The decay of the Ricci curvature at a real hypersurface singularity.

Brian Smyth

University of Notre Dame

Abstract: The Ricci curvature at an isolated singularity of an immersed hypersurface exhibits a local behaviour echoing a global property of the Ricci curvature on a complete hypersurface in euclidean space (i.e., the Efimov theorem [12], that $\sup Ric \geq 0$). This local behaviour takes the form of a near universal bound on the decay of the Ricci curvature at a simple singularity (eg. a cone singularity) in a real hypersurface in $\mathbb{R}^{m+1}, m \geq 3$ (Theorem 1). The bound is dimension-free, sharp and may be universal. If the bound is not universal, such exceptions as occur must be even-dimensional hypersurfaces with profiles (intersections with small spheres centred at the singularity) which are topological spheres admitting — by a canonical construction (Remark 3, §4) — Riemannian metrics with positive curvature operator. Thus any exceptional singularity would be topologically trivial and if, as long suspected, there are no exotic spheres with positive curvature operator then it would also be differentiably trivial.

§1. Introduction

The only universal curvature bound for complete surfaces in euclidean 3-space is $\sup K \geq 0$, where K is the Gauss curvature; this was Efimov's pioneering generalization [4] of Hilbert's theorem [6] that the full Poincaré plane cannot be isometrically immersed in $\mathbb{R}^3$. Our generalization [12] of Efimov's theorem states that $\sup Ric \geq 0$, where Ric denotes the Ricci curvature, for any complete hypersurface in euclidean space whose sectional curvature does not take all real values.

While this phenomenon is global, another simple universal inequality — also dimension-free — begins to emerge when we consider the Ricci curvature at an

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[*] Partially supported by National Science Foundation Grant DMS 00-71729 and the Max-Planck Insitut für Mathematik, Bonn.

Typeset by $\mathcal{AMS}$-TEX

isolated singularity of a hypersurface in $\mathbb{R}^{m+1}$. There is no limit to how positively (sectionally) curved a hypersurface might be near a singularity but the essence of our inequality is that there is a limit to how negatively (Ricci) curved it may be.

The geometry of the singularity is studied by observing the curvature of its *profiles* (i.e., intersections with small spheres centred at the singularity). While the initial interest was in negative Ricci curvature, observations on the principal curvature sets of these profiles (that is, the set of numbers occurring as eigenvalues of the second fundamental form of a profile hypersurface within its sphere) and their connectivity, in the spirit of [12], result in the inequality being transferred into questions in the realm of compact manifolds of positive sectional curvature in euclidean codimension two. Such manifolds have positive curvature operator by Weinstein [13] and so, after some holonomy considerations (Lemma 1), Meyer's work [8] shows that the profiles are rational homology spheres; Chen's inequality on the total curvatures [3] of such compact codimension two manifolds is vital and led Moore [10] and Baldin-Mercuri [1] to their classification as topological spheres, in all dimensions. While Hadamard's theorem [14] guarantees that compact hypersurfaces of positive sectional curvature in euclidean space are differentiable spheres, the differentiable structure of such hypersurfaces in spheres is unknown.

Our result on the singularity is stated in Theorem 1. If there are exceptions arising in Theorem 1, they may yet be as interesting as the inequality itself.

The singularity may be taken to be the origin o, and the length r of the position vector (from the origin) and the oriented angle ϕ that this vector makes with the (oriented) tangent space to the hypersurface give two smooth functions on the hypersurface. The singularities considered are *simple* in that as $r \to 0$

a) $\phi \longrightarrow 0$,
b) $r\nabla\phi \longrightarrow 0$, where $\nabla\phi$ is the gradient of ϕ in the hypersurface metric, and
c) rA is bounded, where A is the second fundamental form of the hypersurface.

Cone singularities (suspensions of compact connected hypersurfaces in the unit sphere $S^m(1)$ over the origin o in $\mathbb{R}^{m+1}$) are obvious examples; in particular, if f_0 is a homogeneous polynomial of degree $k \geq 2$ in $(m+1)$ variables, which has an isolated critical point at o and changes sign, then the hypersurface $f_0 = 0$ is such an example. If f is any analytic function on a neighbourhood of o whose principal part f_0 is as above, then $f = 0$ is a smooth non-conical hypersurface with an isolated simple singularity at o. More generally, let V be a smooth immersed hypersurface in $\mathbb{R}^{m+1}$ with an isolated singularity at o; if V intersects all small spheres $S^m(r)$ tranversally, and we assume that the radial derivative of ϕ is bounded and that the resulting family of hypersurfaces of $S^m(1)$ — obtained by re-normalization — is C^2-convergent as $r \longrightarrow 0$ to a smooth hypersurface in $S^m(1)$, then o is a simple singularity.

Let V be any smooth immersed hypersurface in $\mathbb{R}^{m+1}$ with an isolated simple singularity at the origin. The tangent vectors to V orthogonal to the position vector are called *angular* or *transverse* and the symbol Ric_T stands for the Ricci curvatures of unit vectors X in these angular directions. The singularity is *topologically trivial* when its profiles are topological spheres. It is our main result that Ric_T cannot decay faster than $-\frac{1}{r^2}$ near a simple singularity without an exceptional conjunction in its dimension, topology and geometry.

