplicity at P satisfies $m_P(\mathcal{C}_1, \mathcal{C}_2) \geq s_1 s_2$; with equality when the two curves do not have any common tangent at P. If instead t is the number of common tangents at P, then

$$m_P(\mathcal{C}_1, \mathcal{C}_2) \geq s_1 s_2 + t.$$

2 Hilbert variety: the general framework

Let X be a complex projective irreducible variety. Let $\mathrm{Pic}_0(X) \subset \mathrm{Pic}(X)$ denote the topologically trivial line bundles contained in $\mathrm{Pic}(X)$. Set $\mathbf{N}(X) := \mathrm{Pic}(X)/\mathrm{Pic}_0(X) \otimes_{\mathbb{Z}} \mathbb{C}$. The Euler characteristic map

$$\chi : \mathrm{Pic}(X) \to \mathbb{Z},$$

defined by $L \mapsto \chi(L)$, gives rise to a polynomial function

$$p : \mathbf{N}(X) \to \mathbb{C}.$$

Note that $\mathbf{N}(X) \cong \mathbb{A}^{\rho}_{\mathbb{C}}$, where $\rho := \rho(X)$ is the Picard number of X. Via this isomorphism, if $\mathbf{N}(X) = \langle L_1, \dots, L_\rho \rangle$ with $L_1, \dots, L_\rho \in \mathrm{Pic}(X)$ and writing $\mathcal{L} = \sum_{i=1}^{\rho} x_i L_i \in \mathbf{N}(X)$, $x_i \in \mathbb{C}$, the image

$$p(\mathcal{L}) = p(x_1, \dots, x_\rho)$$

is the evaluation in $\mathcal{L}$ of the polynomial $p \in \mathbb{C}[x_1, \dots, x_\rho]$, when we consider $x_1, \dots, x_\rho$ as complex variables. In other words, for $x_1, \dots, x_\rho$ integers, we consider the Hilbert polynomial

$$\chi(x_1, \dots, x_\rho) := \chi(x_1 L_1 + \cdots + x_\rho L_\rho),$$

and we denote by $p(x_1, \dots, x_\rho)$ the polynomial $\chi(x_1, \dots, x_\rho)$ when we consider $x_1, \dots, x_\rho$ as complex variables.

Let us consider the affine variety $V_X := V(p)$, which is an hypersurface of degree $\dim(X)$ in $\mathbf{N}(X) \cong \mathbb{A}^{\rho}_{\mathbb{C}}$. We say that V_X is the (*affine*) *Hilbert variety associated to X*.

From now on, unless otherwise specified, we will use the word *variety* to mean a normal, Gorenstein complex variety, X.

Write an element $\mathcal{L} \in \mathbf{N}(X)$ as $\mathcal{L} = xK_X + \sum_i y_i \mathcal{L}_i$, with $\mathcal{L}_i \in \mathrm{Pic}(X)$ and $x, y_i \in \mathbb{C}$. Then the mapping

$$\mathcal{L} = xK_X + \sum_i y_i \mathcal{L}_i \mapsto (1-x)K_X - \sum_i y_i(\mathcal{L}_i)$$

defines the *Serre involution*

$$s : \mathbf{N}(X) \to \mathbf{N}(X), \ \ (x_1, x_2, \dots, x_\rho) \mapsto (1 - x_1, -x_2 \dots, -x_\rho).$$

More precisely, for integers x, y_i, look at the Hilbert polynomial $\chi(x, \dots, y_i, \dots) := \chi(xK_X + \sum_i y_i \mathcal{L}_i)$. By Serre duality,

$$\chi(x, \dots, y_i, \dots) = \chi\big(xK_X + \sum_i y_i \mathcal{L}_i\big) =$$
$$= (-1)^{\dim(X)} \chi\big((1-x)K_X - \sum_i y_i \mathcal{L}_i\big) = (-1)^{\dim(X)} \chi(1-x, \dots, -y_i, \dots).$$

According to the above notation, denote by $p(x, \dots, y_i, \dots)$ the polynomial $\chi(x, \dots, y_i, \dots)$ when we consider x, y_i as complex variables. Thus

$$p(x, y_1, \dots, y_{\rho-1}) = (-1)^{\dim(X)} p(1-x, -y_1, \dots, -y_{\rho-1}).$$

Clearly, the Hilbert variety V_X is fixed under the Serre involution s, that is $s(V_X) = V_X$. Moreover the (unique) fixed point of the involution s is $C = \left(\frac{1}{2}, 0, \dots, 0\right) \in \mathbb{A}^{\rho}_{\mathbb{C}}$, and V_X is symmetric with respect to C. We say that C is the *central point* of the Serre involution. Notice that

$$C \in V_X \ \text{ for } \ \dim(X) \ \text{ odd.} \tag{1}$$

Since, for any j-th partial derivative ∂^j, $j \geq 0$,

$$\partial^j p(1-x, -y_1, \dots, -y_{\rho-1}) = (-1)^{\dim(X)+j}\, \partial^j p(x, y_1, \dots, y_{\rho-1}),$$

we conclude that

$$\big(\partial^j p(1-x, -y_1, \dots, -y_{\rho-1})\big)_{|C} = 0 \text{ if } n+j \text{ is odd.} \tag{2}$$

Summarizing we have the following.

Proposition 2.1 *Let V_X be the Hilbert variety of an n-dimensional variety X, and let C be the central point of the Serre involution.*

1. *V_X is symmetric with respect to C;*
2. *For n even, if $C \in V_X$, then V_X is singular at C;*
3. *For any n, if $C \in V_X$ is a point of multiplicity $n-1$, then C is a point of multiplicity n of V_X.*

Proof. It is an immediate consequence of condition (2): take $j = 1$ to get 2), and $j = n-1$ to get 3). Q.E.D.

Let us denote by $\overline{V_X} \subset \overline{\mathbf{N}(X)} (\cong \mathbb{P}^{\rho}_{\mathbb{C}})$ the projective closure of $V_X \subset \mathbf{N}(X)$. We also say that $\overline{V_X}$ is the (*projective*) *Hilbert variety* of (X, L).

Denoting by $[u_0, \dots, u_\rho]$ homogeneous coordinates in $\mathbb{P}^{\rho}_{\mathbb{C}}$, with $xu_\rho = u_0$, $y_i u_\rho = u_{i-1}$, the Serre involution extends to an involution

$$\overline{s} : \overline{\mathbf{N}(X)} \to \overline{\mathbf{N}(X)}, \quad [u_0, u_1, \dots, u_\rho] \mapsto [u_\rho - u_0, -u_1, \dots, -u_{\rho-1}, u_\rho],$$

with the hyperplane at infinity $u_\rho = 0$ consisting of fixed points.

The extended Serre involution acts on $\overline{V_X}$. Hence we can consider the quotient map

$$\overline{V_X} \to \overline{V_X}/\langle\, \overline{s}\, \rangle \hookrightarrow \overline{\mathbf{N}(X)}/\mathbb{Z}_2,$$

which is a degree two morphism ramified along the locus of fixed points of $\overline{s}$, which is given by $\overline{V_X} \cap \{u_\rho = 0\}$ or $\{C\} \cup (\overline{V_X} \cap \{u_\rho = 0\})$ according to whether n is even or odd.

The quotient $\overline{\mathbf{N}(X)}/\mathbb{Z}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{\rho-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{\rho-1}})$ is (isomorphic to) a cone over the Veronese variety $(\mathbb{P}^{\rho-1}, \mathcal{O}_{\mathbb{P}^{\rho-1}}(2))$. Therefore $\overline{\mathbf{N}(X)}/\mathbb{Z}_2$ is singular if $\rho \geq 2$.

2.2 (The Hilbert curve of a quasi-polarized variety) Let L be an ample, or merely nef and big, line bundle on X. The Hilbert polynomial $\chi(x, y) := \chi(xK_X + yL)$, $x, y \in \mathbb{Z}$, arises naturally in the study of polarized varieties (X, L). As usual, denote by $p(x, y)$, sometimes by $p_{(X,L)}(x, y)$, the polynomial $\chi(x, y)$ when we consider x, y as complex variables. Then looking at the zeroes of $p(x, y)$ corresponds to taking a slice of the Hilbert variety V_X by the 2-dimensional vector subspace $\mathbb{C}^2_{(x,y)} \subseteq \mathbf{N}(X)$ ($\mathbb{C}^2_{(x,y)} = \langle K_X, L\rangle$ whenever K_X and L are $\mathbb{C}$-linearly independent). We will also write

$$V_{(X,L)} := \mathbb{C}^2_{(x,y)} \cap V_X,$$

and we will say that the degree $n := \dim(X)$ affine plane curve $V_{(X,L)}$ is the *Hilbert curve of the polarized variety* (X, L).

More generally, we can consider a slice with a vector subspace $\mathbb{C}^2 \subseteq \mathbf{N}(X)$ which is not necessarily generated by line bundles. In particular we can merely assume that

$$\mathbb{C}^2 \cap \mathrm{Pic}(X) \supseteq \mathbb{Z}\langle K_X\rangle.$$

In fact, for any $m \geq 2$, whenever we consider a slice with a vector subspace $\mathbb{C}^m \subseteq \mathbf{N}(X)$, it will be *natural to assume* that

$$K_X \in \mathbb{C}^m. \tag{3}$$

It is worth noting that condition (3) implies that our space $\mathbb{C}^m$ is s-invariant, i.e., $s(\mathbb{C}^m) = \mathbb{C}^m$, this allowing us to consider the action of the Serre involution on $\mathbb{C}^m$, and to use several and remarkable consequences of this fact. In turn, one has

$$V(p_{|\mathbb{C}^m}) = \mathbb{C}^m \cap V(p).$$

Note also that, any time we take a slice of V_X with a vector subspace $\mathbb{C} = \mathbb{C}\langle\mathfrak{L}\rangle \subset \mathbf{N}(X)$ generated by any line bundle $\mathfrak{L}$, then $\mathbb{C} \cap V_X$ consists of k points, where

$$k \leq \max\{s \mid c_1(\mathfrak{L})^s \text{ is not numerically trivial}\}.$$

Moreover, by taking the projective closure, one has

$$\overline{V_X} \cap \overline{\mathbb{C}\langle\mathfrak{L}\rangle} = \dim(X) \text{ points in } \overline{\mathbf{N}(X)} \text{ (counted with multiplicities).}$$

It is just the case to note that if $\dim(X) = 1$, then the Hilbert variety V_X is a point in $\mathbb{C}$, so everything is trivial. We can thus assume $\dim(X) \geq 2$.

