NON-EXISTENCE OF ELLIPTIC SECTORS IN THE PRINCIPAL FOLIATIONS OF SURFACE THEORY

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Abstract. The character of the principal foliations on a smooth surface in $\mathbb{R}^3$ is not well understood, in particular, whether there are geometric or topological restrictions on their singularities (umbilics) distinguishing them from the singularities of arbitrary smooth foliations on an abstract smooth surface. The longstanding Carathéodory Conjecture hypothesizes one such restriction. This conjecture would follow, by Bendixson's formula, from non-existence of elliptic sectors in the principal foliations at any isolated umbilic. Evidence for the nonexistence of such sectors in principal foliations is the main result.

1. Introduction

The principal foliations on a smooth surface in $\mathbb{R}^3$ were of classical geometric interest but their study has implications beyond the geometry of surfaces. Already a century ago Gullstrand [3] noted their connexion with optics. The relevance of their study to the solution of longstanding problems in general relativity and three-dimensional Riemannian geometry emerged recently in the work of Bessières-Lafontaine-Rozoy [1] and its relevance for the solution of a longstanding problem in elasticity is seen in the recent work of Smyth [8]. Of these last two works, the former is linked with our "generic" solution [9] of the Carathéodory Conjecture, a problem which has historically eclipsed the general study of the principal foliations. The Carathéodory Conjecture is first mentioned in Cohn-Vossen's report [2] to the ICM at Bologna in 1928 and its formulation for smooth surfaces appears in Hamburger [4].

The conjecture says that any smooth immersion of the sphere in $\mathbb{R}^3$ must have at least two umbilic points; the (stronger) local form of this conjecture states that the index j of the principal foliations at an umbilic is ≤ 1 and this is the route taken by all attempts. Many of these attempts confine themselves to surfaces analytically immersed in $\mathbb{R}^3$ and the difficulties and gaps attending the analytic side of that endeavour were recently taken up in Ivanov [6], where a proof of the conjecture for the analytic case is presented. The conjecture is open for smooth surfaces.

Looking at the local Carathéodory Conjecture again, it clearly posits a special topological behaviour for the principal foliations near an isolated umbilic, a behaviour not enjoyed by arbitrary smooth foliations near an isolated singularity. Seventy-five years after Cohn-Vossen's announcement [2], it is still not clear what might be the topological or geometric source for such an expectation. So it may be time to revisit the topic.

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In this spirit, Lazarovici [7] was the first to note the fundamental result that there exist smooth singular foliations of a neighbourhood of the origin in $\mathbb{R}^2$ (of every topological index) no diffeomorph of which occur as principal foliations on any smooth surface in $\mathbb{R}^3$. The smooth singular foliations in his examples all have an elliptic sector. Is the presence of an elliptic sector at an isolated singularity in a smooth foliation on a smooth surface an obstruction to its realization as a principal foliation?

Bendixson's formula for the index of an isolated singularity of a smooth foliation says that

$$j = 1 + \frac{e-h}{2},$$

where e and h are the number of elliptic and hyperbolic sectors of the foliation at the singularity [5]. Thus one attractive way of accounting for Carathéodory's Conjecture would be the non-existence of elliptic sectors in the principal foliations at an isolated umbilic on any smooth surface in $\mathbb{R}^3$. Such a conjecture is not only stronger than the Carathéodory Conjecture but of more geometric interest.

No example of an elliptic sector occurring in a principal foliation is known to the author and the spirit of our main result is that if elliptic sectors occur they must be complicated. The method of proof here is direct and simple. The key idea in the proof is that *the presence of an elliptic sector in the principal foliations results in the appearance of an integral equation whose solvability is more or less equivalent to Codazzi's equation on that sector* (see §2). Our result, Theorem 1, is a direct consequence of this integral equation.

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2. An integral equation for principal elliptic sectors

Our interest here is the following:

> **The elliptic sector conjecture:** *At an isolated umbilic in a smooth surface in* $\mathbb{R}^3$ *the principal foliations cannot have an elliptic sector.*

The first hint of the existence of such a result came in the work of Laurentiu Lazarovici [7], who showed that no diffeomorph of the standard dipole foliation (an example with $j = 2$) could occur as a principal foliation on any surface in $\mathbb{R}^3$. His proof is rather technical using a standard local normalization of the principal foliations — under the Gauss map — as Hessian foliations in $\mathbb{R}^2$ and then applying Riemann's method in hyperbolic p.d.e to these Hessian foliations combined with strong convexity conditions to clinch the result. Our approach here will be much simpler, depending critically on Codazzi's equation and being carried out on the surface itself; it also gives a far stronger result.