Theorem 1. *Let V^m be a smooth immersed hypersurface in $R^{m+1}, m \geq 3$, with an isolated simple singularity at o. Then either*

$$\sup_U r^2 Ric_T \geq -1$$

for every neighbourhood U of o or else all of the following hold: m is even, the singularity is topologically trivial, each profile admits a metric with positive curvature operator, and $\lim_{r\to 0} rH$ does not exist, where H is the mean curvature of V.

The inequality is sharp in all dimensions $m \neq 4$.

The inequality of the theorem may be universal and no exceptions are known. However, if $\sup_U r^2 Ric_T < -1$ for a simple singularity then the profiles are odd-dimensional topological spheres; if, further, the hypersurface is embedded the profiles are differentiable spheres [9]; the possibility that exotic spheres might occur as profiles is engaging since, by Theorem 1, they would admit canonically constructed Riemannian metrics with positive curvature operators and not even a metric of positive sectional curvature is yet known to exist on an exotic sphere. We note here that under the assumption that the inequality of Theorem 1 is violated, the Gauss map of any profile (of sufficiently small radius) induces a metric of positive curvature on that profile.

It follows that, for simple singularities, the inequality $\sup_U r^2 Ric_T \geq -1$ holds always in odd-dimensional hypersurfaces and in any hypersurface where the mean curvature is bounded in a neighbourhood of the singularity; in particular, it always holds for minimal or constant mean curvature hypersurfaces; even in the best studied case of minimal cones (see [11], for example) the inequality of Theorem 1 is new.

The inequality of Theorem 1 is sharp for $m \neq 4$. For each $m \neq 4$ there are Clifford cones with $r^2 Ric_T \equiv -1$ with non-spherical profiles $S^p \times S^q$ for $p, q \geq 2$ or

$p = q = 1$; note $p + q = m - 1$ (Remark 2, §4). In dimension $m = 4$ it can easily be shown that there are no *cones* satisfying the Ricci identity $r^2 Ric_T \equiv -1$ and the bound in Theorem 1 may not be optimal in just this dimension. In the case $m = 2$ the singularities are automatically topologically trivial.

The work is organized as follows: §2 relates the geometry of the hypersurface to that of its profiles at an isolated singularity; in §3 the singularity is assumed simple and this allows a comparison of the Ricci curvatures of the hypersurface with those of its profiles; in §4 we state and prove Theorem 2 on the Ricci curvatures of compact hypersurfaces of the unit sphere; the essential ingredients of this result are the observations on the connectivity of the principal curvature set for hypersurfaces in the sphere and the inequality of Chen [3]. Theorem 1 follows from Theorem 2 by the remarks at the end of §3. At the end of §4 we show how the method can be adapted to *complete* hypersurfaces in the unit sphere give a form of Efimov's Theorem in the sphere.

It is a pleasure to thank the Max Planck Institut für Mathematik in Bonn where this work was done. The work I benefited from conversations with Gudlaugur Thorbergsson and Fuguan Fang at MPIM-Bonn and I would like to thank them again here.

§2. Calculations at a singularity

Let V be a closed connected subset of $\mathbb{R}^{m+1}$ containing the origin o such that $(V - \{o\}) \cap B(2\varepsilon)$ is the image of a smooth m-manifold N^m under a proper smooth immersion $x : N^m \longrightarrow \mathbb{R}^{m+1}$; the open ball and sphere of radius r about o in $\mathbb{R}^{m+1}$ are denoted $B(r)$ and $S^m(r)$, respectively. We call V a *smooth hypersurface with an isolated singularity at o.* The considerations which follow may be applied on each component of N so that we may assume N connected. Nothing is lost in assuming N orientable and selecting a unit normal field η along the immersion x. The second fundamental form of the immersion x is then defined by $X\eta = -x_*(AX)$ for all vectors X tangent to N. The metric induced on N from the euclidean metric $< \, , \, >$ on $\mathbb{R}^{m+1}$ will also be denoted by the same symbol and its connexion by ∇.

The distance $r(p) = |x(p)|$ may be considered as a smooth function on N and the oriented angle $\phi(p)$ that $x(p)$ makes with the hyperplane $x_*(N_p)$ is another smooth function on N with values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The decomposition

$$x = x_*(x^T) + < x, \eta > \eta,$$

defines a vector field x^T on N, called the tangential component of x. Denoting the gradient of a function f in the induced metric by ∇f, we see that $x^T = r\nabla r$. Thus the above equation may be rewritten, in terms of r and ϕ, as

$$x = r\{x_*(\nabla r) + \sin\phi\eta\}.$$

From now on we assume transversality of the immersion x to the spheres $S^m(r)$, for small r. This is equivalent to saying that ∇r is nonvanishing (or $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) for such r ; this is guaranteed by the first property of the singularity (see §1). Writing $e = \frac{\nabla r}{|\nabla r|}$ the above equation may be rewritten