2.3 (The degenerate case) Consider a polarized pair as above, and assume that $K_X = \lambda L$ for some $\lambda \in \mathbb{Q}$. Even in this case we can consider the polynomial

$$p(x, y) = \chi(xK_X + yL),$$

defining a plane curve, which we call the *degenerate Hilbert curve*, say Γ_0, of (X, L).

Note that such a curve can be not a slice of type $\mathbb{C}^2 \cap V_X$ with $\mathbb{C}^2$ a vector subspace of $\overline{\mathbf{N}(X)}$. In fact, writing $t := \lambda x + y$,

$$p(x, y) = \wp(t) \in \mathbb{C}[t]$$

is a polynomial of degree $n := \dim(X)$ in t and its zeros correspond to the slice $\mathbb{C}_{(t)} \cap V_X$. Moreover, Γ_0 is the union of n parallel lines, ℓ_j, of equation $\lambda x + y - t_j = 0$, where t_j are the roots of $\wp(t)$, $j = 1, \ldots, n$. We refer to this situation as the "degenerate case".

The configuration of such lines ℓ_j is symmetric with respect to the point $(\frac{1}{2}, 0)$ (i.e., the central point of the "Serre involution" $s : \mathbb{C}_{(x,y)} \to \mathbb{C}_{(x,y)}$ defined by $(x, y) \mapsto (1 - x, -y)$. According to (1), if n is odd, one of that lines passes through it.

E.g., consider $(X, L) = (\mathbb{P}^n, L)$. Then

$$p_{(\mathbb{P}^n, L)}(x, y) = \chi(xK_X + yL) = \chi(\mathcal{O}_{\mathbb{P}^n}(-x(n + 1) + ay)),$$

where $L = \mathcal{O}_{\mathbb{P}^n}(a)$ for some integer a. Set $t := -x(n + 1) + ay$, so that

$$\chi(xK_X + yL) = \chi(\mathcal{O}_{\mathbb{P}^n}(t)) = h^0(\mathcal{O}_{\mathbb{P}^n}(t)) = \binom{n+t}{t} = \frac{1}{n!}(t + n) \cdots (t + 1).$$

Thus the Hilbert polynomial of $(\mathbb{P}^n, L)$ can be written in the form

$$p_{(\mathbb{P}^n, L)}(x, y) = \wp(t) = \frac{1}{n!} \prod_{i=1}^{n} (t + i), \quad i = 1, \ldots, n.$$

We have the following numerical interpretation of the degenerate case.

Lemma 2.4 *Let L be an ample line bundle on the variety X, of dimension $n \geq 2$. Assume that there exists a smooth surface S given by the transversal intersection of $n-2$ effective divisors of $|L|$. Then $dd_2 = d_1^2$ if and only if $K_X = \lambda L$ for some $\lambda \in \mathbb{Q}$.*

Proof. By the Hodge index theorem, the assumption $(K_{X|S})^2(L_S)^2 = (K_{X|S} \cdot L_S)^2$ implies that $K_{X|S} - \lambda L_S$ is numerically trivial for some $\lambda \in \mathbb{Q}$. Since the restriction map $H^2(X,\mathbb{Z}) \to H^2(S,\mathbb{Z})$ is injective by Lefschetz's theorem, we get $K_X = \lambda L$. Q.E.D.

2.5 The Hilbert curve of products. Assume that the variety X is a product, $X = X_1 \times X_2$, and let $\pi_i : X \to X_i$ be the projections on the two factors, $i = 1,2$. Set $L_1 \boxtimes L_2 := \pi_1^* L_1 \otimes \pi_2^* L_2$, where $L_i \in \mathrm{Pic}(X_i)$. By Künneth formulas one has $\chi(L_1 \boxtimes L_2) = \chi(L_1)\chi(L_2)$, so that we have, if x, y are complex variables and with clear meaning of notation,

$$p(x,y) := \chi(xL_1 \boxtimes yL_2) = \chi(xL_1)\ \chi(yL_2) := p_{X_1}(x)\ p_{X_2}(y).$$

Thus the Hilbert variety is reducible, and one has

$$V(p) = \mathbf{N}(X_1) \times V(p_{X_2}) \cup V(p_{X_1}) \times \mathbf{N}(X_2).$$

Note also that

$$\mathbf{N}(X_1) \times \mathbf{N}(X_2) \subseteq \mathbf{N}(X_1 \times X_2),$$

with equality if either $h^1(\mathcal{O}_{X_1}) = 0$ or $h^1(\mathcal{O}_{X_2}) = 0$. In fact, "we work" in $\mathbf{N}(X_1) \times \mathbf{N}(X_2)$ since $K_{X_1 \times X_2} := K_{X_1} \boxtimes K_{X_2} \in \mathbf{N}(X_1) \times \mathbf{N}(X_2)$.

Assume L_1, L_2 ample. Then $L := L_1 \boxtimes L_2$ is nef and big, and consider the (quasi)-polarized pair (X,L). We have

$$xK_X \otimes yL = (xK_{X_1} \otimes yL_1) \boxtimes (xK_{X_2} \otimes yL_2)$$

and, for $i \geq 0$, Künneth's formula yields

$$h^i(xK_X \otimes yL) = \sum_k h^{i-k}(xK_{X_1} \otimes yL_1) h^k(xK_{X_2} \otimes yL_2).$$

Thus

$$\chi(xK_X \otimes yL) = \chi(xK_{X_1} \otimes yL_1)\ \chi(xK_{X_2} \otimes yL_2).$$

Therefore

$$p(x,y) := \chi(xK_X \otimes yL) = p_{X_1}(x,y)\ p_{X_2}(x,y),$$

giving examples of reducible Hilbert curves.

In particular, if $X = C_1 \times \cdots \times C_n$ is the product of $n = \dim(X)$ smooth curves, then the Hilbert curve $V_{(X,L)}$ is union of n lines.

Example 2.6 Consider the product $X = \mathcal{C} \times \mathcal{C} \times Y$, for some curve $\mathcal{C}$ and some $(n-2)$-fold Y. With the usual notation, let $L := \pi_1^* A \otimes \pi_2^* A \otimes \pi_3^* M$ for some line bundles A and M on $\mathcal{C}$ and Y respectively (choose A, M to avoid the trivial case $L = \lambda K_X$, $\lambda \in \mathbb{Q}$). Then the Hilbert curve of (X,L) contains a non-reduced line coming from the first two factors (compare with (4.6)).

Example 2.7 (The Hilbert variety of $\mathcal{C} \times \mathcal{C}$ for $\mathcal{C}$ a very general curve) Let $X = \mathcal{C} \times \mathcal{C}$, where $\mathcal{C}$ is a very general curve of genus $g \geq 2$ (i.e., its isomorphism class does not belong to a countable union of proper subvarieties of the moduli space of curves of genus g, defined by certain rationality conditions for the period matrix). According to [3, note at pp. 285–286], $\mathrm{Pic}(X)$ is generated by the classes of the two factors $E = \mathcal{C} \times \{x\}$, $F = \{x\} \times \mathcal{C}$ ($x \in \mathcal{C}$) and the diagonal Δ. We know that $E^2 = F^2 = 0$, $\Delta^2 = 2 - 2g$ and $E \cdot F = E \cdot \Delta = F \cdot \Delta = 1$. Recall that $\chi(\mathcal{O}_X) = 1 - 2g + g^2 = (1-g)^2$. Moreover, $K_X \equiv (2g-2)(E+F)$. By the Riemann–Roch theorem we thus see that the Hilbert variety of X is the affine quadric surface $V_X \subset \mathbb{C}^3$ of equation

$$x_1x_2 + x_1x_3 + x_2x_3 + 2(1-g)x_3^2 + (1-g)(x_1 + x_2 + 2x_3) + (1-g)^2 = 0,$$

with respect to coordinates x_1, x_2, x_3 induced by the basis $\{E, F, \Delta\}$ of $\mathbf{N}(X)$. It is immediate to check that V_X is a quadric cone with vertex $(g-1, g-1, 0)$, corresponding to the numerical class of $\frac{1}{2}K_X$ (the central point of the Serre involution).

Now, let $L \in \mathrm{Pic}(X) \setminus \langle K_X \rangle$ be any ample line bundle. It is clear that the Hilbert curve $V_{(X,L)}$ of the polarized surface (X, L) is the slice of V_X with the 2-dimensional vector subspace of $\mathbf{N}(X)$ generated by the numerical classes of K_X and L. Since this is an affine plane containing the vertex of V_X it turns out that $V_{(X,L)}$ consists of two lines.

Let us emphasize the fact that this happens for any L, i.e., not only for the line bundles of the form $L = L_1 \boxtimes L_2$, where $L_i \in \mathrm{Pic}(\mathcal{C})$, $i = 1, 2$.

3 The Hilbert curve

Let X be an n-dimensional projective variety ($n \geq 2$), and let L be an ample (or merely nef and big) line bundle on X. Let $\Gamma := V_{(X,L)}$ be the degree n affine Hilbert curve of the polarized pair (X, L), and let $\overline{\Gamma}$ be its projective closure in $\mathbb{P}^2$, where $[x, y, z]$ denote homogeneous coordinates. Let $\ell_\infty : z = 0$ be the line at infinity. The Serre involution $s : \mathbb{A}^2_{(x,y)} \to \mathbb{A}^2_{(x,y)}$ extends to the projective transformation $\overline{s} : \mathbb{P}^2 \to \mathbb{P}^2$ defined by $[x, y, z] \mapsto [z - x, -y, z]$.

We make the **blanket assumption** that the numerical classes of L and K_X are linearly independent in the vector space $N^1(X)$.

According to (2.1), 1), the curve Γ is symmetric with respect to the central point $C = \left(\frac{1}{2}, 0\right)$ of the Serre involution. Therefore, it is useful to have the expression of the defining equation $p(x, y) = 0$ of Γ in the new coordinates $u = x - \frac{1}{2}$ and $v = y$.

For instance, for $n = 1$ we have

$$p\left(\frac{1}{2} + u, v\right) = d_1 u + dv.$$

For $n = 2$,

$$p\left(\frac{1}{2} + u, v\right) = \frac{1}{2}(d_2u^2 + 2d_1uv + dv^2) + \frac{1}{8}(8\chi(\mathcal{O}_X) - d_2). \tag{4}$$

Note that there are no linear terms in that expression. In particular, according to (2.1), 3) if C belongs to the conic Γ, then it is a double point of Γ.

