

Problems for Mathematics 790

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March 17, 2010

6 March 18, 2010: Due March 25, 2010

Problem 6.1 Consider the heat equation

$$u_t = u_{xx}$$

for $(x, t) \in [0, 1] \times [0, \infty)$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = x(1 - x)$. For a positive integer M , and with $h = 1.0/11$ and $k = 1.0/M$, you want to find an approximate solution $u_{i,j}$ using finite differences on the $10 \times M$ grid $(x_i, t_j) = (ih, jk)$ for $i = 1, \dots, 10$ and $j = 1, \dots, M$. Setting

1. $x_0 = 0$, $x_{11} = 1$, and $t_0 = 0$;
2. $u_{0,j} = u_{11,j} = 0$ for all j ; and
3. $u_{i,0} = ih(1 - ih)$

you do this by computing

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2} (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

for j from 1 to M .

Compute the infinity norm of the vector $(u_{1,M}, \dots, u_{10,M})$ for $M = 20$, for $M = 100$, for $M = 200$, and for $M = 250$.

5 February 18, 2010: Due February 25, 2010

Problem 5.1 Assume you wish to find the eigenvalues and eigenvectors for an $n \times n$ complex matrix A , i.e., you wish to solve $Ax - \lambda x = 0$ where x is an $n \times 1$ vector. This is a polynomial system in the $n + 1$ variables, x_1, \dots, x_n and λ . There are only n equations, but the eigenvectors are only well-defined

up to a nonzero multiple. The simplest way of making this into a system with the same number of equations as variables is to add an extra random linear equation in the x variables, e.g.,

$$\begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n - \lambda x_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n - \lambda x_n \\ b_1x_1 + \cdots + b_nx_n - 1 \end{bmatrix} = 0.$$

We have here n quadratic equations and 1 linear equation in $n + 1$ variables giving a total degree of 2^n . We expect n solutions most of the time, so we have to track $2^n - n$ extra paths using this approach. For a size 10,000 matrix, this means we must track $2^{10,000} \approx 10^{3,010}$ extra paths.

A different approach presented in articles by T.Y. Li, Michigan State University uses bihomogeneity. Since x is only well-defined up to nonzero multiple, we consider it as a point in \mathbb{P}^{n-1} . Homogenizing with respect to λ we get the system $Ax\lambda_0 - \lambda x = 0$ on $\mathbb{P}^{n-1} \times \mathbb{P}^1$ in the homogeneous variables $[x_1, \dots, x_n]$ and $[\lambda_0, \lambda]$, i.e.,

$$\begin{bmatrix} a_{11}x_1\lambda_0 + \cdots + a_{1n}x_n\lambda_0 - \lambda x_1 \\ \vdots \\ a_{n1}x_1\lambda_0 + \cdots + a_{nn}x_n\lambda_0 - \lambda x_n \end{bmatrix} = 0.$$

So we have n equations on $\mathbb{P}^{n-1} \times \mathbb{P}^1$ each of bidegree $(1, 1)$.

1. What is the number of paths we need to follow?
2. If B is a diagonal $n \times n$ matrix with all eigenvalues different from one another and 0, what are the solutions of the system on $\mathbb{P}^{n-1} \times \mathbb{P}^1$ associated to $Bx - \lambda x = 0$.

Extra Credit: Write down a homotopy based on the system in 2) as a start system to find the solutions of the system $Ax\lambda_0 - \lambda x = 0$ on $\mathbb{P}^{n-1} \times \mathbb{P}^1$.

4 February 4, 2010: Due February 18, 2010

Problem 4.1 Write a continuation solver for constant stepsize h to solve a polynomial system

$$f(z) = \begin{bmatrix} f_1(z_1, z_2) \\ f_2(z_1, z_2) \end{bmatrix} = 0$$

with $\deg f_i = d_i$ for $i = 1, 2$. Use the homotopy

$$H(t, z) = (1 - t) \begin{bmatrix} f_1(z_1, z_2) \\ f_2(z_1, z_2) \end{bmatrix} + \gamma t \begin{bmatrix} z_1^{d_1} - 1 \\ z_2^{d_2} - 1 \end{bmatrix}$$

as t goes from 1 to 0 with γ a random complex number. Use Euler's method and one Newton correction at each step. Using your solver, solve the systems

$$f(z) = \begin{bmatrix} x^2 + y^2 - 1 \\ \frac{x^2}{16} + \frac{y^2}{9} - 1 \end{bmatrix} = 0$$

and

$$f(z) = \begin{bmatrix} xy \\ (x-1)(y-1) \end{bmatrix} = 0$$

for stepsizes $h = 0.2$, $h = 0.1$, and $h = 0.01$.

3 January 28, 2010: Due February 4, 2010

Problem 3.1 Given the polynomial $p(x, y) = y(y - x)^2(y - 2x) + 6(y - x)x^2 - 8x^2 + 1$

1. Homogenize the polynomial to a homogeneous polynomial $q(x, y, z)$.
2. Compute the solutions at infinity of $p(x, y) = 0$, i.e., compute the solutions to $q(x, y, 0) = 0$.
3. From the solutions to part 2) compute the slopes of the lines asymptotic to the curve.

2 January 21, 2010: Due January 28, 2010

Problem 2.1 What is the discriminant of $p(x) = x^3 + ax^2 + bx + c$? (I.e., compute the polynomial $D(u, v, w)$, such that $D(a, b, c) = 0$ if and only if $x^3 + ax^2 + bx + c$ has a multiple zero.)

Problem 2.2 Use resultants to find the common zeros of $x^2 + y^2 - 10 = 0$ and $\frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$.

1 January 14, 2010: Due January 21, 2010

Problem 1.1 Write a continuation solver for constant stepsize h to solve a polynomial $p(z) = 0$ of degree d . Use the homotopy

$$H(t, z) = (1 - t)p(z) + \gamma t(z^d - 1)$$

as t goes from 1 to 0 with γ a random complex number. Use Euler's method and one Newton correction at each step. Using your solver, solve $z^5 - 15 * z + 11 = 0$ for stepsizes $h = 0.2$, $h = 0.1$, and $h = 0.01$.