$$x = r\{|\nabla r| x_*(e) + \sin\phi\eta\},$$

from which we conclude that $\nabla r = \cos\phi e$ and

$$x = r\{\cos\phi x_*(e) + \sin\phi\eta\}. \tag{1}$$

Then the level sets

$$M^{m-1}(r) = \{p \in N|\ |x(p)| = r\}$$

are a compact oriented embedded hypersurfaces in N. By transversality and connectedness of N, these hypersurfaces are connected for sufficiently small r. The unit vector field $e = \frac{\nabla r}{|\nabla r|}$ on N is orthogonal to this family of hypersurfaces in N and the one-parameter family of local diffeomorphisms generated by the vector field $E = \frac{-e}{|\nabla r|}$ determines a diffeomorphism between any two hypersurfaces in the family; note that the derivative of r in the direction E is $E(r) \equiv -1$. Writing $M^{m-1} = M^{m-1}(\varepsilon)$, for ε sufficiently small, and keeping in mind the identifications set up by the flow of E, we obtain a family of *profile immersions*

$$x_r = x\big|_{M^{m-1}(r)} : M^{m-1} \longrightarrow S^m(r).$$

for each small r. If ξ_r denotes the oriented unit normal field to the immersion x_r in $S^m(r)$ and B_r its second fundamental form then the corresponding *normalized profile immersion*

$$x_r^0 = \frac{1}{r} x_r : M^{m-1} \longrightarrow S^m(1)$$

has the same unit normal field ξ_r and second fundamental form $B_r^0 = rB_r$.

The distribution $T = e^\perp$ on N^m coincides with the tangent space to $M^{m-1}(r)$ at each point of N and its elements will be called *transverse* vectors in N.

The normal field to the profile hypersurfaces

$$x_r = x\big|_{M^{m-1}(r)} : M^{m-1} \longrightarrow S^m(r).$$

is given by

$$\xi = \cos\phi\ \eta - \sin\phi\ x_*(e). \tag{2}$$

Differentiating (1) with respect to any vector X tangent to N and taking η-components yields the equation

$$\nabla\phi = -Ae - \frac{\sin\phi}{r}e, \tag{3}$$

and the tangential components along N give

$$\frac{1}{r}X_T = -\sin\phi\ (AX)_T + \cos\phi\ \nabla_X e, \tag{4}$$

where transverse components are denoted by the subscript T. Taking $X = e$ in this equation we have

$$\nabla_e e = \tan\phi\ (Ae)_T. \tag{5}$$

If now X is transverse a similar calculation on (2) gives

$$B_r X = \cos\phi\ (AX)_T + \sin\phi\ \nabla_X e. \tag{6}$$

Using (3) and eliminating between (4) and (6) for transverse vectors X we have

$$\begin{aligned} \cos\phi\ B_r X &= (AX)_T + \frac{\sin\phi}{r}X, \\ &= AX + X(\phi)e + \frac{\sin\phi}{r}X \end{aligned} \tag{7}$$

and similarly

$$\sin\phi\ B_r X = -\frac{\cos\phi}{r}X + \nabla_X e. \tag{8}$$

§3. The Ricci curvature near an isolated singularity

Throughout this section we will assume that N meets the spheres $S^m(r)$ transversally, for small r. By Gauss' equation, the Ricci curvature of N in the direction of any unit tangent vector X is

$$S(X,X) = < [(Tr\ A)A - A^2]X, X >, \tag{9}$$

where Tr denotes trace.

In particular, it follows from (3) that the Ricci curvature $S(e,e)$ in the radial direction satisfies

$$\begin{aligned} r^2S(e,e) &= r^2 < [(Tr\ A)A - A^2]e, e >, \\ &= -rTrA < r\nabla\phi + \sin\phi e, e > -||r\nabla\phi + \sin\phi e||^2 \end{aligned} \tag{10}$$

Thus $\lim_{r\to 0} r^2S(e,e) = 0$ for simple singularities. Next we turn to examine the Ricci curvature in transverse directions.

From now on X will be a transverse vector at a distance r from o (i.e. tangent to $M^{m-1}(r)$). Then, by (7),

$$rAX = r\cos\phi\ B_rX - \sin\phi\ X - rX(\phi)e, \tag{11}$$

where B_r is the second fundamental form of the immersion x_r. By (3) and (11) it follows that

$$Tr\ (rA) = r\cos\phi\ Tr\ B_r - m\sin\phi - re(\phi). \tag{12}$$

Furthermore, from (11), for any unit transverse vector X

$$\begin{aligned} < r^2A^2X, X > =&< rAX, rAX > \\ &= r^2\cos^2\phi < B_r^2X, X > + \sin^2\phi + r^2X(\phi)^2 \\ &- 2r\sin\phi\cos\phi < B_rX, X > . \end{aligned} \tag{13}$$

It now follows from (9)-(13) that for any transverse unit vector X at distance r from o we have

$$\begin{aligned} r^2S(X,X) = r^2\cos^2\phi\ &< [(TrB_r)B_r - B_r^2]X,\ X > \\ &- r\cos\phi\{(m-2)\sin\phi + re(\phi)\} < B_rX, X > \\ &- r\sin\phi\cos\phi\ TrB_r \\ &- r^2X(\phi)^2 + re(\phi)\sin\phi + (m-1)\sin^2\phi. \end{aligned} \tag{14}$$