For $n=3$, let us assume that $|L|$ contains a smooth surface, S (e.g., X smooth and L ample and spanned). Then the expression above becomes:

$$p\left(\frac{1}{2}+u,v\right)=\frac{1}{6}(d_3u^3+3d_2u^2v+3d_1uv^2+dv^3)+ \\ -\frac{1}{24}\big((48\chi(\mathcal{O}_X)+d_3)u+(d_2+2d_1+2d-2e(S))v\big), \quad (5)$$

where $e(S)$ stands for the topological Euler characteristic of S.

Note that there are no terms of second degree in the expression above. This means that the tangent line t_C has intersection multiplicity 3 with $\overline{\Gamma}$ at C, i.e., the central point C is a flex of $\overline{\Gamma}$. More generally, if n is odd and $m_C(t_C,\overline{\Gamma})=2r$ then $m_C(t_C,\overline{\Gamma})=2r+1$.

We first characterize the remarkable case when the Hilbert curve splits into lines.

Theorem 3.1 *Let (X,L) be an n-dimensional polarized variety, $n\geq 2$. Assume that the Hilbert curve Γ of (X,L) has an n-fold point, P. Then P coincides with the central point $C=\left(\frac{1}{2},0\right)$ of the Serre involution and $p(x,y)$ factors as a product of linear factors $\prod_{i=1}^{n}\big(\alpha_i(x-\frac{1}{2})+\beta_i y\big)$. Moreover,*

$$p(x,y)=\frac{1}{n!}\left[d_n\left(x-\frac{1}{2}\right)^n+\binom{n}{1}d_{n-1}\left(x-\frac{1}{2}\right)^{n-1}y+\cdots+\binom{n}{n-1}d_1\left(x-\frac{1}{2}\right)y^{n-1}+dy^n\right].$$

In particular, $\chi(\mathcal{O}_X)=\left(-\frac{1}{2}\right)^n\frac{d_n}{n!}$.

Proof. Let $s:\mathbb{A}^2_{(x,y)}\to\mathbb{A}^2_{(x,y)}$ be the Serre involution. If $P=(x,y)$ is an n-fold point of Γ, then $s(P)=(1-x,-y)$ is an n-fold point of $s(\Gamma)$. Since $s(\Gamma)=\Gamma$, we conclude that $s(P)=P$, that is $P=C=\left(\frac{1}{2},0\right)$. Thus Γ consists of n lines $\ell_i:\alpha_i\big(x-\frac{1}{2}\big)+\beta_i y=0$, all passing through C. So, up to scaling $[\alpha_1,\beta_1]$, we can write

$$p(x,y)=\prod_{i=1}^{n}\big(\alpha_i(x-\frac{1}{2})+\beta_i y\big).$$

Then, $p(x,y)$ is a homogenous polynomial of degree n in $x-\frac{1}{2}$, y, i. e.,

$$p(x,y)=a_0\left(x-\frac{1}{2}\right)^n+a_1\left(x-\frac{1}{2}\right)^{n-1}y+\cdots+a_{n-1}\left(x-\frac{1}{2}\right)y^{n-1}+a_ny^n. \quad (6)$$

Homogenizing (6) and intersecting with the line ℓ_∞ we obtain that

$$p(x,1,0)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n.$$

On the other hand, taking into account the expression of $\chi(xK_X+yL)$, we have

$$p(x,1,0)=\frac{1}{n!}(xK_X+L)^n=\frac{1}{n!}\left(d_nx^n+\binom{n}{1}d_{n-1}x^{n-1}+\cdots+\binom{n}{n-1}d_1x+dy^n\right).$$

By comparing the two expressions we thus obtain

$$a_i=\frac{1}{n!}\binom{n}{i}d_{n-i} \qquad \text{for } i=0,\ldots,n.$$

This identifies the coefficients in (6) and gives the claimed expression of $p(x,y)$. In particular, we get $p(0,0) = a_0 \left(-\frac{1}{2}\right)^n = \left(-\frac{1}{2}\right)^n \frac{d_n}{n!}$. Then the final assertion follows simply recalling that $p(0,0) = \chi(\mathcal{O}_X)$. Q.E.D.

As to the behavior of the Hilbert curve at infinity, let us prove a useful general fact.

Lemma 3.2 *Notation as above. Let L be an ample line bundle on the variety X, of dimension $n \geq 2$. Assume that there exists a smooth surface S given by the transversal intersection of $n-2$ effective divisors of $|L|$. Let $\overline{\Gamma}$ be the projective Hilbert curve of (X,L). Assume that we are not in the degenerate case. Then $m_P(\overline{\Gamma}, \ell_\infty) < n$ for each point $P \in \ell_\infty$.*

Proof. Let $p(x,y,z)$ be the homogeneous polynomial defining $\overline{\Gamma}$ in $\mathbb{P}^2$. Restricting to ℓ_∞ and letting $y=1$, we can write

$$p(x,1,0) = \frac{d}{n!}\left[\frac{d_n}{d}x^n + \binom{n}{1}\frac{d_{n-1}}{d}x^{n-1} + \binom{n}{2}\frac{d_{n-2}}{d}x^{n-2} + \cdots + \binom{n}{n-1}\frac{d_1}{d}x + 1\right].$$

Assume that

$$m_P(\overline{\Gamma}, \ell_\infty) = n \tag{7}$$

for some $P \in \ell_\infty$. This is equivalent to saying that $p(x,1,0) = \frac{d}{n!}(kx+1)^n$, for some $k \in \mathbb{C}$. Thus, for $j = 0, \ldots, n$, it must be $k^j = \dfrac{d_j}{d}$. For $j = 0,1,2$ we get

$$1 = 1;\;\; k = \frac{d_1}{d};\;\; k^2 = \frac{d_2}{d} = \frac{d_1^2}{d^2}$$

respectively. Whence $dd_2 = d_1^2$, so we are done by Lemma (2.4). Q.E.D.

In particular, it follows that any point $P \in \ell_\infty$ cannot be a point of multiplicity n for $\overline{\Gamma}$ (i.e., Γ can consist of n parallel lines only in the degenerate case). Moreover, if $\overline{\Gamma}$ is smooth at a point $P \in \ell_\infty$, then P cannot be a contact point of order n, i.e., an n-osculating point. In fact, as we show below, $\overline{\Gamma}$ cannot even be tangent to ℓ_∞ at P.

Let us recall a straightforward fact we need in the proof of the theorem below.

Lemma 3.3 *Let $\mathcal{C}$ be an irreducible curve, $P \in \mathcal{C}$ a smooth point, and let $\sigma : \mathcal{C} \to \mathcal{C}$ be an involution such that $\sigma(P) = P$. Then either the differential of σ at P is the multiplication by -1 or σ is the identity map.*

Theorem 3.4 *Let (X,L) be an n-dimensional polarized pair, $n \geq 2$. Suppose that the Hilbert curve $\overline{\Gamma}$ is smooth. Then $\overline{\Gamma}$ meets the line $\ell_\infty : z = 0$ in n distinct points A_i, $i = 1, \ldots, n$. Moreover, the line joining the central point C with A_i is tangent to $\overline{\Gamma}$ at the points A_i for every $i = 1, \ldots, n$.*

Proof. Since $\overline{\Gamma}$ is smooth, we can apply Lemma (3.3) to $\mathcal{C} = \overline{\Gamma}$ and $\sigma = \overline{s}_{|\overline{\Gamma}}$, concluding that the differential of $\overline{s}$ is the multiplication by -1 on the tangent space $T_A(\overline{\Gamma})$ of $\overline{\Gamma}$ at any point $A \in \overline{\Gamma}$. Now suppose by contradiction that $A \in \ell_\infty$ and that $\overline{\Gamma}$ is tangent to ℓ_∞ at A. Then the projective closure of $T_A(\overline{\Gamma})$ coincides with ℓ_∞ and we know that $\overline{s}$ induces the identity on ℓ_∞. This leads to a contradiction. Therefore $\overline{\Gamma}$ cannot be tangent to ℓ_∞.

To prove the second assertion, let $A = A_i$ for any $i = 1, \dots, n$, and consider the tangent line ℓ given by the projective closure of $T_A(\overline{\Gamma})$. Since $\overline{\Gamma}$ is fixed by $\overline{s}$, the line ℓ is fixed as well by $\overline{s}$. Since the lines fixed by $\overline{s}$ are only ℓ_∞ and the lines through the central point C, we conclude that $C \in \ell$. Q.E.D.

3.5 Characterizing smooth surfaces with reducible Hilbert curve. Let X be a smooth surface polarized by a (very) ample line bundle L. Suppose that the numerical classes of L and K_X are linearly independent in the vector space $N^1(X)$. In particular this rules out minimal surfaces of Kodaira dimension $\kappa(X) = 0$, as well as multi-canonical (anti-multi-canonical) pairs from our considerations.

According to the general definition, the Hilbert curve Γ of (X, L) is the affine conic of equation

$$p(x, y) = \chi(xK_X + yL) = \frac{1}{4}\big(2d_2x^2 + 4d_1xy + 2dy^2 - 2d_2x - 2d_1y + 4\chi(\mathcal{O}_X)\big) = 0.$$

As a matrix for Γ we can take

$$A = \begin{pmatrix} 2d_2 & 2d_1 & -d_2 \\ 2d_1 & 2d & -d_1 \\ -d_2 & -d_1 & 4\chi(\mathcal{O}_X) \end{pmatrix}.$$

Note that

$$\begin{vmatrix} 2d_2 & 2d_1 \\ 2d_1 & 2d \end{vmatrix} = 4(d_2d - d_1^2) < 0$$

by the Hodge index theorem, due to the assumption that $\mathrm{rk}\langle K_X, L\rangle = 2$. This implies that, when irreducible, our Γ is a hyperbola with center $C = (\frac{1}{2}, 0)$, the central point of the Serre involution, and asymptotes with slopes $(-d_1 \pm \sqrt{d_1^2 - d_2d})/d$. Moreover, computing the determinant of A we see that

$$\det(A) = 2(d_2 - 8\chi(\mathcal{O}_X))\big(d_1^2 - d_2d\big) = 0.$$

Therefore Γ is reducible, and consisting of two distinct lines through C, if and only if (compare with (4))

$$d_2 = 8\chi(\mathcal{O}_X). \tag{8}$$

Let us describe pairs (X, L) characterized by condition (8). Note that (8) does not involve any polarization. Hence, once X is known, we can take for L any (very) ample line bundle whose numerical class does not belong to the ray generated by K_X. We proceed case-by-case according to the Kodaira dimension.

