Homework Problems 9
Due Monday, December 1, 2008

In Problems 1, 2, and 3, let B_t with $t \geq 0$ denote a Brownian motion with filtration $\mathcal{F}_t := \sigma(B_s \text{ for s} \geq 0)$.

Problem 1 *Show that for each* $\theta \geq 0$

$$X_t := e^{\left(\theta B_t - \frac{\theta^2 t}{2}\right)}$$

is a Martingale relative to the filtration $\mathcal{F}_t$ for t ≥ 0, *i.e., show that* $E(|X_t|) < \infty$ *and that for* $t \geq s$, $E(X_t|\mathcal{F}_s) = X_s$.

Problem 2 *Expanding*

$$e^{\left(\theta B_t - \frac{\theta^2 t}{2}\right)} = \sum_{j=0}^{\infty} H_j(t, B_t) \frac{\theta^j}{j!},$$

it follows from Problem 1 that for each $j \geq 0$, *the stochastic process* $H_j(t, B_t)$ *is a Martingale relative to* $\mathcal{F}_t$ for t ≥ 0. *Compute* $H_j(t, B_t)$ *for* $j \leq 5$.

Problem 3 *Write down the integral that computes the function*

$$\mathfrak{p}(t) := P(-1 \leq B_t \leq 1).$$

Plot $\mathfrak{p}(t)$ *when* $0 \leq t \leq 1$.

Problem 4 *Let* N_t *for* $t \geq 0$ *denote the Poisson process with parameter* $\lambda = 1$, *i.e., the Poisson process with*

$$P(N_t = k) = \frac{t^k}{k!} e^{-t}.$$

Define the process $X_t := N_t - t$ *for* $t \geq 0$. *Show that* X_t *and* $X_t^2 - t$ *are Martingales relative to the filtration*

$$\mathcal{F}_t := \sigma(N_s \text{ for s} \geq 0).$$

Problem 5 *Using the notation of Problem 4, show that for each* $\theta \geq 0$

$$E\left(e^{(\theta N_t + t - te^{\theta})}|\mathcal{F}_s\right) = e^{(\theta N_s + s - se^{\theta})}$$

for $t \geq s \geq 0$.