Because of the scale change in replacing the r-profile immersion x_r by its normalization $x_r^0 = \frac{1}{r}x_r$, the vector $X_r^0 = r\ X$ is a unit vector with respect to the x_r^0-induced metric $g_r^0 = \frac{1}{r^2} <,>$ and the second fundamental forms are related by $B_r^0 = rB_r$. Thus

$$\begin{aligned} r^2S(X,X) = \cos^2\phi g_r^0 &\left(\left[(TrB_r^0)B_r^0 - (B_r^0)^2\right] X_r^0, X_r^0\right) - \sin\phi\cos\phi\ TrB_r^0 \\ &- \{(m-2)\sin\phi + re(\phi)\}\cos\phi g_r^0(B_r^0X_r^0, X_r^0) \\ &- r^2X(\phi)^2 + re(\phi)\sin\phi + (m-1)\sin^2\phi. \end{aligned} \tag{15}$$

From Gauss' equation for hypersurfaces in the unit sphere, the metric expression in the first term on the right-hand side is $S^0_r(X^0_r, X^0_r) - (m-2)$ where S^0_r is the Ricci tensor of the metric g^0_r. After a re-arrangement of terms, (15) takes the form

$$\begin{aligned} r^2 S(X,X) + 1 = {} & \cos^2\phi\{S^0_r(X^0_r, X^0_r) - (m-3)\} - \sin\phi\cos\phi\ TrB^0_r \\ & - \cos\phi\{(m-2)\sin\phi + re(\phi)\}g^0_r(B^0_r X^0_r, X^0_r) \\ & - r^2 X(\phi)^2 + re(\phi)\sin\phi + m\sin^2\phi. \end{aligned} \tag{16}$$

Recalling the definition of a simple singularity in the introduction we note from (3) that, in the presence of (a), condition (b) means $rAe \longrightarrow 0$ as $r \longrightarrow 0$. In the presence of (a) and (b), condition (c) is equivalent to $B^0_r = rB_r$ being bounded independent of r, by (11). From now on the singularity is assumed simple and then, from (11), the operators B^0_r have a bound independent of r.

Assume that the curvature inequality of Theorem 1 is violated. Then there exists $k > 0$ such that for all unit transverse vectors X tangent to N (in some neighbourhood of o) the Ricci curvature of N satisfies

$$r^2 S(X,X) + 1 < -2k^2.$$

Then it follows from (16) and the fact that the singularity is simple that, for all r sufficiently small, the normalized profile $x^0_r : M^{m-1} \longrightarrow S^m(1)$ has Ricci curvature satisfying

$$(*) \qquad Ric < (m-3) - k^2.$$

Thus, by (16), for a simple hypersurface singularity violating the curvature inequality of Theorem 1, the normalized profiles x^0_r in $S^m(1)$ satisfy the inequality (*) for r sufficiently small. Hypersurfaces of the sphere satisfying this condition are the subject of Theorem 2 in §4 and Theorem 1 will follow from Theorem 2 and the following remark on writing $n = m - 1 \geq 2$.

Remark 1: Note from (11) that

$$\cos\phi\ TrB^0_r = rTrA + r\ e(\phi) + m\sin\phi. \tag{17}$$

Thus if the singularity is simple and $\lim_{r\to 0} rTrA = l$ exists — in particular, if the mean curvature is bounded — then the normalized profile x^0_r has mean curvature arbitrarily close to a constant, for r sufficiently small. However, when the inequality of Theorem 1 is violated, we will see in §4 that the normalized profiles have mean curvature of total variation bounded away from zero, with a bound independent of

r; hence, when the inequality of Theorem 1 is violated, $\lim_{r\to 0} rH$ does not exist, where H is the mean curvature function of the hypersurface N.

§4. A Ricci inequality for hypersurfaces in the sphere.

In arriving at the extension of Efimov's Theorem to complete hypersurfaces of euclidean space, we introduced in [12] the principal curvature set of the hypersurface (i.e., the set of real numbers taken as values of the principal curvatures). It was our observations on the behaviour of the principal curvature set under parallel deformations which provided the essential ingredient of that extension — together with the Hadamard-Sacksteder classification theorem for complete hypersurfaces in $\mathbb{R}^{n+1}$ with non-negative sectional curvature (see Wu [14]). However, it must be emphasised that there is no similar characterization - neither as to diffeomorphy nor extrinsic geometry - of complete (or even compact) positive sectional curvature hypersurfaces in spheres. But, for the purposes here, this lack will be compensated for by an inequality of Chen [3] on the total curvatures of compact submanifolds M^n of $\mathbb{R}^{n+2}$ with positive sectional curvature (which, by Weinstein's [13] codimension two result, is equivalent to positive curvature operator).

Let $f : M^n \longrightarrow S^{n+1}(1)$ be a smooth immersion of a smooth compact orientable n-dimensional manifold M^n into the unit sphere, $n \geq 2$. If ξ is a unit normal field to M^n in $S^{n+1}(1)$ along this immersion and B is its second fundamental form then

$$\Lambda = \{\lambda \in \mathbb{R} | \lambda \text{ is an eigenvalue of } B(p) \text{ for some } p \in M\}$$

is *the principal curvature set of the immersion* f.