If $f : M \longrightarrow \mathbb{R}^3$ is a smooth immersion of a smooth oriented surface M in $\mathbb{R}^3$, we denote by g the induced metric, ξ the oriented unit normal field along the immersion f, and A the second fundamental form of ξ; A is defined by $X\xi = -f_{*_p}(A_pX)$ for all vectors $X \in M_p$, the tangent space to M at p, where f_{*_p} is the differential of f at p. The inner product space (M_p, g_p) has orientation coming from M and counterclockwise rotation through $\frac{\pi}{2}$ defines the complex structure J_p. The metric connexion of g is denoted ∇ and the important equation for us here is Codazzi's equation

$$(\nabla_X A)Y - (\nabla_Y A)X = 0.$$

$H = \frac{1}{2}TrA$ and $K = detA$ denote the mean curvature and the Gauss curvature, respectively, and we denote by $\sigma = 2\sqrt{H^2 - K}$ the square root of the discriminant of A.

On the complement of the umbilic set the eigenvalues of A are denoted by λ and μ, with $\lambda > \mu$. The eigenspaces of A give two smooth, mutually orthogonal foliations (not necessarily orientable) on the complement of the umbilic set. These are the principal foliations of the immersed surface; the foliation determined by λ will be denoted $\mathcal{F}^+$ and that determined by μ will be denoted $\mathcal{F}^-$.

Let p_0 be an isolated umbilic of the immersion f and E an elliptic sector of $\mathcal{F}^+$ at p_0. Then E is bounded by a leaf of $\mathcal{F}^+$, born and dying at p_0, with no other umbilic within E. Hence $\mathcal{F}^+$ is orientable on E. We orient $\mathcal{F}^+$ so that it gives the standard orientation to the boundary ∂E. The orientation of $\mathcal{F}^+$ is given by a unit field e_1 on E and $e_2 = Je_1$ orients $\mathcal{F}^-$ on E. The curvatures of the oriented leaves of $\mathcal{F}^+$ are given by $g(\nabla_{e_1} e_1, Je_1) = k_1$ and those of $\mathcal{F}^-$ by $g(\nabla_{e_2} e_2, Je_2) = k_2$.

Each $p \in E$ determines a unique oriented leaf segment of $\mathcal{F}^+$ which is born at p_0 and ends at p; this is denoted $C^+(p)$. Its length $l_1(p)$ is a positive *extended* real-valued function on E. Similarly $C^-(p)$ is the unique oriented leaf of $\mathcal{F}^-$ which begins at p and dies at p_0 and its length is $l_2(p)$. The functions l_1 and l_2 are extended real-valued functions on E. The region $L(p)$ with positively oriented boundary $C^+(p) + C^-(p)$ is called *the lens determined by p*.

If in the Codazzi equation above we replace X by e_1 and Y by e_2, we obtain from the definition of the curvatures k_1 and k_2 above

$$e_1(\mu) = (\lambda - \mu)k_2$$

and

$$e_2(\lambda) = (\lambda - \mu)k_1.$$

Let p be any point in the elliptic sector E. Integrating the first of these equations along the oriented leaf $C^+(p)$ of the $\mathcal{F}^+$-foliation through p which is born at p_0 we have

$$\mu(p) - \mu(p^+) = \int_{C^+(p^+,p)} (\lambda - \mu)k_2 ds,$$

where p^+ is any point on $C^+(p)$ between p_0 and p and the integration on the right takes place along the subarc $C^+(p^+, p)$ of $C^+(p)$ from p^+ to p. By continuity we have

$$\mu(p) - \mu(p_0) = \int_{C^+(p)} (\lambda - \mu)k_2 ds,$$

where the improper integral on the right converges. If instead we had integrated the second of the Codazzi equations above we would, in like manner, arrive at the equation