Let $\kappa(X) = -\infty$ and let $\eta : X \to X_0$ be a birational morphism from X to a minimal model X_0. Then $d_2 = K_{X_0}^2 - t$, where t is the number of blowing–ups η factors through. Moreover, $\chi(\mathcal{O}_X) = 1 - q$ and $K_{X_0}^2 = 8(1 - q)$, where $q = h^1(\mathcal{O}_{X_0}) = h^1(\mathcal{O}_X)$. Thus condition (8) becomes

$$8(1 - q) - t = d_2 = 8\chi(\mathcal{O}_X) = 8(1 - q).$$

This happens if and only if $t = 0$, i.e., $X = X_0$. Therefore if $\kappa(X) = -\infty$ condition (8) holds if and only if X is a $\mathbb{P}^1$-bundle over a smooth curve of any genus.

Let $\kappa(X) = 0$. Then X is not minimal according to our assumption, hence $d_2 < 0$. On the other hand, $\chi(\mathcal{O}_X) \geq 0$, X being non-ruled. Therefore equality (8) cannot occur.

Let $\kappa(X) = 1$. Then $d_2 \leq 0$ with equality if and only if X is minimal. Since $\chi(\mathcal{O}_X) \geq 0$ condition (8) holds if and only if X is a minimal surface with $\chi(\mathcal{O}_X) = 0$ (i.e., X is an elliptic quasi-bundle, in Serrano's terminology [5, Prop. (4.2)]). Note that in this case Γ has equation

$$y(2d_1x + dy - d_1) = 0,$$

hence the x-axis is a component of Γ. On the other hand this fact occurs only in this case and for elliptic $\mathbb{P}^1$-bundles. Actually, it requires that $\chi(\mathcal{O}_X) = 0$ and this cannot happen if $\kappa(X) = 2$.

Finally, let $\kappa(X) = 2$. We have $d_2 = K^2_{X_0} - t$ again, where $t \geq 0$ is the number of blowing-ups factoring a birational morphism from X to its minimal model X_0. So, equality (1) implies that $K^2_{X_0} = 8\chi(\mathcal{O}_{X_0}) + t$. Recalling the Miyaoka–Yau inequality $K^2_{X_0} \leq 9\chi(\mathcal{O}_{X_0})$, we see that $t \leq \chi(\mathcal{O}_X)$ and X is obtained by a sequence of t blowing-ups from a minimal surface X_0 sitting in the corner $8\chi(\mathcal{O}_X) \leq K^2 \leq 9\chi(\mathcal{O}_X)$.

Note also that, by Lemma (2.4), the conic Γ is a double line (equivalently, Γ is singular at infinity) if and only if $dd_2 = d_1^2$.

It is worth mentioning that the fake quadric X (i.e., the surface of general type homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) gives an example. Indeed, since $p_g(X) = q(X) = 0$, the exponential exact sequence yields $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$, whence $\rho(X) = 2$. Moreover the corner above is densely populated according to a result of the third author [6]. Thus there are surfaces with Picard number ≥ 2 providing further examples.

4 Cubic Hilbert curves

Let X be an n-dimensional projective variety ($n \geq 2$), and let L be an ample (or merely nef and big) line bundle on X. We keep the notation as in §3.

The case $n = 3$ is of special interest. We discuss here several properties of the cubic $\overline{\Gamma}$, starting with the singular case.

Note that the affine Hilbert curve Γ cannot split in three general lines, due to the fact that Γ is symmetric with respect to the central point C by (2.1), 1).

Next, note that if Γ is singular at a point $P = (x_0, y_0)$, $P \neq C = \left(\frac{1}{2}, 0\right)$, then it is also singular at $P' = (1 - x_0, -y_0)$, again by the symmetry with respect to C. In particular, Γ is reducible since it contains the line $\langle P, P' \rangle$. On the other hand, by (2.1), 3), Γ cannot be singular at C unless it is reducible. Thus, if Γ is singular, it must be reducible.

Example 4.1 Let $X \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(4, 4)|$ be a smooth hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ and let $L = \mathcal{O}_X(1, 2)$. In this case we can see that

$$p_{(X,L)}(x, y) = (2x + 3y - 1)(2x^2 + 6xy + 4y^2 - 2x - 3y + 8).$$

Then Γ splits into a line passing through C, and an irreducible conic with center C.

Now, we look at singularities at infinity. Let

$$p(x, y, z) = \frac{(xK_X + yL)^3}{6} - \frac{K_X \cdot (xK_X + yL)^2}{4} z + \mathrm{O}(z^2)$$

be the equation of $\overline{\Gamma} \subset \mathbb{P}^2{}_{[x,y,z]}$, with $z = 0$ defining the line at infinity ℓ_∞. Then the points $[x, y, 0]$ of $\overline{\Gamma}$ satisfy the condition

$$(xK_X + yL)^3 = 0.$$

On the other hand if $[x, y, 0]$ is a singular point of $\overline{\Gamma}$ its coordinates have to annihilate the partial derivatives of $p(x, y, z)$ with respect to x and y. This gives the further conditions

$$K_X \cdot (xK_X + yL)^2 = L \cdot (xK_X + yL)^2 = 0.$$

Hence we get

$$(xK_X + yL)^3 = x^3 d_3 + 3x^2 y d_2 + 3xy^2 d_1 + y^3 d = 0, \tag{9}$$

$$K_X \cdot (xK_X + yL)^2 = x^2 d_3 + 2xyd_2 + y^2 d_1 = 0, \tag{10}$$

$$L \cdot (xK_X + yL)^2 = x^2 d_2 + 2xyd_1 + y^2 d = 0. \tag{11}$$

In particular we see that

$$[0, 1, 0] \notin \overline{\Gamma}, \tag{12}$$

since otherwise (9) gives $d = L^3 = 0$, contradicting ampleness.

Assume $d_1 = K_X \cdot L^2 \neq 0$. From (10) and (11) we get

$$x^2 dd_3 + 2xyd_2 d = d_1 d_2 x^2 + 2xyd_1^2.$$

Recalling (12), we obtain $(d_1 d_2 - dd_3)x = 2(d_2 d - d_1^2)y$, which leads to the conclusion that

$$\left[1, \frac{d_1 d_2 - dd_3}{2(d_2 d - d_1^2)}, 0\right] \tag{13}$$

is the only singular point of the cubic $\overline{\Gamma}$ on the line $\ell_\infty : z = 0$.

On the other hand, if $d_1 = 0$ condition (9) follows from (10) and (11), and by (12), condition (10) gives $\dfrac{y}{x} = -\dfrac{d_3}{2d_2}$. This leads to the same conclusion as above.

So we get the following numerical characterization for the cubic Γ to have a singular point at infinity. For an explicit example, see (6.5).

Proposition 4.2 *The Hilbert curve $\overline{\Gamma}$ has a singular point (whose coordinates are given by (13)) on the line $\ell_\infty : z = 0$ if and only if*

$$d \left(\frac{d_1 d_2 - dd_3}{2(d_2 d - d_1^2)} \right)^2 + 2 \left(\frac{d_1 d_2 - dd_3}{2(d_2 d - d_1^2)} \right) d_1 + d_2 = 0.$$

Proof. Use (11) combined with (13). Q.E.D.

Recall that by Lemma (3.2), if $|L|$ contains a smooth surface, the cubic $\overline{\Gamma}$ can have a double point at most on the line ℓ_∞, unless we are in the degenerate case.

Relation (5) allows us to specialize Theorem (3.1) to smooth 3-folds. (The equivalence in the statement below follows immediately from Noether's formula $e(S) + K_S^2 = 12\chi(\mathcal{O}_S)$, after noting that $2e(S) = d_2 + 2d_1 + 2d = (K_X + L)^2 \cdot L + d = K_S^2 + d$.)

Proposition 4.3 *Let (X, L) be a 3-dimensional polarized variety. Assume that $|L|$ contains a smooth surface S. Further assume that we are not in the degenerate case. Then the Hilbert curve of (X, L) has a triple point if and only if*

$$48\chi(\mathcal{O}_X) + d_3 = 0 \quad \textit{and} \quad 2e(S) = d_2 + 2d_1 + 2d \ (\textit{i.e.}, K_S^2 = 8\chi(\mathcal{O}_S) - d/3).$$

Proof. It simply follows from Theorem (3.1) recalling expression (5). Q.E.D.

Example 4.4 Let $X = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ be the product of three smooth curves $\mathcal{C}_i$, $i = 1, 2, 3$, and let $L = L_1 \boxtimes L_2 \boxtimes L_3$ for some ample line bundles $L_i \in \mathrm{Pic}(\mathcal{C}_i)$, $i = 1, 2, 3$. Then the Hilbert curve of (X, L) has a triple point, so that the (quasi)-polarized pair (X, L) gives an example as in (4.3). Indeed, by (2.5), we know that the Hilbert curve $\Gamma = \ell_1 \cup \ell_2 \cup \ell_3$ of (X, L) is split into three lines ℓ_i, where ℓ_i is the Hilbert curve of the polarized pair $(\mathcal{C}_i, L_i)$, $i = 1, 2, 3$. By (1), each line ℓ_i passes through the central point C of the Serre involution.

Remark 4.5 As already noted as a comment on relation (5), if the central point C is a smooth point of $\overline{\Gamma}$, then the tangent line to $\overline{\Gamma}$ at C is an inflectional tangent; hence C is a flex of $\overline{\Gamma}$. Furthermore, suppose that $\overline{\Gamma}$ is smooth. Then by Theorem (3.4) we know that $\overline{\Gamma}$ meets the line at infinity in three distinct points A_i, $i = 1, 2, 3$. Moreover, the line joining C with A_i is tangent to $\overline{\Gamma}$ at A_i for every i. Combining this with Abel's theorem on elliptic integrals we can identify C and the A_i's as the zero and the points of order 2 of the group structure of $\overline{\Gamma}$.

Remark 4.6 (The non-reduced case) Assume that the Hilbert curve is not reduced. Then by Lemma (3.2) we see that $\overline{\Gamma}$ cannot split in three coinciding lines. Since the affine Hilbert curve is symmetric with respect to the central point $C = \left(\frac{1}{2}, 0\right)$ we thus conclude that $\overline{\Gamma}$ has equation of the form

$$\overline{\Gamma} : \left(a\left(x - \frac{z}{2}\right) + by\right)\left(a'\left(x - \frac{z}{2}\right) + b'y\right)^2 = 0,$$

for some complex coefficients a, b, a', b'. Since $\left[1, -\frac{a'}{b'}, 0\right]$ is the only singular point on the line $\ell_\infty : z = 0$, we know by (13) that it must be

$$\frac{a'}{b'} = \frac{d_1 d_2 - d d_3}{2(d_2 d - d_1^2)}.$$

Furthermore the pair (X, L) satisfies the numerical conditions expressed by Proposition (4.2) and (if L is spanned by global sections) by Proposition (4.3) as well.