We will show $\sup_M Ric \geq n-2$ for any compact hypersurface of the sphere — with the exception of at most some odd-dimensional topological spheres — and this inequality is optimal in that there are non-spherical hypersurfaces of every dimension $n \neq 3$ with $Ric \equiv n-2$. Such exceptional topological spheres as occur must admit Riemannian metrics of positive sectional curvature, as we will see below, and this construction is natural.

Our proof involves considerations on the connectivity of the set Λ. While Λ might have n connected components in general, the final lemma shows Λ is connected for convex hypersurfaces in spheres (that is, hypersurfaces with sectional curvature $K \geq 1$ in the unit sphere or, equivalently, those hypersurfaces which lie to one side of each tangent great hypersphere [2]).

Theorem 2. *Let* $f : M^n \longrightarrow S^{n+1}(1)$ *be a smooth immersion of a smooth compact orientable* n*-manifold in the unit sphere* $S^{n+1}(1), n \geq 2$. *Then either*

$$\sup_M Ric \geq n-2$$

or else each of the following must hold:

(i) n *is odd,*

(ii) M^n *is a topological sphere,*

(iii) the Gauss map immerses M^n *with positive curvature operator in* $S^{n+1}(1)$ *with image in no closed hemisphere and disconnected principal curvature set, and*

(iv) the total variation of the mean curvature of the immersion exceeds $4c \log c$, *where* $c^2 = (n-1) - \sup_M Ric > 1$.

In particular, by (iv), $\sup_M Ric \geq n-2$ *for any compact minimal or constant mean curvature hypersurface in* $S^{n+1}(1)$.

Remark 2: For each $n \neq 3$ the inequality is sharp. If $p+q = n, p \geq 2, q \geq 2$ then the Clifford cone X^{n+1} in $\mathbb{R}^{n+2}$ defined by

$$(q-1)(x_1^2 + \cdots + x_{p+1}^2) - (p-1)(y_1^2 + \cdots + y_{q+1}^2) = 0$$

has normalized profile $S^p(\sqrt{\frac{p-1}{n-2}}) \times S^q(\sqrt{\frac{q-1}{n-2}})$ in $S^{n+1}(1)$ with Riccci curvature $\equiv n-2$. For $n = 2$ the Clifford cone

$$x_1^2 + x_2^2 - y_1^2 - y_2^2 = 0$$

has normalized profile the flat torus $S^1(\sqrt{\frac{1}{2}}) \times S^1(\sqrt{\frac{1}{2}})$ in $S^3(1)$. In dimension $n = 3$ it can easily be checked that there are no hypersurfaces satisfying the identity $Ric \equiv n-2$.

Proof of Theorem 2. From Gauss' equation the Ricci curvatures of the induced metric on M are the values of the quadratic form defined by $(Tr\ B)B - B^2 + (n-1)I$ on unit vectors tangent to M.

Let us assume

$$(**) \qquad \sup_M Ric < n-2$$

and write $c^2 = (n-1) - \sup_M Ric > 1$. It follows that at each point x of M the eigenvalues of B lie outside the interval $\left(\frac{H-\sqrt{H^2+4c^2}}{2}, \frac{H+\sqrt{H^2+4c^2}}{2}\right)$ where $H(x) = Tr\ B(x)$ is the mean curvature function evaluated at x. Clearly $0 \notin \Lambda$ and, by compactness of M and the continuity of the principal curvatures, the set Λ is compact and so must miss a neighbourhood of o in $\mathbb{R}$. If Λ misses $\mathbb{R}^+$ or $\mathbb{R}^-$ all principal curvatures are of one sign and it follows easily from Gauss' equation that $Ric > n-1$, contradicting our assumption above; hence Λ has at least two components, separated by o. This is the first important information carried by the Ricci curvature hypothesis $(**)$.

The normal variation $f_t = \cos t\ f + \sin t\ \xi$ of the immersion f is an immersion of M in $S^{n+1}(1)$ when $k = \cot t \notin \Lambda$ and its induced metric is given by the expression $g_t(X,Y) = \sin^2 t < (kI - B)X, (kI-B)Y >$. Its second fundamental form with respect to the unit normal field $\xi_t = \cos t\ \xi - \sin t\ f$ is

$$B_t = \frac{kB + I}{kI - B}$$

where quotient notation is used for inverses. The principal curvature set Λ_t of the immersion f_t of M in $S^{n+1}(1)$ is obtained by applying the obvious Moebius transformation to Λ, and so has the same connectivity as Λ.

Now the Ricci curvature hypothesis $(**)$ implies that $\hat{f} = f_{\frac{\pi}{2}}$ is an immersion with second fundamental form $\hat{B} = -B^{-1}$ and so the same principal directions as the immersion f. The metric $\hat{g}$ induced by $\hat{f}$ has sectional curvature $1 + \frac{1}{\lambda_i\lambda_j}$ on the plane spanned by any pair of principal directions $\{e_i, e_j\}$ of the immersion f, where $Be_i = \lambda_i e_i$. When λ_i and λ_j are of the same sign, this sectional curvature is > 1. When λ_i and λ_j have opposite signs, they lie on either side of the interval $\left(\frac{H-\sqrt{H^2+4c^2}}{2}, \frac{H+\sqrt{H^2+4c^2}}{2}\right)$, so that $\lambda_i\lambda_j \leq -c^2$ and therefore $1+\frac{1}{\lambda_i\lambda_j} \geq 1 - \frac{1}{c^2} > 0$. Hence the curvature operator of the metric $\hat{g}$ is strictly positive definite. Since $\hat{f} = \xi$ immerses M in $\mathbb{R}^{n+2}$ (i.e., codimension 2) this is equivalent to the positivity of all sectional curvatures, by Weinstein [13].

