Homework Problems 7
Due Wednesday, October 29, 2008

In Problems 1, 2, and 4 let $(\Omega, \mathcal{F}, P)$ denote the probability triple associated to the tossing of a fair coin, i.e., $\Omega := \{H, T\}^\infty$ with

$$P(A \times \{H, T\}^\infty) = P(A) = \frac{\text{number of elements in } A}{2^k}$$

for any set $A \in \{H, T\}^k$.

Let Z_k denote the random variable on Ω with $Z_k(\omega) = 1$ or -1 depending on whether the k-th coordinate of $\omega \in \Omega$ is H or T. Then we have the random walk $X_0, X_1, X_2, \ldots$ starting at j defined by $X_0 = j$ and $X_k = X_{k-1} + Z_k$ for any integer $k > 0$.

Let $\mathcal{F}_k = \sigma(Z_0, \ldots, Z_k)$.

Problem 1 *Show that $Y_k := X_k^2 - k$ is a Martingale adapted to the filtration*

$$\{\mathcal{F}_k | k \geq 0\}.$$

Problem 2 *Show that for each $x \in \mathbb{R}$,*

$$Y_k := \frac{e^{xX_k}}{\left(\frac{e^x + e^{-x}}{2}\right)^k}$$

is a Martingale adapted to the filtration $\{\mathcal{F}_k | k \geq 0\}$.

Problem 3 *Assuming Problem 2, and not worrying about convergence, show that for a fixed integer $n \geq 0$, the sequence consisting of the coefficients of x^n for $Y_0, Y_1, Y_2, \ldots$ is a Martingale adapted to the filtration $\{\mathcal{F}_k | k \geq 0\}$. Write down these Martingales for $n = 0, 1, 2, 3$, e.g., for $n = 2$, the Martingale is $X_k^2 - k$ for $k = 0, 1, 2, \ldots$.*

Problem 4 *Problem 5 of Homework 6 may be restated in terms of a Markov Chain.*

1. *Draw the finite-state machine with transition probabilities describing Aladdin's actions and their outcomes.*

2. *Write down the transition matrix associated to this finite-state machine.*

Have A Good Fall Break!