Note that by the above, taking into account expression (5), the coefficients a and b can be expressed in terms of the invariants d, d_1, d_2, d_3.

The following discussion leads to exhibit a nontrivial class of polarized threefolds whose Hilbert curves are non-reduced cubics.

Let $n = 3$ and let (X, L) be a quadric fibration over a smooth curve B via a morphism $\varphi : X \to B$. Then $\mathcal{E} := \varphi_* L$ is a vector bundle of rank 4 on B. Set $P := \mathbb{P}_B(\mathcal{E})$, let $p : P \to B$ be the bundle projection and consider the tautological line bundle ξ of $\mathcal{E}$ on P. Then X embeds fiberwise inside P (i.e., $\varphi = p_{|X}$) as a divisor $X \in |2\xi - p^*\mathcal{B}|$ for some

$\mathcal{B} \in \mathrm{Pic}(B)$. Moreover, $L = \xi_X$. Set $e = \deg \mathcal{E}$, $b = \deg \mathcal{B}$ and recall that the number of singular fibers of φ is (e.g. see [2, p. 83], but note that our b is $-b$ in [2])

$$\delta = 2e - 4b \tag{14}$$

Letting $\mathcal{A} := K_B + \det \mathcal{E} - \mathcal{B}$, we get by adjunction

$$K_X = (K_P + 2\xi - p^*\mathcal{B})_X = (-2\xi + p^*\mathcal{A})_X = -2L + \varphi^*\mathcal{A}.$$

This allows us to compute the following invariants of (X, L), where q is the genus of B.

$$d = L^3 = \xi_X^3 = \xi^3(2\xi - p^*\mathcal{B}) = 2e - b;$$

$$d_1 = K_X L^2 = (-2\xi + p^*\mathcal{A})\xi^2(2\xi - p^*\mathcal{B}) = 4(q-1) - 2e;$$

$$d_2 = K_X^2 L = (-2\xi + p^*\mathcal{A})^2\xi(2\xi - p^*\mathcal{B}) = -16(q-1) + 4b;$$

$$d_3 = K_X^3 = (-2\xi + p^*\mathcal{A})^3(2\xi - p^*\mathcal{B}) = 48(q-1) + 8e - 16b.$$

Moreover, $\chi(\mathcal{O}_X) = 1-q$. This can be computed by the formula $\chi(\mathcal{O}_X) = \frac{1}{24}c_1(T_X)c_2(T_X)$, where T_X is the tangent bundle of X, recalling the tangent–normal bundle sequence

$$0 \to T_X \to T_P|_X \to [2\xi - p^*\mathcal{B}] \to 0$$

and with the help of the following two standard exact sequences on P:

$$0 \to T_{P/B} \to T_P \to p^*T_B \to 0,$$

$$0 \to \mathcal{O}_P \to p^*\mathcal{E}^\vee \otimes \xi \to T_{P/B} \to 0.$$

Finally, suppose that $|L|$ contains a smooth surface S. Note that $\varphi|_S$ makes (S, L_S) a conic fibration over B. In particular, S is ruled, hence $\chi(\mathcal{O}_S) = 1 - q$. Moreover, by adjunction we get $K_S = (K_X + L)_S = (-\xi + p^*\mathcal{A})_S$, and so,

$$K_S^2 = (-\xi + p^*\mathcal{A})^2(2\xi - p^*\mathcal{B})\xi = -8(q-1) - 2e + 3b.$$

Therefore Noether's formula gives

$$e(S) = -4(q-1) + 2e - 3b.$$

Proposition 4.7 *Let (X, L) be a 3-dimensional quadric fibration over a smooth curve B and suppose that $|L|$ contains a smooth surface S. Let Γ be the Hilbert cubic curve associated to (X, L). Then the following facts are equivalent:*

1. *Γ has a triple point;*
2. *Γ is non-reduced (in fact consisting of a line with multiplicity 2 plus another line, the two lines meeting at the center of the Serre involution);*
3. *X has no singular fibres.*

Proof. We can confine to prove that 1) $\Rightarrow$ 3) $\Rightarrow$ 2), the implication 2) $\Rightarrow$ 1) being obvious. Let φ, e and b be as before. By Proposition 4.3, taking into account the above computations we see that Γ has a triple point if and only if $e = 2b$. But, according to (14) this is equivalent to φ having no singular fibers, i.e., condition 3). Now, let $e = 2b$. Then, recalling (5), a direct check shows that the equation of Γ, expressed in the coordinates $u = x - \frac{1}{2}$ and $v = y$, becomes

$$p(u, v) = \frac{1}{6}(d_3u^3 + 3d_2u^2v + 3d_1uv^2 + dv^3) = \frac{1}{4}(2u - v)^2\big(8(q - 1)u + ev\big) = 0.$$

This proves that Γ is non-reduced. Q.E.D.

We made the blanket assumption of considering pairs not in the degenerate case. Accordingly, Γ cannot consist of a line with multiplicity 3. Note that if 1) holds, then $e = \frac{2}{3}d$, hence $e > 0$. Looking at the above equation we thus see that $q = 0$ with $e = 4$ can occur only if our quadric fibration (X, L) is in the degenerate case. Actually, in this situation we have $\mathcal{A} = \mathcal{O}_{\mathbb{P}^1}$, hence $K_X = -2L$. In particular, if $\mathcal{E}$ is ample, then necessarily $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 4}$. Thus $P = \mathbb{P}^3 \times \mathbb{P}^1$, $\xi = \mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(1, 1)$, $X \in |\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(2, 0)|$, so that $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with $L = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(1, 1, 1)$.

4.8 The j-invariant. Suppose that $n \geq 2$ and $\overline{\Gamma}$ is smooth. A natural question is about moduli. Of course, if $n = 2$, $\overline{\Gamma}$ is a conic and there is nothing to say. So, let $n \geq 3$.

Proposition 4.9 *Let X be smooth variety of dimension $n \geq 3$ with Picard number $\rho(X) = 2$. Then for any two ample (or merely nef and big) line bundles $L_1, L_2 \in \mathrm{Pic}(X) \setminus \langle K_X\rangle$ the corresponding Hilbert curves Γ_1, Γ_2 are equivalent up to an affinity.*

Proof. This simply follows from the fact that $\{K_X, L_1\}$ and $\{K_X, L_2\}$ are two bases of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Q.E.D.

In particular, if $n = 3$ and $\rho(X) = 2$ it follows that the two plane cubics $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are projectively equivalent for any $L_1, L_2 \in \mathrm{Pic}(X) \setminus \langle K_X\rangle$. Hence, if they are smooth, they have the same j-invariant.

Example 4.10 Inside $\mathbb{P}^2 \times \mathbb{P}^2$ consider a smooth hypersurface $X \in |\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(h, k)|$, where h, k are positive integers. Note that $\rho(X) = 2$ by Lefschetz theorem, any line bundle on X being induced by $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(m, n)$ for some integers m, n. According to Proposition (4.9), for any line bundle $L = \mathcal{O}_X(m, n)$ with m, n positive integers such that $m(k - 3) \neq n(h - 3)$ the projective Hilbert curve $\overline{\Gamma}_{(h,k)}$ of (X, L) has the same j-invariant.

E.g., for $(h, k) = (2, 3)$, a computation carried out by using `algcurves` of MAPLE 11 package shows that $\overline{\Gamma}_{(2,3)}$ is smooth with j-invariant $j = \frac{702595369}{72900}$.

On the other hand, varying h, k, even keeping m, n fixed, we can see that the j-invariant varies. Here is a list of values obtained by using MAPLE 11 program, in the case $(m, n) = (1, 1)$.

(h,k)	$j = j(\overline{\Gamma}_{(h,k)})$
$(2,3)$	$\frac{702595369}{72900}$
$(2,4)$	$\frac{148176}{25}$
$(2,5)$	$\frac{611960049}{122500}$
$(3,4)$	$\frac{5203798902289}{57153600}$
$(3,5)$	$\frac{20034997696}{455625}$
$(4,5)$	$\frac{4102915888729}{9000000}$

In the computation process, the program warns us if Γ is reducible. For instance, for $(h,k,m,n) = (4,5,1,2)$ we have $K_X = L$, hence Γ consists of three parallel lines according to (2.3). Also, for $(h,k,m,n) = (2,2,2,3)$, the program warns us that Γ is reducible. In this case, $L \notin \langle K_X \rangle$. However $K_X + 3L \approx p_1^* \mathcal{O}_{\mathbb{P}^2}(-1)$ and $K_X + 2L \approx p_2^* \mathcal{O}_{\mathbb{P}^2}(1)$, where p_1, p_2 are the restrictions to X of the projections of $\mathbb{P}^2 \times \mathbb{P}^2$ on the two factors. In fact, Γ is the union of two parallel lines with a third line according to Theorem (6.1) (taking $\frac{a}{b} = 3$ and 2 respectively).

It thus follows from Proposition (4.9) that $\Gamma_{(2,2)}$ is the union of two parallel lines with a third line for every $L \in \mathrm{Pic}(X) \setminus \langle K_X \rangle$. Note that this cannot be deduced directly from Theorem (6.1) if $L = \mathcal{O}_X(m,n)$ with $(m,n) \neq (2,3)$.

5 Image of the Hilbert curve in $\mathbb{P}^3$

Let (X,L) be an n-dimensional polarized variety, and let Γ be the Hilbert curve of (X,L). As usual, denote by $\overline{\Gamma}$ the projective closure of Γ in $\mathbb{P}^2$, where $[x,y,z]$ are homogeneous coordinates, with $\ell_\infty : z = 0$ the line at infinity. Let $\overline{s} : \mathbb{P}^2 \to \mathbb{P}^2$ be the Serre involution, and let $\overline{\Gamma}/\langle\, \overline{s}\,\rangle := \gamma$.