$$\lambda(p_0) - \lambda(p) = \int_{C^-(p)} (\lambda - \mu)k_1 ds,$$

where $C^-(p)$ is the oriented leaf of the $\mathcal{F}^-$-foliation through p which dies at p_0. Since p_0 is an umbilic, adding these last two equations we obtain

$$-(\lambda - \mu)(p) = \int_{C^+(p)} (\lambda - \mu)k_2 ds + \int_{C^-(p)} (\lambda - \mu)k_1 ds,$$

with all the improper integrals on the right converging whether the the arcs $C^+(p)$ and $C^-(p)$ are rectifiable or not. Thus in the presence of an elliptic sector E in the principal foliations on a smooth surface, Codazzi's equation gives rise to an integral equation (with no rectifiability assumptions) for $\sigma = \lambda - \mu$ which must hold for each $p \in E$.

Lemma 1. *Let p_0 be an isolated umbilic of a smooth surface smoothly immersed in $\mathbb{R}^3$. If there exists an elliptic sector E in one of the principal foliations at p_0, then for every lens $L(p)$ in E we have, with the notation above, the integral equation*

$$-\sigma(p) = \int_{C^+(p)} \sigma k_2 ds + \int_{C^-(p)} \sigma k_1 ds,$$

where $\sigma = \lambda - \mu = 2\sqrt{H^2 - K}$ and both improper integrals in the equation converge.

In particular, if C^+ (resp. C^-) is a full leaf of the $\mathcal{F}^+$-foliation (resp. $\mathcal{F}^-$-foliation) in an elliptic sector of that foliation then $\int_{C^+} \sigma k_2 ds = 0$ (resp. $\int_{C^-} \sigma k_1 ds = 0$).

We remark that the lemma holds more generally. Let A be a smooth symmetric operator satisfying Codazzi's equation on a smooth Riemannian 2-manifold with the eigenfoliations of A having an isolated singularity at p_0. If an eigenfoliation of A has an elliptic sector at p_0 then the same integral equation holds for the square root of the discriminant of A.

Conversely suppose we are given a smooth singular foliation on a Riemannian 2-manifold with an isolated singularity at p_0 and an elliptic sector E at that singularity. Orienting the leaves in E by the unit vector field e_1 which gives the standard orientation to the boundary of E, the oriented orthogonal field is denoted e_2. For any $p \in E$ the leaves $C^+(p)$ and $C^-(p)$ are as defined above. Now suppose σ is any continuous solution of the above integral equation which is smooth *inside* E — so that the improper integrals appearing there are implicitly convergent. We then define

$$\lambda(p) = -\int_{C^-(p)} \sigma k_1 ds + C,$$

and

$$\mu(p) = \int_{C^+(p)} \sigma k_2 ds + C,$$

where C is any real constant and note the identity $\sigma = \lambda - \mu$ from the integral equation. If we define

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with respect to the frame field $\{e_1, e_2\}$ on E, then A is a smooth symmetric operator on the Riemannian 2-manifold and satisfies the Codazzi equation inside E and A is uniquely determined by σ to within a constant multiple of the identity. Any umbilics of A inside E are automatically removable and there are none if σ does not vanish inside E. Also σ is the square root of the discriminant of the smooth symmetric tensor field A inside E which has the given foliation as an eigenfoliation and satisfies Codazzi's equation inside E

Since $\sigma > 0$ on the elliptic sector E there is an immediate contradiction from the lemma *if k_1 and k_2 are non-negative on some lens* $L(p)$ in this elliptic sector, thereby giving Lazarovici's result (Main Theorem, [7]). A closer analysis of our integral equation provides us with a more general result:

Theorem 1. *Let p_0 be an isolated umbilic of a smooth surface smoothly immersed in $\mathbb{R}^3$. If there exists an elliptic sector E in one of the principal foliations at p_0, then for every lens $L(p)$ in E either*

(a) $\sup_{L(p)}\{l_1, l_2\} = \infty$

or

(b) $\inf_{L(p)}\{k_1, k_2\} = -\infty$.

