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A BOUND FOR THE DIMENSION OF THE AUTOMORPHISM GROUP OF A HOMOGENEOUS COMPACT COMPLEX MANIFOLD

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ABSTRACT. Let X be a homogeneous compact complex manifold, and let $\mathrm{Aut}(X)$ be the complex Lie group of holomorphic automorphisms of X. Examples show that $\dim \mathrm{Aut}(X)$ can grow exponentially in $n = \dim X$. In this note it is shown that

$$\dim \mathrm{Aut}(X) \leq n^2 - 1 + \binom{2n-1}{n-1}$$

when $n \geq 3$. Thus, $\dim \mathrm{Aut}(X)$ is at most exponential in n. The proof relies on an upper bound for the dimension of the space of sections of the anticanonical bundle, $K_Y^* = \det T_Y$, of a homogeneous projective rational manifold Y of dimension m: $\dim H^0(Y, K_Y^*) \leq \binom{2m+1}{m}$.

1. Introduction

Let X be a connected compact complex manifold. Then $\mathrm{Aut}(X)$ is a complex Lie group acting holomorphically on X [2]. If X is homogeneous under $\mathrm{Aut}(X)$, we may identify X with a coset space G/H where $G = \mathrm{Aut}^0(X)$ is the connected component of the identity of $\mathrm{Aut}(X)$ and H is the closed complex subgroup fixing a point in X. One simple measure of the degree of homogeneity of X is given by the dimension of its automorphism group, $d = \dim \mathrm{Aut}(X)$. It is natural to ask how large d can be relative to the dimension n of X and to seek those homogeneous compact complex manifolds X of a fixed dimension n for which d is a maximum.

If X can be equivariantly embedded in complex projective space, then standard arguments involving Lie's theorem imply that the radical of G acts trivially on X and hence G is semisimple. Moreover, a maximal compact subgroup $K \subset G$ acts transitively, $X = K/L$, $L = K \cap H$, and the isotropy representation of L is an embedding into the unitary group $U(n)$. Therefore, $d = \dim_{\mathbb{C}} G = \dim_{\mathbb{R}} K = 2n + \dim_{\mathbb{R}} L \leq 2n + n^2$. More generally, if $X = G/H$ is Kähler, then $X = Y \times Z$ where Y is homogeneous and admits an equivariant embedding into projective space, and Z is a compact complex torus [3]. It follows that the same estimate on d holds: $d \leq n(n+2)$. Furthermore, it is not hard to verify that equality occurs only in the case where X is a complex projective space, $X = \mathbb{P}^n$. Thus, the question about the maximum of d for a fixed n is completely answered in the Kähler case.

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The non-Kähler case does not yet have such complete answers. It has been known for some time that for each n there is a theoretical upper bound for d; see [1, p. 99]. Examples show that an upper bound for d must be at least exponential in n [7]. The construction of these examples relies on several delicately balanced factors, including the existence of certain uniform discrete subgroups, which make it seem unlikely that a precise maximum d will be found for each n. Nevertheless, in this note we establish the following explicit estimate for d when $n \geq 3$:

$$d \leq n^2 - 1 + \binom{2n-1}{n-1}.$$

Stirling's formula reveals that $\binom{2n-1}{n-1}$ asymptotically approaches $2^{2n-1}/\sqrt{(n-1)\pi}$. Therefore, d is at most exponential in n.

The idea of the proof of this estimate is to examine the restrictions placed on a Levi decomposition $G = R \cdot S$ by the normalizer fibration $X \to Y$. The base Y is homogeneous under the semisimple factor S and can be equivariantly embedded in projective space, thus providing the estimate $\dim S \leq m(m+2)$ where $m = \dim Y$. The group N that acts transitively on the fiber normalizes H^0, contains the radical R of G, and has a unimodular quotient, N/H^0. These facts have strong consequences for the weights of the adjoint representation of S on the Lie algebra of R. The highest weights of this representation are shown to be bounded above (coordinate-wise) by the weight μ_Y associated to the anticanonical bundle of Y, $K_Y^* = \det T_Y$. The result of [8] provides an upper bound for the dimension of the irreducible representation $V(\mu_Y)$ with highest weight μ_Y: $\dim V(\mu_Y) = \dim H^0(Y, K_Y^*) \leq \binom{2m+1}{m}$. This estimate then leads to an upper bound for $\dim R$ which, along with the estimate for $\dim S$, yields the given upper bound for d.