Make the change of coordinates $[x,y,z] \mapsto [x - \frac{z}{2}, y, z]$, so that the central point becomes $C = [0,0,1]$, and consider the map

$$\Phi : \mathbb{P}^2 \to \mathbb{P}^3_{[T_0,T_1,T_2,T_3]} \text{ defined by } \big[x - \frac{z}{2}, y, z\big] \mapsto \big[(x - \frac{z}{2})^2, (x - \frac{z}{2})y, y^2, z^2\big]. \tag{15}$$

We have the following commutative diagram

$$\begin{tikzcd} \overline{\Gamma} \subset \mathbb{P}^2 \arrow[r, "v"] \arrow[d] \arrow[dr, "\Phi"] & S \subset \mathbb{P}^5 \arrow[d] \\ \gamma \arrow[r] & \mathcal{Q} \subset \mathbb{P}^3, \end{tikzcd} \tag{16}$$

where $v : [x - \frac{z}{2}, y, z] \mapsto [(x - \frac{z}{2})^2, (x - \frac{z}{2})y, (x - \frac{z}{2})z, y^2, yz, z^2]$ is the Veronese embedding, and $S \to \mathcal{Q}$ is the two-to-one morphism obtained by projection of the Veronese surface S from the line $x_0 = x_1 = x_3 = x_5 = 0$ onto the quadric cone $\mathcal{Q} \cong \mathbb{P}^2/\langle\, \overline{s}\,\rangle \subset \mathbb{P}^3$ of equation $T_0T_2 - T_1^2 = 0$.

Express Φ locally around C in affine coordinates as $(x-1/2,y)\mapsto\big((x-\frac{1}{2})^2,(x-\frac{1}{2})y,y^2\big)$. Then the Jacobian matrix

$$\begin{pmatrix} 2\big(x-\frac{1}{2}\big) & y & 0 \\ 0 & x-\frac{1}{2} & 2y \end{pmatrix}$$

has rank 1 if and only if $y=0$, $x=\dfrac{1}{2}$, that is Φ is ramified at the central point C.

Similarly, fix a point on ℓ_∞, e.g., $[0,1,0]$, and take (x,z) as local coordinates around it. Then Φ expresses locally as $(x,z)\mapsto\big((x-\frac{z}{2})^2,x-\frac{z}{2},z^2\ \big)$. Therefore the Jacobian matrix

$$\begin{pmatrix} 2\big(x-\frac{z}{2}\big) & 1 & 0 \\ -\big(x-\frac{z}{2}\big) & -\frac{1}{2} & 2z \end{pmatrix}$$

has rank 1 if and only if $z=0$.

These local computations show the following simple property.

Proposition 5.1 *Let (X,L) be an n-dimensional polarized variety. Consider the map $\Phi:\mathbb{P}^2\to\mathcal{Q}\subset\mathbb{P}^3$ defined as in (15). Then Φ is a two-to-one immersion outside of the central point C of the Serre involution and the line $\ell_\infty: z=0$.*

As we observed, $\gamma=\overline{\Gamma}/\langle\overline{s}\rangle$ is a twisted curve contained in quadric cone $\mathcal{Q}\subset\mathbb{P}^3$. Moreover, $\deg(\gamma)=n$ by construction. A further property holds true.

Proposition 5.2 *Let (X,L) be an n-dimensional polarized variety. Assume that the Hilbert curve $\overline{\Gamma}$ of (X,L) is smooth. Then γ is a smooth Castelnuovo's curve in $\mathbb{P}^3$.*

Proof. Let $\widetilde{\gamma}$ be a desingularization of γ. Then we have a commutative diagram

$$\begin{array}{ccc} \overline{\Gamma} & \longrightarrow & \widetilde{\gamma} \\ & \searrow & \downarrow \\ & & \gamma \end{array}$$

where $\overline{\Gamma}\to\widetilde{\gamma}$ is a two-to-one map and $\widetilde{\gamma}\to\gamma$ is a one-to-one map.

First, assume that n is odd. Then we know from (1) that the central point C of $\overline{s}$ belongs to $\overline{\Gamma}$. Therefore the map $\overline{\Gamma}\to\widetilde{\gamma}$ is ramified along the $n+1$ points $\{C,\ell_\infty\cap\overline{\Gamma}\}$. Thus, Hurwitz's theorem yields

$$2(g(\overline{\Gamma})-1)=2(2g(\widetilde{\gamma})-2)+n+1.$$

Since $g(\overline{\Gamma})=\dfrac{(n-1)(n-2)}{2}$, we find $g(\widetilde{\gamma})=\dfrac{1}{4}(n^2-4n+3)$, which equals Castelnuovo's bound $g_{\max}$ for odd degree n curves in $\mathbb{P}^3$.

If n is even, we know by (2.1), 2) that $C\notin\overline{\Gamma}$ since $\overline{\Gamma}$ is smooth. Then the map $\overline{\Gamma}\to\widetilde{\gamma}$ is ramified along the n points $\{\ell_\infty\cap\overline{\Gamma}\}$. The same argument as above gives now $g(\widetilde{\gamma})=\dfrac{n^2}{4}-n+1$, which equals $g_{\max}$ for n even.

Let $g(\gamma)$ be the arithmetic genus of γ. Since $g(\widetilde{\gamma})\le g(\gamma)\le g_{\max}$, we thus conclude that $g(\widetilde{\gamma})=g(\gamma)$, which implies that $\gamma\cong\widetilde{\gamma}$ is a smooth Castelnuovo's curve in $\mathbb{P}^3$, as claimed. Q.E.D.

6 Fibrations and singular points of the Hilbert curve at infinity

In this section we show how the existence of some fibrations on a variety X forces the Hilbert curve to have lines as components. In order to have better statements we allow here line bundles on X slightly more general than in (2.2).

Theorem 6.1 *Let X be a smooth n-dimensional smooth variety, and let $\varphi : X \to Y$ be a morphism onto a normal variety Y of dimension* $\dim(Y) < \dim(X)$. *Let L be a φ-nef and φ-big line bundle on X, and assume that for coprime positive integers a, b, $K_X + \frac{a}{b}L = \varphi^*A$ for some $\mathbb{Q}$-line bundle A on Y. Then $\chi(xK_X + yL) = 0$ for all integers x, y belonging to the $a-1$ parallel lines $ax - by - i = 0$ for $i = 1, \ldots, a-1$. In particular,*

$$p(x, y) = \prod_{i=1}^{a-1}(ax - by - i)R(x, y),$$

for some degree $n - a + 1$ factor $R(x, y)$ (so that the projective Hilbert curve $\overline{\Gamma} \subset \mathbb{P}^2$ of (X, L) has a point of multiplicity at least $a - 1$ at $[b, a, 0]$).

Proof. Choose positive integers α, β such that $b\alpha - \beta b = 1$. Let $\mathfrak{L} := \beta K_X + \alpha L$. By using Lemma [1, (1.5.6)] we can "remove denominators", letting us to conclude that $\mathfrak{L}$ is φ-nef and φ-big. Then Kawamata–Viehweg vanishing theorem [4, Theorem 1-2-3] applies to give, for any integer $t > 0$,

$$R^j\varphi_*(K_X + t\mathfrak{L}) = 0, \quad \text{for } j > 0. \tag{17}$$

We claim that

$$\varphi_*(K_X + t\mathfrak{L}) = 0, \quad \text{for } 1 \leq t \leq a - 1. \tag{18}$$

To see this, it is enough to show that the restriction $(K_X + t\mathfrak{L})_F$ to any fiber F of φ is the opposite of an ample line bundle on F. In fact, write

$$(K_X + t\mathfrak{L})_F = \left((1 + t\beta)K_X + t\alpha L)_F\right) = (1 + t\beta)\left(K_X + \frac{t\alpha}{1 + t\beta}L\right)_F.$$

Since $K_X + \frac{a}{b}L$ restricts trivially to F, it suffices to show that

$$\frac{t\alpha}{1 + t\beta} < \frac{\alpha}{\beta},$$

or, equivalently,

$$t(a\beta - \alpha b) + a = -t + a < 0.$$

This is fact true, proving the claimed assertion (18).

By combining (17) and (18), the Leray spectral sequence gives, for each $j \geq 0$ and $1 \leq t \leq a - 1$,

$$H^j(X, K_X + t\mathfrak{L}) = H^j(Y, \varphi_*(K_X + t\mathfrak{L})) = 0.$$

Now, set $x = 1 + t\beta$, $y = t\alpha$, so that $K_X + t\mathfrak{L} = xK_X + yL$. Thus any such integers x, y satisfy the condition $p(x, y) = \chi(xK_X + yL) = 0$.

Rewriting the relation $b\alpha - \beta a = 1$ as $ax - by - (a - t) = 0$, we see that for each integer $i := a - t = 1, \ldots, a - 1$, the line of equation $ax - by - i = 0$ is contained in Γ, so we are done. Q.E.D.

Example 6.2 Consider $X = \mathbb{P}^2 \times \mathbb{P}^3$ and $L = \mathcal{O}_X(1,1)$. Let p_i, $i = 1,2$, be the projections on the two factors. Then $K_X + 3L = \mathcal{O}_X(0,-1) = p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)$ as well as $K_X + 4L = \mathcal{O}_X(1,0) = p_1^*\mathcal{O}_{\mathbb{P}^2}(1)$. Thus the projective Hilbert curve $\overline{\Gamma}$ is a plane quintic having a double point at $[1,3,0]$ and a triple point at $[1,4,0]$.

Remark 6.3 Slightly different versions of Theorem (6.1) above allow singularities on the variety X, but require more restrictive assumptions on the line bundle L. Precisely, the same conclusion as in Theorem (6.1) holds true in the following cases.

a) X is an n-dimensional variety with terminal singularities, there is a Zariski open subset $U \subset Y$ such that $\varphi^{-1}(U)$ is smooth, and L is a φ-semiample and φ-big line bundle on X.

b) X is an n-dimensional variety with terminal singularities, and L is a φ-ample line bundle on X.

In both cases the proof runs parallel to that of Theorem (6.1). In case a) to get the same assertion as in (17), we have to combine the fact that $R^j\varphi_*(K_X + s\mathcal{L})$ is torsion free by [4, Theorem 1-2-7] with the fact that it is zero on U. In case b) one has simply to replace the use of [4, Theorem 1-2-3] with [4, Theorem 1-2-5].

It is worth noting that Theorem (6.1) applies in particular to the case when the canonical bundle K_X is not nef, L is an ample line bundle on X and $\varphi : X \to Y$ is the *nefvalue morphism* of (X, L), that is φ is defined by $|m(bK_X + aL)|$ for $m \gg 0$ and coprime positive integers a, b. In this case $\tau := a/b$ is said to be the *nefvalue* of (X, L).

Considering the nefvalue morphism allows us to describe a further property of the Hilbert curve (not covered by Theorem (6.1) when $\tau = 1/b$).