Lemma 1. *The restricted holonomy group of a connected hypersurface M in S^{n+1} is either $SO(n)$ or else the hypersurface is locally a product of a pair of round hyperspheres in complementary orthogonal subspaces and its holonomy splits as a product of orthogonal groups.*

Proof. To see this we first note that the assumption of holonomy distinct from $SO(n)$ implies that the Lie algebra generated by the curvature transformations $R_x(X,Y)$ at any point x of M is not the full Lie algebra of all skew-symmetric transformations of the tangent space at x. If $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of M_x diagonalising B_x, with $B_x e_i = \lambda_i e_i$ for each i, then by Gauss' equation, $R_{ij} = R(e_i, e_j) = (1 + \lambda_i\lambda_j)E_{ij}$, where E_{ij} is the skew-symmetric transformation given by

$$E_{ij}(e_s) = \delta_{js}e_i - \delta_{is}e_j.$$

Under the assumption that the curvature transformations do not generate the full Lie algebra of all skew-symmetric transformations of the tangent space M_x — and so some E_{ij} must not occur among the generators — we must have $1 + \lambda_i\lambda_j = 0$ for some distinct pair of principal curvatures. If there is a third λ_k distinct from these, then $1 + \lambda_i\lambda_k \neq 0$ and $1 + \lambda_j\lambda_k \neq 0$ and it is easily computed that

$$[R_{ik}, R_{jk}] = -(1 + \lambda_i\lambda_k)(1 + \lambda_j\lambda_k)E_{ij},$$

so that E_{ij} is in the holonomy algebra; this contradiction shows that there are precisely two distinct principal curvatures $\lambda > 0$ and $\mu < 0$. Thus the above holonomy assumption means

$$B_x = \lambda I_p + \mu I_q$$

relative to some orthogonal decomposition of the tangent space into a pair of orthogonal subspaces of dimensions p and q with $p+q = n$. By continuity the integers p and q are independent of x and, by Codazzi's equation for B, λ and μ are independent of x and nonzero. It is now easy to see that M is a piece of the standard Clifford product of round spheres $S^p(\frac{1}{\lambda}) \times S^q(\frac{1}{|\mu|})$ in $\mathbb{R}^{n+2}$ and so has holonomy $SO(p) \times SO(q)$. This ends the proof of Lemma 1.

Returning now to our hypersurface $\hat{f}$ of the earlier paragraph since the induced metric has strictly positive sectional curvature, Lemma 1 implies its holonomy is $SO(n)$. Since the curvature operator is even strictly positive definite, as noted earlier, and the holonomy is $SO(n)$, Meyer's theorem [8] guarantees that the universal cover of M is a rational homology sphere.

For a compact submanifold M^n immersed in $\mathbb{R}^{n+p}$ the height function corresponding to $a \in S^{n+p-1}(1)$ is a Morse function for almost all a and we denote by $\mu_k(a)$ the number of nondegenerate critical points of index k of each such Morse function. The *total curvature of index k*, first introduced by Kuiper [7], is given by

$$\tau_k = \frac{1}{\mathrm{Vol}S^{n+p-1}(1)} \int_{S^{n+p-1}(1)} \mu_k(a).$$

When M^n is immersed with positive sectional curvature in euclidean space with codimension $p = 2$ (as with the immersion $\hat{f} = \xi$ in the last paragraph) Chen [3] established the inequality

$$\tau_1 + \tau_2 + \cdots + \tau_{n-1} < \tau_0 + \tau_n$$

among the total curvatures. In the circumstance of $SO(n)$ holonomy, Moore [10] used Chen's inequality and the fact that the universal cover of M is a rational homology sphere to show that M^n itself must be a homotopy sphere in dimensions $n \geq 5$ and so a topological sphere for $n \neq 3$ and 4. But in these latter dimensions Baldin-Mercuri [1] produced Morse functions with only two critical points so that M^n is a topological sphere in all dimensions, and so (ii) of Theorem 2 holds.

However, the Ricci curvature hypothesis $(**)$ means Λ is not connected at 0, and so it follows that the tangent bundle of M splits into the subbundles on which B is positive definite and negative definite. Since such a splitting is impossible for an even dimensional sphere, (i) is proved.