2. Preliminaries

As in the introduction, let $X = G/H$ be a connected compact complex manifold homogeneous with respect to $G = \mathrm{Aut}^0(X)$. Let $G = R \cdot S$ be a Levi decomposition of G into its radical R and a semisimple complex Lie group S. Let $N = N_G(H^0)$ be the normalizer in G of the connected component of the identity of H. Let $Y = G/N$ be the base of the normalizer fibration, $G/H \to G/N$. Then $Y = S/P$, where $P = S \cap H$ is a parabolic subgroup of S. The fiber $Z = N/H = (N/H^0)/(H/H^0)$ is the quotient of a complex Lie group by a uniform discrete subgroup; see [9], [3].

We use German letters, $\mathfrak{g}$, $\mathfrak{h}$, etc., to denote the Lie algebras of the Lie groups G, H, etc.

Let T be a maximal torus of S. Let $\Phi \subset \mathfrak{t}^*$ denote the roots of S with respect to T and let $\{\alpha_1, \dots, \alpha_\ell\} \subset \Phi$ be a system of simple roots. Let Φ^+ denote the subset of positive roots, i.e., those that are positive integral combinations of the simple roots. The negative roots are denoted by $\Phi^- = -\Phi^+$. For any root $\alpha \in \Phi$, let $e_\alpha \in \mathfrak{s}$ be the corresponding root vector, $[x, e_\alpha] = \alpha(x)e_\alpha$ for all $x \in \mathfrak{t}$. We let B denote the Borel subgroup of S generated by T and the negative root groups $\exp \mathbb{C}e_\alpha$, for all $\alpha \in \Phi^-$. We may assume that P contains B.

Let $\lambda_1, \dots, \lambda_\ell$ be the fundamental dominant weights of S defined by $\langle \lambda_i, \alpha_j \rangle = 2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$ where $(\ ,\ )$ denotes the Killing form. Any weight $\mu \in \mathfrak{t}^*$ can be written $\mu = \sum_{i=1}^{\ell} \langle \mu, \alpha_i \rangle \lambda_i$. The irreducible representation of S with highest weight μ is denoted by $V(\mu)$. If V is a finite-dimensional T-module we denote by $\chi(V) \in \mathfrak{t}^*$ the character of V, i.e., the sum of the weights of V.

Let $P = R_P \cdot S_P$ be a Levi decomposition of the parabolic subgroup P. We let Φ_P denote the subsystem of roots of S_P and let $\Phi_P^+ = \Phi_P \cap \Phi^+$. Let I denote the subset of indexes, $I \subset \{1, \dots, \ell\}$, such that $\Phi_P^+ \cap \{\alpha_1, \dots, \alpha_\ell\} = \{\alpha_i\}_{i \in I}$. The conjugacy class of P is uniquely determined by I and any such choice of indexes is associated to a parabolic subgroup of S.

We define $\Phi_Y^+ = \Phi^+ \setminus \Phi_P^+$. These roots clearly coincide with the negatives of the roots of R_P. Since T_Y is generated at the identity coset by the root vectors $e_\alpha \in \mathfrak{s}$ for $\alpha \in \Phi_Y^+$, the anticanonical bundle $K_Y^* = \bigwedge^m T_Y$, $m = \dim Y$, is the homogeneous line bundle associated to the weight

$$\mu_Y = \sum_{\alpha \in \Phi_Y^+} \alpha = -\chi(\mathfrak{r}_P). \tag{1}$$

The weight μ_Y is dominant: $\langle \mu_Y, \alpha_i \rangle > 0$ for $i \notin I$, and $\langle \mu_Y, \alpha_i \rangle = 0$ for $i \in I$. In particular, K_Y^* is a very ample line bundle and μ_Y is orthogonal to the roots Φ_P^+. A simple formula for determining the coefficients $\langle \mu_Y, \alpha_i \rangle$ can be found in [6]. An important component in the proof of the main theorem is the following estimate on $\dim V(\mu_Y) = \dim H^0(Y, K_Y^*)$.