Proposition 6.4 *Let (X, L) be an n-dimensional polarized variety, $n \geq 2$. Assume that K_X is not nef, let $\tau = u/v$ be the nefvalue of (X, L) and let $\varphi : X \to Y$ be the nefvalue morphism of (X, L). If $\dim(\varphi(X)) \leq n-2$, then the projective Hilbert curve $\overline{\Gamma}$ is singular at the point $[1, \tau, 0]$.*

Proof. Let

$$p(x,y,z) = \frac{(xK_X + yL)^n}{n!} - \frac{K_X \cdot (xK_X + yL)^{n-1}}{2(n-1)!}z + \mathrm{O}(z^2)$$

be the equation of $\overline{\Gamma} \subset {\mathbb{P}^2}_{[x,y,z]}$, with $z = 0$ defining the line at infinity ℓ_∞. Then the points $[x, y, 0]$ of $\overline{\Gamma}$ satisfy the condition

$$(xK_X + yL)^n = 0. \tag{19}$$

Computing the singularities at infinity we have therefore to consider the restriction to ℓ_∞ of the equations

$$\frac{\partial(xK_X + yL)^n}{\partial x} = 0 \quad \text{and} \quad \frac{\partial(xK_X + yL)^n}{\partial y} = 0,$$

or else

$$\frac{K_X \cdot (xK_X + yL)^{n-1}}{(n-1)!} = 0 \quad \text{and} \quad \frac{L \cdot (xK_X + yL)^{n-1}}{(n-1)!} = 0. \tag{20}$$

This shows that if $[x,y,0] \in \overline{\Gamma}$ and $(xK_X + yL)^{n-1}$ is a numerically trivial cycle, then $[x,y,0]$ is a singular point of $\overline{\Gamma}$. Note that whenever $\dim(\varphi(X)) \leq n-2$, then $p(v,u,0) = 0$, and conditions (19), (20) are satisfied by $(x,y) = (v,u)$.

Notice that if $\dim(\varphi(X)) = n-1$ the above argument shows that $[1,\tau,0] \in \overline{\Gamma}$. Q.E.D.

Example 6.5 (Scrolls over curves) With the notation as in (6.1), assume that $\varphi : X \to Y$ is a scroll over an m-dimensional variety Y. Then $a = n-m+1$, $b = 1$, so that

$$p(x,y) = \prod_{i=1}^{n-m} \big((n-m+1)x - y - i\big) R(x,y), \tag{21}$$

for some degree m factor $R(x,y)$. Writing $x = u - \frac{1}{2}$, $v = y$, we get the symmetric expression (in terms of j)

$$p(x,y) = \prod_{j=-(n-m-1)}^{n-m-1} (2(n-m+1)u - 2v + j)\, R\left(u + \frac{1}{2}, v\right),$$

where j satisfies the condition $j = n-m+1-2i$ (hence in particular $j \neq 0$ if $n-m$ is even).

In the special case when Y is a curve ($m = 1$) the expression (21) becomes

$$p(u,v) = \frac{1}{2}\Big[\prod_{j=-(n-2);\, n-j \text{ even}}^{n-2} (2nu - 2v + j) R\left(u + \frac{1}{2}, v\right)\Big].$$

E.g., for $n = 3$,

$$p(u,v) = \frac{1}{2}(6u - 2v + 1)(6u - 2v - 1) R\left(u + \frac{1}{2}, v\right).$$

Let us compute the factor $R(u + \frac{1}{2}, v)$ in the special case when $X = \mathbb{P}(\mathcal{E})$ for an ample rank 2 vector bundle on a smooth curve Y of genus g, with bundle projection $\pi : X \to Y$.

Let $L := \xi$ be the tautological line bundle of $\mathcal{E}$ on X. Then, since $K_X \approx -3\xi + \pi^*(K_X + \det \mathcal{E})$, we get

$$d = L^3 = \xi^3 = \deg \mathcal{E};$$

$$d_1 = K_X \cdot L^2 = (-3\xi + \pi^*(K_X + \det \mathcal{E})) \cdot \xi^2 = 2g - 2 - 2d;$$

$$\begin{aligned} d_2 &= K_X^2 \cdot L = \big(-3\xi + \pi^*(K_X + \det \mathcal{E})\big)^2 \cdot \xi \\ &= \big(9\xi^2 - 6\pi^*(K_X + \det \mathcal{E}) \cdot \xi\big) \cdot \xi = 3d - 12(g-1); \end{aligned}$$

$$\begin{aligned} d_3 &= K_X^3 = \big(-3\xi + K_X + \pi^*(K_X + \det \mathcal{E})\big)^3 \\ &= -27\xi^3 + 27\xi^2 \cdot \pi^*(K_X + \det \mathcal{E}) = 54(g-1). \end{aligned}$$

Let S be a smooth member in $|L|$. Then $\chi(\mathcal{O}_X) = 1 - g = \chi(\mathcal{O}_S)$ and $e(S) = 4(1-g)$. A numerical computation carried out by using a MAPLE 11 package, leads for the Hilbert curve Γ of (X, L) to the expression

$$\begin{aligned} p(x,y) &= \frac{d_3}{6}x^3 + \frac{d_2}{2}x^2y + \frac{d_1}{2}xy^2 + \frac{d}{6}y^3 + \\ &- \frac{d_3}{4}x^2 - \frac{d_2}{2}xy - \frac{d_1}{4}y^2 + \\ &+ \frac{d_3}{12}x - 2\chi(\mathcal{O}_X)x + \frac{d_2}{12}y + \frac{1}{12}e(S)y - \frac{d_1}{12}y - \frac{d}{12}y + \chi(\mathcal{O}_X), \end{aligned}$$

and, putting $u = x - \frac{1}{2}$, $y = v$,

$$\begin{aligned} p(u,v) = \frac{1}{24}(-1+2v-6u)(1+2v-6u)(dv+6u(g-1)) \\ = \frac{1}{24}(6u-2v+1)(6u-2v-1)(6(g-1)u+dv). \quad (22) \end{aligned}$$

Look at Y polarized by an ample line bundle $\mathfrak{L}$. Then

$$\chi(xK_Y + y\mathfrak{L}) = x(2g-2) + y\deg\mathfrak{L} + 1 - g = \left(x - \frac{1}{2}\right)(2g-2) + y\deg\mathfrak{L}.$$

Consider the $\mathbb{Q}$-line bundle $\mathfrak{L}$ defined by $3\mathfrak{L} := \det\mathcal{E}$. Then $3\deg\mathfrak{L} = d$, and therefore the third linear factor $6(g-1)u + dv$ in (22) satisfies the relation

$$6(g-1)u + dv = 3\chi(xK_Y + y\mathfrak{L}).$$

This leads to the natural question of understanding the meaning of the residual degree $n-u+1$ factor $R(x,y)$ as in (6.1), 1) in terms of Hilbert polynomials of some polarization (possibly with rational coefficients) on Y.

More generally, the above example suggests the following

Problem 6.6 Let (X, L) be a polarized manifold with non-nef canonical bundle and nef-value $\tau = \frac{a}{b}$. Assume that the nefvalue morphism $\varphi : X \to Y$ has smooth lower dimensional image Y. Then by (6.1) we know that

$$p(x,y) = \prod_{i=1}^{a-1}(ax - by - i)R(x,y).$$

Rephrase better Problem 1)

1) Is the polynomial $R(x,y)$ interpretable in terms of the geometry of Y and φ?

2) Notice that if (X, L) is a scroll over Y with projection φ, then $\deg R(x,y) = \dim(Y)$. In this case, is there any nef and big $\mathbb{Q}$-line bundle $\mathfrak{L}$ on Y such that

$$R(x,y) = c\,\chi(xK_Y + yk\mathfrak{L}),$$

where k is an integer such that $k\mathfrak{L} \in \mathrm{Pic}(Y)$, and c is a constant?

7 Serre-invariant curves

Let $\mathbb{A}^2 = \mathbb{A}^2{}_{(x,y)}$, $\mathbb{P}^2 = \mathbb{P}^2{}_{[x,y,z]}$, and let $s : \mathbb{A}^2 \to \mathbb{A}^2$, $\overline{s} : \mathbb{P}^2 \to \mathbb{P}^2$ be the Serre involutions defined in §3.

It is natural to consider a family of plane curves larger than that one of Hilbert curves; namely the family of curves that are invariant under the Serre involution.

Let $\mathcal{C}$ be a possibly non-reduced curve on $\mathbb{P}^2$ (respectively $\mathbb{A}^2$) of given degree d. We say that $\mathcal{C}$ is a *Serre-invariant curve* if $\overline{s}(\mathcal{C}) = \mathcal{C}$ (respectively $s(\mathcal{C}) = \mathcal{C}$). The Serre involution acts on $\mathcal{C}$, so that we can consider the quotient $\mathcal{C}/\langle\, \overline{s}\,\rangle$ and identify Serre-invariant curves with their images on the quadric cone $\mathcal{Q} = \mathbb{P}^2/\langle\, \overline{s}\,\rangle \subset \mathbb{P}^3$.

Clearly a Hilbert curve of a d-dimensional polarized variety is a Serre-invariant curve of degree d.

A noteworthy property is that Serre-invariant curves are in fact zero sets of polynomials with the same Serre-invariance as the Hilbert polynomial.

Let $C = \left(\frac{1}{2}, 0\right)$ be the central point of the Serre involution s. Let $\mathcal{C}$ be a possibly non-reduced curve on $\mathbb{A}^2$, i.e., a possibly non-reduced Cartier divisor on $\mathbb{A}^2$.

Claim 7.1 *Let $\mathcal{C}$ be a Serre-invariant curve on $\mathbb{A}^2$, defined by a polynomial $f(x, y)$ of degree d. Then*

$$f(x, y) = (-1)^d f(1 - x, -y).$$

Proof. Since $s(\mathcal{C}) = \mathcal{C}$, and $\mathcal{C}$ is defined by a single polynomial up to multiplication by a constant, we know that $f(s(x, y)) = \lambda f(x, y)$ for some constant $\lambda \neq 0$. Thus

$$f(s^2(x, y)) = \lambda f(s(x, y)) = \lambda^2 f(x, y).$$

But $s^2(x, y) = (x, y)$, so that $\lambda^2 = 1$, or $\lambda = \pm 1$.

To determine λ it is enough to compare a non-zero monomial of maximal degree d, say $cx^a y^{d-a}$, of $f(x, y)$ with its corresponding monomial in $f(s(x, y))$.

If $a = 0$, then our term is cy^d. Hence clearly $c(-y)^d = (-1)^d cy^d$, giving $\lambda = (-1)^d$.

If $f(x, y)$ does not contain the term y^d, then $a > 0$. Thus

$$f(s(x, y)) = f(1 - x, -y) = f(-x, -y) + \cdots ,$$

where "$\cdots$" means terms of degree $< d$. Therefore the corresponding monomial of $cx^a y^{d-a}$ in $f(s(x, y))$ is

$$c(-x)^a(-y)^{d-a} = (-1)^d cx^a y^{d-a},$$

so that $\lambda = (-1)^d$ once again. Q.E.D.