Let $a \in \mathbb{R}^{n+2}$ be any unit vector and consider its height function $h =< f, a >$ on M. Its Hessian operator

$$\mathrm{Hess}_h : X \longrightarrow \nabla_X(\nabla h),$$

where ∇h is the gradient of h computed with respect to the induced metric g on M, is easily found to be

$$\mathrm{Hess}_h =< \xi, a > B - < f, a > I.$$

If now a compact immersed hypersurface lies in the closed hemisphere with pole $a \in S^{n+1}(1)$, then from the above formula for the Hessian operator $h =< f, a >$ on M we see that its second fundamental form is semi-definite at points where the height function has an absolute minimum. Recalling that the second fundamental forms of the immersions f and $\hat{f} = \xi$ are, respectively, B and $-B^{-1}$ — and therefore both have principal curvature sets split by 0 — neither of the images $f(M)$ or $\xi(M)$ can be contained in a closed hemisphere. Together with the earlier remarks on the sectional curvature of the immersion $\hat{f} = \xi$ this completes (iii).

Let the range of the function H on M be $[\alpha, \beta]$. From the Ricci curvature hypothesis $(**)$, we see that $\Lambda \cap \left(\frac{-c^2}{\psi(\beta)}, \psi(\alpha)\right) = \emptyset$, where ψ is the function defined by $\psi(x) = \frac{x+\sqrt{x^2+4c^2}}{2}$. It is easily seen that $\frac{\psi(\beta)}{\psi(\alpha)} < e^{\frac{\beta-\alpha}{2c}}$ for all α and β. Assume $\beta - \alpha \leq 4c \log c$. Then it follows that $\frac{\psi(\beta)}{\psi(\alpha)} < c^2$ and from the previous property of Λ there exists $k > 0$ such that $\left[-\frac{1}{k}, k\right]$ does not meet Λ. Now choose $t > 0$ such that $\cot t = k$ and it follows that f_t is an immersion with $\Lambda_t \subset \mathbb{R}^-$, i.e., f_t has strictly negative definite second fundamental form. By do Carmo and Barbosa [2], f_t immerses M as a strictly convex hypersurface in $S^{n+1}(1)$, i.e., as the boundary of a compact convex set contained in an open hemisphere of $S^{n+1}(1)$. Since Λ —

and therefore Λ_t — is known to be disconnected from $(**)$, as we saw earlier, we obtain a contradiction to $\beta - \alpha \leq 4c \log c$ once we prove the following lemma and (iv) is then proved.

Lemma 2. *A complete convex hypersurface of* $S^{n+1}(1)$ *has connected principal curvature set.*

Proof. Let $F : M^n \longrightarrow S^{n+1}(1)$ be a complete convex hypersurface. Its second fundamental form B, with respect to a global unit normal field ξ, may be assumed positive semi-definite. From Gauss' equation the Ricci curvature is positive and bounded away from zero, so that M is compact by Myer's theorem. Assuming the principal curvature set Λ is not connected, we may choose $t > 0$ such that $k = cot\ t \notin \Lambda$ and splits Λ. It is now easy to see that the immersion F_t has a nowhere singular and nowhere definite second fundamental form B_t. Since F is convex we may choose a unit vector $a \in \mathbb{R}^{n+2}$ such that the height function $h =< F, a >$ is Morse and has only two critical points.

The Hessian operator of h is given by

$$\text{Hess}_h =< \xi, a > B - < F, a > I$$

and from the relations between F_t, ξ_t and B_t and F, ξ and B we easily calculate Hess_{h_t} (where $h_t =< F_t, a >$) with respect to the F_t - induced metric as

$$\text{Hess}_{h_t} = (k^2 + 1) \sin t \cdot \frac{\text{Hess}_h}{(kI - B)}$$

Now h_t has the same critical set as h and by the previous equation neither of these two points is a maximum or minimum of h_t since k splits Λ and, in consequence, $kI - B$ is nowhere definite. This contradiction ends the proof of the lemma, and so, of Theorem 2 also.

Remark 3: Let V be a smooth immersed hypersurface in $\mathbb{R}^{m+1}$ with an isolated singularity at o with V intersecting all small spheres $S^m(r)$ tranversally and $x : N^m \longrightarrow \mathbb{R}^{m+1}$ the corresponding map introduced earlier in §2. The smooth functions r and ϕ, the unit vector field e and the profile Gauss map ξ were defined there also. Then the "dual" map $\tilde{x} = r\xi = r\{\cos\phi\eta - \sin\phi x_*(e)\}$ from N^m to $\mathbb{R}^{m+1}$ is everywhere orthogonal to $\tilde{\eta} = x_*(e)$. If the second fundamental form of each profile hypersurface of x is nonsingular then $\tilde{x}$ is an immersion with a simple singularity at o and the corresponding entities for the immersion $\tilde{x}$ are $\tilde{r} = r$ and $\tilde{\phi} = \phi$ and

$\tilde{e} = e$ and $\tilde{x}_*(\tilde{e}) = \eta$. Now if the simple singularity defined by x violates the bound in Theorem 1 then its profiles have nonsingular second fundamental forms and, by the proof of Theorem 2 above, a metric with positive curvature operator is obtained canonically on each profile by restricting the immersion $\tilde{x}$ restricted that profile,

$$\tilde{x}_r = \big|_{M^{m-1}(r)} : M^{m-1}(r) \longrightarrow S^m(r).$$

An Efimov theorem for complete hypersurfaces of a sphere is now easily developed from the argument of Theorem 2, above.