Theorem 1 ([8])**.** *Let Y be a homogeneous projective rational manifold of dimension m. Then*

$$3^m \leq \dim H^0(Y, K_Y^*) \leq \binom{2m+1}{m}$$

with equality in the lower bound if and only if Y is a flag manifold and equality in the upper bound if and only if Y is complex projective space.

3. A bound for $\dim \mathrm{Aut}(X)$

We retain the notation of the previous section. In particular, $X = G/H$ is a homogeneous compact complex manifold; $G = R \cdot S$ is a Levi decomposition with R the radical of G and S semisimple; $Y = G/N = S/P$ is the base of the normalizer fibration $G/H \to G/N$; $N = N_G(H^0) = R \cdot P$; P is a parabolic subgroup of S with Levi decomposition $P = R_P \cdot S_P$; I is the set of indexes that give the generators $\{\alpha_i\}_{i \in I}$ of the positive roots of S_P; μ_Y denotes the weight of the anticanonical bundle K_Y^* of Y, and $\chi(V)$ denotes the character of an S-module V. The following lemma describes a well-known property of $\mathfrak{sl}_2$-modules (see, e.g., [1, p. 90]).

Lemma 1. *Let V be a finite-dimensional $\mathfrak{sl}_2$-module, and let W be a proper subspace of V consisting of weight spaces. Assume that W is invariant under $e_{-\alpha}$ where α is a positive root. Then for any highest weight λ of V,*

$$0 \leq \lambda(x_\alpha) \leq \chi(V/W)(x_\alpha).$$

Proof. Let $\lambda_1 = \lambda$, and let $V = V(\lambda_1) \oplus \cdots \oplus V(\lambda_t)$ be a decomposition of V into irreducible representations with highest weights $\lambda_1, \dots, \lambda_t$. Let $W = W_1 \oplus \cdots \oplus W_t$, $W_i = W \cap V(\lambda_i)$, $1 \leq i \leq t$, be the corresponding decomposition of W. Since W_i is invariant under $e_{-\alpha}$, there is a nonnegative integer k_i such that the weights of $V(\lambda_i)/W_i$ are λ_i, $\lambda_i - \alpha$, $\dots$, $\lambda_i - k_i\alpha$. Cancelling any negative terms with corresponding positive terms we find there is a nonnegative integer $k_i' \leq k_i$ such that

$$\chi(V(\lambda_i)/W_i)(x_\alpha) = \lambda_i(x_\alpha) + (\lambda_i(x_\alpha) - 2) + \cdots + (\lambda_i(x_\alpha) - 2k_i')$$

with each term $\lambda_i(x_\alpha) - 2j \geq 0$ for $j = 0, \ldots, k_i'$. Therefore,

$$\chi(V/W)(x_\alpha) = \sum_{i=1}^{t} \chi(V(\lambda_i)/W_i)(x_\alpha) \geq \chi(V(\lambda_1)/W_1) \geq \lambda_1(x_\alpha) \geq 0.$$

□

The next proposition provides crucial information about the weights of the representation of S on $\mathfrak{r}$.

Proposition 1.

a) $\chi(\mathfrak{h}) = -\mu_Y$.
b) $\chi(\mathfrak{r}/\mathfrak{r} \cap \mathfrak{h}) = \mu_Y + \chi(\mathfrak{r}_P \cap \mathfrak{h})$.
c) $\chi(\mathfrak{r}/\mathfrak{r}\cap\mathfrak{h})$ *is a dominant weight whose coefficients satisfy* $\langle \chi(\mathfrak{r}/\mathfrak{r}\cap\mathfrak{h}), \alpha_i \rangle = 0$ *for* $i \in I$ *and* $0 \leq \langle \chi(\mathfrak{r}/\mathfrak{r} \cap \mathfrak{h}), \alpha_i \rangle \leq \langle \mu_Y, \alpha_i \rangle$ *for* $i \notin I$.