Remark 7.2 With the notation as above, break up $\mathcal{C}$ as $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_m$, where $\mathcal{C}_\mu$ is the union of all multiplicity μ components. Then $s(\mathcal{C}_\mu) = \mathcal{C}_\mu$, and so $\mathcal{C}_\mu$ and $\left(\mathcal{C}_\mu\right)_{\text{red}}$ are also Serre-invariant curves. We thus conclude that if D is an irreducible and reduced component of $\mathcal{C}$ that contains the central point $\left(\frac{1}{2}, 0\right)$, and if $\deg(D)$ is even, then D is singular at $\left(\frac{1}{2}, 0\right)$ (compare with (2.1)).

Let us point out some consequences of Claim (7.1) (compare with (2) and (2.1), 2)).

1. If d is odd, then
$$\left(\frac{\partial}{\partial x}\right)^s \left(\frac{\partial}{\partial y}\right)^t f(x,y)_{|C} = 0$$
for all non-negative integers s, t with $s+t$ even.

2. If d is even, then
$$\left(\frac{\partial}{\partial x}\right)^s \left(\frac{\partial}{\partial y}\right)^t f(x,y)_{|C} = 0$$
for all non-negative integers s, t with $s+t$ odd.

3. The central point of the Serre involution belongs to a smooth Serre-invariant curve of degree d if and only if d is odd.

Denote by $\mathcal{V}_d \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ the linear subsystem of smooth Serre-invariant curves of degree d and identify the group $\mathcal{A}$ of affinities of $\mathbb{A}^2{}_{(x,y)}$ with the subgroup of $\mathrm{PGL}(3;\mathbb{C})$ fixing ℓ_∞. Let G be the subgroup of $\mathcal{A}$ defined by
$$G := \{g \in \mathcal{A} \mid g \circ \overline{s} = \overline{s} \circ g\}.$$

We have the following result.

Theorem 7.3 *Let G and $\mathcal{V}_d$ be as above. Then*

1. $\dim(G) = 4$*;*

2. $\dim(\mathcal{V}_d) = \frac{(d+2)^2}{4} - 1$ *for d even, and* $\dim(\mathcal{V}_d) = \frac{(d+1)(d+3)}{4} - 1$ *for d odd.*

Proof. Let
$$A := \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M := \begin{pmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{pmatrix}$$
be the matrices of $\mathrm{PGL}(3;\mathbb{C})$ associated to the Serre involution and to any affinity $g \in G$ respectively. Then from the equality
$$\begin{pmatrix} -a & -b & 1-c \\ -a' & -b' & -c' \\ 0 & 0 & 1 \end{pmatrix} = AM = MA = \begin{pmatrix} -a & -b & a+c \\ -a' & -b' & a'+c' \\ 0 & 0 & 1 \end{pmatrix}$$
we get $2c = 1-a$, $c' = -2a'$. Therefore
$$M = \begin{pmatrix} -a & b & \frac{1-a}{2} \\ -a' & -b' & -2a' \\ 0 & 0 & 1 \end{pmatrix},$$
so that $\dim(G) = 4$.

Let $\mathbb{F}_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ be the Hirzebruch surface of invariant $e = 1, 2$. In the following we denote by E_e and f_e a section of self-intersection $E_e^2 = -e$ and a fiber of the bundle projection $\mathbb{F}_e \to \mathbb{P}^1$, $e = 1, 2$, respectively. The two-to-one quotient map

$\Phi : \mathbb{P}^2 \to \mathcal{Q}$ defined in §5 induces a double cover $\alpha : \mathbb{F}_1 \to \mathbb{F}_2$ via the commutative diagram

$$\begin{tikzcd} \mathbb{F}_1 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathbb{F}_2 \arrow[d, "\pi"] \\ \mathbb{P}^2 \arrow[r, "\Phi"] & \mathcal{Q}, \end{tikzcd}$$

where $\beta : \mathbb{F}_1 \to \mathbb{P}^2$ is the blowing-up at the central point C of the Serre involution, and $\pi : \mathbb{F}_2 \to \mathcal{Q}$ is the minimal desingularization of $\mathcal{Q}$. Since Φ is branched at the vertex and along a plane section of $\mathcal{Q}$, we get that α is branched along E_2 and a smooth section belonging to $|E_2 + 2f_2| = |\pi^*\mathcal{O}_{\mathcal{Q}}(1)|$. Note that $2E_1 = \alpha^*E_2$ and $f_1 = \alpha^* f_2$.

Now, let $\mathcal{C} \subset \mathbb{P}^2$ be a smooth Serre invariant curve of degree d. First assume that d is even. Then $\mathcal{C}$ does not pass through the central point, so $\widetilde{\mathcal{C}} := \beta^{-1}(\mathcal{C}) \in |d(E_1 + f_1)|$. The curve $\widetilde{\mathcal{C}} \subset \mathbb{F}_1$ is the pull back via α of a smooth curve $\mathcal{C}' \in |a(E_2 + 2f_2)|$ for some integer a. Since $\mathcal{C}^2 = \widetilde{\mathcal{C}}^2 = 2\mathcal{C}'^2$, we find $d^2 = 2(2a^2)$. Thus $2a = d$, so $\mathcal{C}' \in |\frac{d}{2}(E_2 + f_2)|$.

Therefore counting the (smooth) curves on $\mathbb{F}_2$ which pull back to $\widetilde{\mathcal{C}}$ on $\mathbb{F}_1$, we see that they form a family of dimension $h^0\big(\mathbb{F}_2, \frac{d}{2}\big(E_2 + f_2\big)\big) - 1$. In turn, because of the commutativity of the above diagram, one has

$$\dim(\mathcal{V}_d) = h^0\left(\mathbb{F}_2, \frac{d}{2}\big(E_2 + f_2\big)\right) - 1.$$

Recall that $E_2 + 2f_2$ is the tautological line bundle of $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ on $\mathbb{F}_2$, so that

$$\pi_*\left(\frac{d}{2}(E_2 + f_2)\right) = S^{d/2}(\mathcal{E}),$$

where the r-th symmetric power of the vector bundle $\mathcal{E}$ is

$$S^r(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(2r). \tag{23}$$

For $r = d/2$, expression (23) yields

$$h^0(\mathbb{P}^1, S^{d/2}(\mathcal{E})) = 1 + 3 + 5 + \cdots + (d+1),$$

the sum of the first odd integers $\leq d+1$. Thus

$$\begin{aligned} h^0(\mathbb{P}^1, S^{d/2}(\mathcal{E})) &= \sum_{m=1}^{d+1} m \; - \; 2\left(\sum_{m=1}^{d/2} m\right) \\ &= \frac{(d+1)(d+2)}{2} - 2\,\frac{\frac{d}{2}\big(\frac{d}{2}+1\big)}{2} = \frac{(d+2)^2}{4}, \end{aligned}$$

giving the desired result for d even.

Assume now d odd. In this case $\mathcal{C}$ passes through the central point of the Serre involution, and its proper transform $\widetilde{\mathcal{C}} = \beta^*(\mathcal{C}) - E_1$ belongs to $|(d-1)(E_1 + f_1) + f_1|$.

One has $\widetilde{\mathcal{C}} = \alpha^*(\mathcal{C}')$ for some smooth curve $\mathcal{C}' \in |a(E_2 + 2f_2) + bf_2|$, $a, b \in \mathbb{Z}$. Since $\widetilde{\mathcal{C}} \cdot f_1 = 2(\mathcal{C}' \cdot f_2)$ we have $2a = d - 1$. Moreover $2 = (2E_1) \cdot \widetilde{\mathcal{C}} = 2(E_2 \cdot \mathcal{C}') = 2b$ gives $b = 1$. Thus $\mathcal{C}' \in |\frac{d-1}{2}\big(E_2 + 2f_2\big) + f_2|$.

Arguing as above we have

$$\dim(\mathcal{V}_d) = h^0\left(\mathbb{F}_2, \frac{d-1}{2}(E_2 + 2f_2) + f_2\right) - 1,$$

where now

$$h^0\left(\mathbb{F}_2, \frac{d-1}{2}(E_2 + 2f_2) + f_2\right) = h^0(\mathbb{P}^1, S^{(d-1)/2}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(1)).$$

Taking into account expression (23) for $r = \frac{d-1}{2}$ yields

$$h^0(\mathbb{P}^1, S^{(d-1)/2}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(1)) = 2\big(1 + 2 + 3 + \cdots + \frac{d+1}{2}\big).$$

Therefore

$$h^0(\mathbb{P}^1, S^{(d-1)/2}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(1)) = 2\,\frac{\frac{d+1}{2}\big(\frac{d+1}{2} + 1\big)}{2} = \frac{(d+1)(d+3)}{4},$$

and the theorem is proved. Q.E.D.

Remark 7.4 (The case $d = 3$) Note that in the cubics case, the difference $\dim \mathcal{V}_3 - \dim G = 1$ is the dimension of the moduli space $\mathbb{A}^1_{\mathbb{C}}$ of (smooth) complex elliptic curves. This agrees with the discussion on the j-invariant of Hilbert curves of polarized threefolds given in §4.

References

[1] M.C. Beltrametti and A.J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, Expositions in Mathematics, 16, W. de Gruyter, 1995.

[2] T. Fujita Classification of polarized manifolds of sectional genus two, in *Algebraic Geometry and Commutative Algebra*, I, in Honor of M. Nagata, (1987), 73–98, Kinokuniya, Tokyo.

[3] Ph. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.

[4] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the Minimal Model Problem, *Algebraic Geometry*, Sendai, 1985, Advanced Studies in Pure Mathematics 10 (1987) 283–360.

[5] F. Serrano, Fibrations on algebraic surfaces, *Geometry of Complex Projective Varieties*, Proceedings Cetraro, June 1990, Seminars and Conferences, n. 9, Mediterranean Press (1993), 289–301.

[6] A.J. Sommese, On the density of ratios of Chern numbers of algebraic surfaces, Math. Ann. 268 (1984), 207–221.

1993.

M.C. Beltrametti
Dipartimento di Matematica
Via Dodecaneso 35
I-16146 Genova, Italy
beltrame@dima.unige.it

A. Lanteri
Dipartimento di Matematica "F. Enriques"
Università di Milano
Via C. Saldini 50
I-20133 Milano, Italy
lanteri@mat.unimi.it

A.J. Sommese
Department of Mathematics
Notre Dame, Indiana, 46556, U.S.A.
sommese@nd.edu
http://www.nd.edu/∼sommese