Let $f : M^n \longrightarrow S^{n+1}(1)$ *be a smooth immersion of a smooth complete orientable* n*-manifold into the unit sphere* $S^{n+1}(1)$ *(*$n \geq 3$*) with sectional curvature bounded away from* $-\infty$*. Then either* $\sup_M Ric \geq n-2$ *or else* M^n *diffeomorphic to* $\mathbb{R}^n$ *or else* n *is odd and* M^n *is homeomorphic to a sphere.*

The proof is as follows. Assume that the Ricci curvature of the induced metric fails to satisfy the inequality in the statement. Then $c^2 = (n-1) - \sup_M Ric > 1$. The compact case is already taken care of by Theorem 2 and, if H is bounded, the proof in Theorem 2 works verbatim to show that M must be an odd-dimensional topological sphere. Thus we will be assuming that H is unbounded from here on.

By Gauss' equation the Ricci condition amounts to saying that at each point x of M the eigenvalues of the second fundamental form B lie outside the interval $\left(\frac{H-\sqrt{H^2+4c^2}}{2}, \frac{H+\sqrt{H^2+4c^2}}{2}\right)$ where $H = Tr\ B$ is the mean curvature function. In particular $0 \notin \Lambda$ and the numbers p and q of positive and negative principal curvatures, which satisfy $p+q=n$, are independent of x and neither is zero. Note that since the sectional curvature K is bounded below, the assumption $sup_M Ric < n-2$ on the Ricci curvature implies that K must be bounded above as well.

If $\sup H = \infty$ then $p=1$, since otherwise there is a sequence of points along which sectional curvatures approach $+\infty$; similarly if $\inf H = -\infty$ then $q=1$. Since $n \geq 3$, H cannot take all real values. Thus, by the preceding remarks, after the appropriate choice of unit normal we may assume H bounded below, and $\sup H = +\infty$, so that $p=1$ and $q \geq 2$.

From the first of these last two conditions on H and the Ricci curvature condition, $\inf \Lambda^+ = b > 0$ and from the second that $\sup \Lambda^+ = +\infty$. Then $\sup \Lambda^- = 0$, since otherwise — using Gauss' equation — we would have $\inf K = -\infty$. Furthermore $\inf \Lambda^- = -a$, where a is finite and positive, since again $\inf K = -\infty$ otherwise.

Now choosing t so that $0 < k = \cot t < b$ it is easily checked that the immersion $f_t : M^n \longrightarrow S^{n+1}(1)$ has induced metric $g_t \geq cg$ for some positive constant c and so the metric g_t is complete. Furthermore the sectional curvatures of the metric g_t on the principal planes $\{e_i, e_j\}$ of the original immersion are all positive. Indeed the sectional curvature of this principal plane in the metric g_t is $\frac{(1+k^2)(1+\lambda_i\lambda_j)}{(k-\lambda_i)(k-\lambda_j)}$ and since $0 < k = \cot t < b$ this is positive when λ_i and λ_j have the same sign; it is likewise positive when λ_i and λ_j have opposite signs since then $1+\lambda_i\lambda_j < 1-c^2 < 0$ by the Ricci condition. Thus g_t is both complete and has positive definite curvature operator, when $0 < k = \cot t < b$. By the theorem of Gromoll-Meyer [5], M must be diffeomorphic to $\mathbb{R}^n$.

Remark 4: The extension of the above result to $n = 2$, that is, whether the Gaussian curvature of a complete surface in the three sphere must satisfy $\sup K \geq 0$, is open (i.e., does Efimov's theorem hold for complete surfaces in the 3-sphere?).

References

1. Y. Y. Baldin and F. Mercuri, *Isometric immersions in codimension two with non-negative curvature*, Math. Z. **173** (1980), 111 - 117.
2. M. do Carmo and F. W. Warner, *Rigidity and convexity of hypersurfaces in spheres*, J. Diff. Geom. **4** (1970), 133-144.
3. C. S. Chen, *On tight isometric immersions in codimension two*, Amer. J. Math. **94** (1972), 974-990.
4. N. V. Efimov, *Hyperbolic problems in the theory of surfaces*, Proc. Int. Congress Math. Moscow 1966 (Amer. Math. Soc. Translation) **70** (1968), 26-38.
5. D. Gromoll and W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. **90** (1969), 75-90.
6. D. Hilbert, *Über Flächen von konstanter Gausscher Krümmung*, Trans. AMS **2** (1901), 87-99.
7. N.H. Kuiper, *Minimum total absolute curvatures for immersions*, Invent. Math. **10** (1970), 209-238.
8. D. Meyer, *Sur les variétés riemanniennes a operateur de courbure positif*, C.R. Acad. Sci. Paris **272** (1971), 482-485.
9. J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, 1965.
10. J. D. Moore, *Codimension two submanifolds of positive curvature*, Proc. AMS **70** (1978), 72-74.
11. J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. **88** (1968), 62-105.
12. B. Smyth and F. Xavier, *Efimov's theorem in dimension greater than 2*, Invent. Math. **90** (1987), 443-450.
13. A. Weinstein, *Positively curved n-manifolds in* $\mathbb{R}^{n+2}$, J. Diff. Geom. **4** (1970), 1-4.
14. H. Wu, *The spherical images of convex hypersurfaces*, J. Diff. Geom. **9** (1974), 279-290.