Proof. a) Since H/H^0 is a uniform discrete subgroup of N/H^0, the latter group is unimodular (see, e.g., [5]). Thus, $\chi(\mathfrak{n}/\mathfrak{h}) = 0$ and $\chi(\mathfrak{h}) = \chi(\mathfrak{n})$. Since $\mathfrak{n} = \mathfrak{r} + \mathfrak{r}_P + \mathfrak{s}_P$ with $\chi(\mathfrak{r}) = 0$ ($\mathfrak{r}$ is an S-module), $\chi(\mathfrak{s}_P) = 0$, and $\chi(\mathfrak{r}_P) = -\mu_Y$ by (1), we see that $\chi(\mathfrak{h}) = -\mu_Y$ (compare with Lemma 1 in [1, p. 96]).

b) On the other hand, $\chi(\mathfrak{h}) = \chi(\mathfrak{r} \cap \mathfrak{h}) + \chi(\mathfrak{r}_P \cap \mathfrak{h}) + \chi(\mathfrak{s}_P \cap \mathfrak{h})$. Since $\mathfrak{s}_P \cap \mathfrak{h}$ is an ideal in $\mathfrak{s}_P$, it is either trivial or equals $\mathfrak{s}_P$, and hence its character is zero. Therefore, using a), $\chi(\mathfrak{r}/\mathfrak{r} \cap \mathfrak{h}) = -\chi(\mathfrak{r} \cap \mathfrak{h}) = -\chi(\mathfrak{h}) + \chi(\mathfrak{r}_P \cap \mathfrak{h}) = \mu_Y + \chi(\mathfrak{r}_\mathfrak{p} \cap \mathfrak{h})$.

c) Because $\mathfrak{r}_P \cap \mathfrak{h}$ is invariant under $\mathfrak{s}_P$, $\chi(\mathfrak{r}_P \cap \mathfrak{h})(x_{\alpha_i}) = 0$ for $i \in I$, and therefore $\chi(\mathfrak{r}/\mathfrak{r} \cap \mathfrak{h})(\alpha_i) = 0$ for $i \in I$. To see that $\chi(\mathfrak{r}/\mathfrak{r} \cap \mathfrak{h})$ is dominant, let α be any positive root and let s_α be the Lie algebra isomorphic to $\mathfrak{sl}_2$ generated by e_α, $e_{-\alpha}$ and $x_\alpha = [e_\alpha, e_{-\alpha}]$. Since $\mathfrak{r} \cap \mathfrak{h}$ is invariant under $e_{-\alpha}$, we obtain from Lemma 1 that $\chi(\mathfrak{r}/\mathfrak{r} \cap \mathfrak{h})(x_\alpha) \geq 0$, and hence $\chi(\mathfrak{r}/\mathfrak{r} \cap \mathfrak{h})$ is dominant.

It remains to prove that $\chi(\mathfrak{r}_P \cap \mathfrak{h})$ is a negative dominant weight, i.e., that $\langle \chi(\mathfrak{r}_P \cap \mathfrak{h}), \alpha_i \rangle \leq 0$ for $i \notin I$. Let $J = \{\beta_1, \ldots, \beta_m\}$ be the set of weights of $\mathfrak{r}_P \cap \mathfrak{h}$. Then $\chi(\mathfrak{r}_P \cap \mathfrak{h})$ is the weight of $x_{\beta_1} \wedge \cdots \wedge x_{\beta_m} \in \bigwedge^m \mathfrak{s}$. For any negative root α, $\mathfrak{r}_P \cap \mathfrak{h}$ is invariant under x_α. Thus, if $\beta_i + \alpha$ is a root, then $\beta_i + \alpha \in J$, so that

$$x_\alpha \cdot x_{\beta_1} \wedge \cdots \wedge x_{\beta_m} = \sum_{i=1}^{m} c_i x_{\beta_1} \wedge \cdots \wedge x_{\beta_i + \alpha} \wedge \cdots \wedge x_{\beta_m} = 0.$$