Proof. Let E be an elliptic sector at p_0 in the principal foliation corresponding to the eigenvalue λ, let $p \in E$ be different from p_0 and let $L(p)$ be the lens determined by p. Assume *both* (a) and (b) are violated for the lens $L(p)$. Since (b) is violated for the lens $L(p)$ there exists a real constant $m > 0$ such that $k_1, k_2 \geq -m$ on $L(p)$. Since σ is non-negative, this means $\sigma k_1, \sigma k_2 \geq -m\sigma$ at each point of $L(p)$. Since p_0 is an isolated umbilic, σ never vanishes on the compact set $L(p)$ other than at p_0 and so attains a positive maximum on $L(p)$ at some point $p' \in L(p)$. Now since $\sigma(p') \geq \sigma$ on $L(p')$ we have $\sigma k_1, \sigma k_2 \geq -m\sigma \geq -m\sigma(p')$ on $L(p')$. Now, since (a) is violated on $L(p)$ it is also violated on the subset $L(p')$. Applying the integral equation of Lemma 1 to the lens $L(p')$ we obtain

$$-\sigma(p') = \int_{C^+(p')} \sigma k_2 ds + \int_{C^-(p')} \sigma k_1 ds \geq -m\sigma(p')l_1(p') - m\sigma(p')l_2(p')$$

where $l_1(p')$ and $l_2(p')$ are finite, giving

$$\sigma(p') \leq m\sigma(p')l(\partial L(p')),$$

where $l(\partial L(p'))$ denotes the length of the boundary $\partial L(p')$ of $L(p')$. Thus,

$$ml(\partial L(p')) \geq 1$$

for the lens $L(p')$. Write $p' = q_1$. In the lens $L(q_1)$ choose arbitrarily an internal point p_1. The function σ on $L(p_1)$ assumes a maximum at a point p_1' in this latter lens. The *same* lower bound $-m$ for the curvatures k_1, k_2 on $L(p)$ holds on $L(p_1')$. Applying the same reasoning as before to this smaller lens we obtain $ml(\partial L(p_1')) \geq 1$. Write $p_1' = q_2$. Iterating this proceedure we obtain a sequence of points $\{q_i\}$ in $L(p)$ converging to p_0 such that $ml(\partial L(q_i)) \geq 1$ for all i. Thus the quantities $l(\partial L(q_i))$ are bounded away from zero for all i. Passing to a subsequence (which we also denote $\{q_i\}$), if necessary, which converges to p_0 we may assume either $l_1(q_i) \geq \frac{1}{2m}$ for all i or $l_2(q_i) \geq \frac{1}{2m}$ for all i. In the first case it follows that $L(p)$ contains in the limit a full leaf of $\mathcal{F}^+$; this contradicts the fact that $L(p)$ is a lens in the elliptic sector E. The other case is similar. □

In working toward the conjecture considered here, an important advance would be the removal of the "positivity" curvature assumptions in the theorem above (i.e., there are no "rectifiable" elliptic sectors). Note that intuition suggests that k_2 trends toward positivity at the beginning of an e_1-leaf. A more proximate goal would be to use the integral equation to derive the non-existence of elliptic sectors for analytically immersed surfaces in $\mathbb{R}^3$. The principal foliations are given by a complex-valued quadratic differential (the Hopf differential) which, in that case, is real analytic and the goal would be to analyze whether (a) or (b) can occur.

References

[1] L. Bessières, J. Lafontaine, L. Rozoy, *Scalar curvature and black holes*, Preprint, February 2003

[2] St. Cohn-Vossen, Argomenti delle Comunicazioni, p.45, Congresso Internazionale dei Matematici, Bologna, 1928.

[3] A. Gullstrand, *Zur Kenntnis der Kreispunkte*, Acta Math. **29**, 59–100 (1905)

[4] H. L. Hamburger, *Beweis einer Carathéodoryschen Vermütung. I*, Ann. of Math. **41**, 63–86 (1940)

[5] P. Hartman, *Ordinary differential equations,* John Wiley, New York, 1964.

[6] V. V. Ivanov *The Analytic Carathéodory Conjecture*, Siberian Math. J. **43**, 251–322 (2002)

[7] L. Lazarovici, *Elliptic sectors in surface theory and the Carathéodory-Loewner conjectures*, J. Differential Geom. **55**, 453–473 (2000)

[8] B. Smyth, *Soliton surfaces in the mechanical equilibrium of closed membranes*, Comm. Math. Phys. To appear (2004)

[9] B. Smyth, F. Xavier, *A geometric estimate for the index of an umbilic point on a smooth surface*, Bull. London Math. Soc. **24**, 176-180 (1992)

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