Therefore, $x_{\beta_1} \wedge \cdots \wedge x_{\beta_m}$ is a lowest weight vector in $\bigwedge^m \mathfrak{s}$ and $\chi(\mathfrak{r}_P \cap \mathfrak{h})$ must be negative dominant. □

Theorem 2. *Let* X *be a homogeneous compact complex manifold of dimension* $n \geq 3$. *Then*

$$\dim \operatorname{Aut}(X) \leq n^2 - 1 + \binom{2n-1}{n-1}.$$

Proof. Let $m = \dim Y$ and $p = \dim Z = n - m$. Since S acts transitively on Y, $\dim S \leq m(m+2)$. To find an estimate for $\dim R$ we decompose $\mathfrak{r}$ into irreducible S-modules, $\mathfrak{r} = V(\lambda_1) \oplus \cdots \oplus V(\lambda_t)$. For any positive root α, Lemma 1 and Proposition 1 imply that $0 \leq \langle \lambda_i, \alpha \rangle = \lambda_i(x_\alpha) \leq \chi(\mathfrak{r}/\mathfrak{r}\cap\mathfrak{h})(x_\alpha) = \langle \chi(\mathfrak{r}/\mathfrak{r}\cap\mathfrak{h}), x_\alpha \rangle \leq \langle \mu_Y, \alpha \rangle$. So $\dim V(\lambda_i) \leq \dim V(\mu_Y)$, for $1 \leq i \leq t$, by the Weyl dimension formula [4]. Therefore, $\dim R = \sum_{i=1}^{t} \dim V(\lambda_i) \leq t \dim V(\mu_Y)$. The number of irreducible components, t, cannot exceed $p = \dim N/H$: If $t > p$, then $V(\lambda_i) \subset \mathfrak{h}$ for at least one i (since $N = R \cdot P$) and the ideal generated by $V(\lambda_i)$ in $\mathfrak{r}$ would be an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$, contradicting the fact that G acts effectively on $X = G/H$ (compare with Lemma 2 in [1, p. 97]). We conclude that $\dim R \leq p \dim V(\mu_Y)$.

Applying Theorem 1, we arrive at the estimate

$$\dim G = \dim S + \dim R \leq m(m+2) + (n-m)\binom{2m+1}{m}. \tag{2}$$

If $m = 0$, then $\dim G = \dim R = n$ and if $m = n$, then $\dim G = \dim S \leq n(n+2)$. For $n \geq 3$ and $1 \leq m \leq n-1$, the maximum of the right-hand side of (2) occurs for $m = n-1$, and this maximum always exceeds $n(n+2)$. Therefore,

$$\dim G \leq (n-1)(n+1) + \binom{2(n-1)+1}{n-1}.$$

□

References

[1] Akhiezer, D., *Lie Group Actions in Complex Analysis*, Vieweg, Braunschweig, 1995. MR **96g:**32051

[2] Bochner, S. and Montgomery, D., *Groups on analytic manifolds*, Ann. Math. **48** (1947), 659–669. MR **9:**174f

[3] Borel, A. and Remmert, R., *Über kompakte homogene Kählersche Mannigfaltigkeiten*, Math. Ann. **145** (1961/1962), 429–439. MR **26:**3088

[4] Fulton, W. and Harris, J., *Representation Theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1991. MR **93a:**20069

[5] Raghanuthan, M. S., *Discrete Subgroups of Lie Groups*, Springer-Verlag, New York, 1972. MR **58:**22394a

[6] Snow, D., *The nef value of homogeneous line bundles and related vanishing theorems*, Forum Math. **7** (1995), 385–392. MR **96a:**14057

[7] Snow, D. and Winkelmann, J., *Compact complex homogeneous manifolds with large automorphism groups*, Invent. Math. **134** (1998), 139–144. MR **99f:**32054

[8] Snow, D., *Bounds for the anticanonical bundle of a homogeneous projective rational manifold*, preprint (`http://www.nd.edu/~snow`).

[9] Tits, J., *Espaces homogènes complexes compacts*, Comment. Math. Helv. **37** (1962/1963), 111–120. MR **27:**4248

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